NP Completeness Reductions

Vertex Cover $3SAT \leq_p VC$

3SAT - the language of all satisfiable boolean formulas in 3CNF

Example

$$(x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3})$$ ← is satisfiable

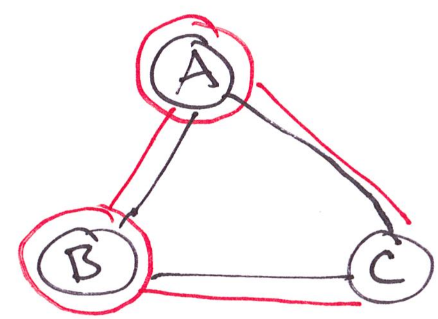
$$(x_1 \vee x_1 \vee x_1) \wedge (\overline{x_1} \vee \overline{x_1} \vee \overline{x_1})$$

↑

is not satisfiable

Vertex Cover

Given a graph, $G = (V, E)$, is there a subset V' of V with no more than k elements such that for every $e = (v, u) \in E$ either $u \in V'$, $v \in V'$ or both



$VC \in NP$

we can check a proposed V' in poly time

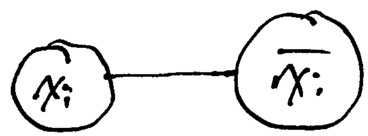
$3SAT \leq_p VC$

- let $U = \{x_1, x_2, \ldots x_n\}$ and $C = \{C_1, C_2, \ldots C_m\}$ describe a 3SAT instance (Boolean formula)

- Construct a graph $G=(V,E)$ and a positive integer $k \le |V|$ such that G has a vertex cover of size $\le k$ iff all clauses in C are satisfiable.

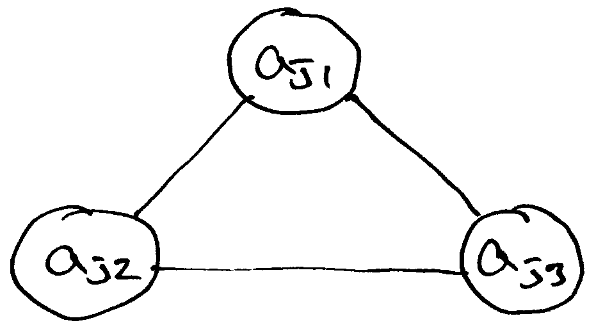
- we construct a number of gadgets
  - truth-setting gadget
  - satisfaction-testing gadget

- for each $x_i \in U$ ~~in~~ we include in G a pair of nodes labelled x_i and $\overline{x_i}$ and a single edge between them



- For each clause $c_j \in C$ we include in G 3 vertices connected by edges labelled with the 3 literals in c_j

$$C_j = (a_{j1} \vee a_{j2} \vee a_{j3})$$

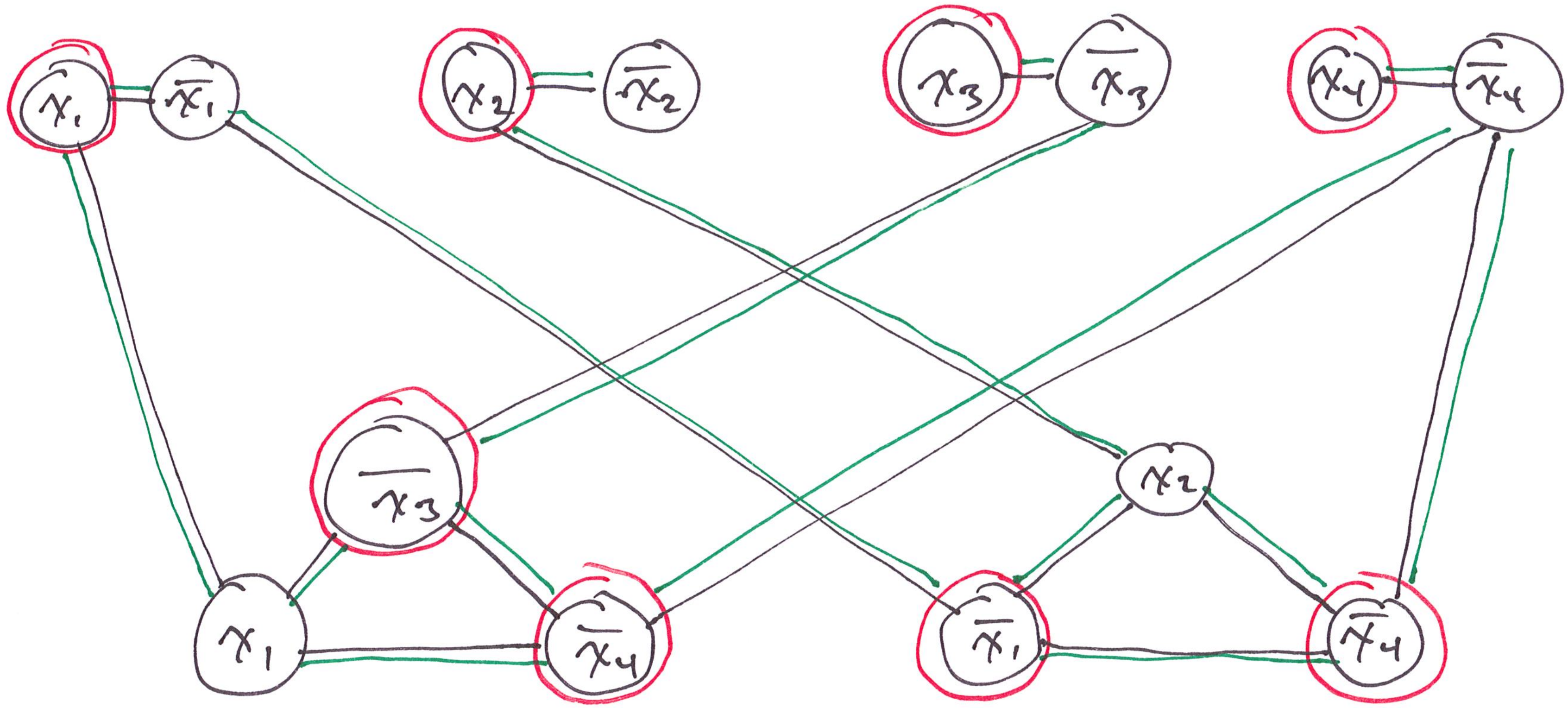


- For each clause $c_j \in C$ where $c_j = (l_1 \vee l_2 \vee l_3)$ and each l_K is a literal, include edges
  $E_{j1} = (a_{j1}, l_1)$, $E_{j2} = (a_{j2}, l_2)$, $E_{j3} = (a_{j3}, l_3)$

- Finally choose $K = n + 2m$

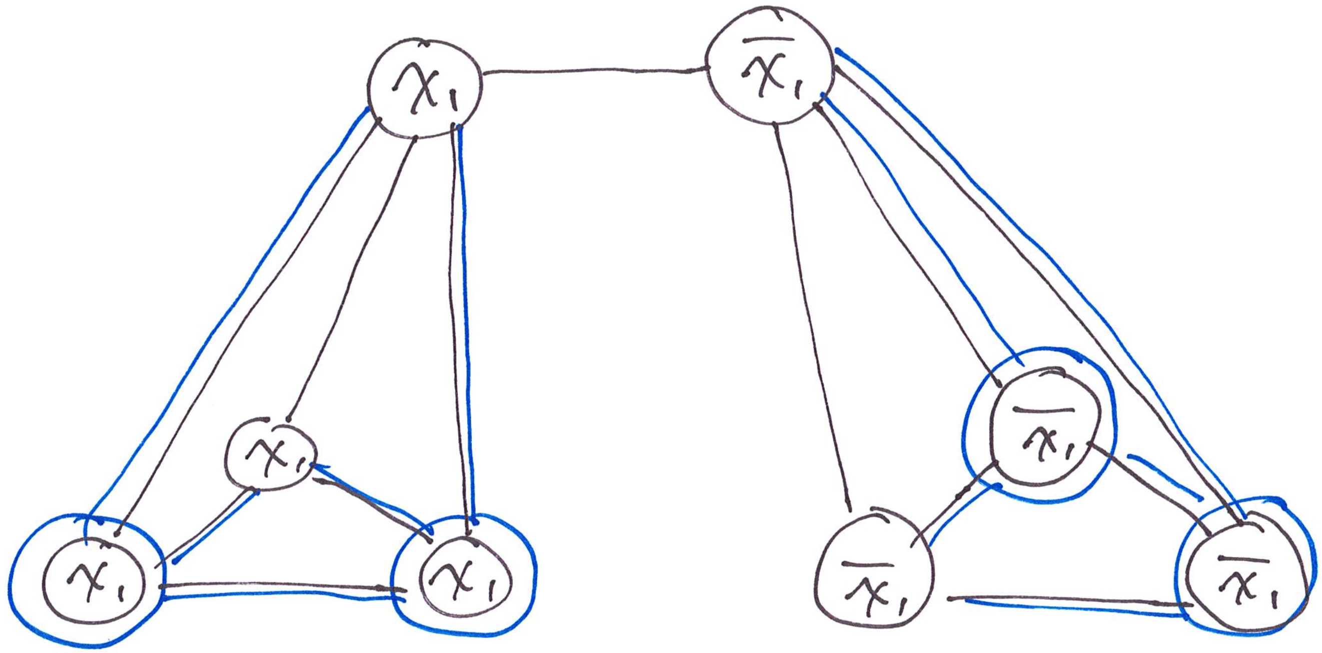
<u>Example</u>: $\Phi = (x_1 \vee \overline{x_3} \vee \overline{x_4}) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_4})$

$k = 4 + 2 \cdot 2 = 8$



Example 2: $\phi = (x_1 \vee x_1 \vee x_1) \wedge (\bar{x}_1 \vee \bar{x}_1 \vee \bar{x}_1)$

$k = 1 + 2 \cdot 2 = 5$



- To show the construction is valid
  - suppose $V' \subseteq V$ is a vertex cover of G with $|V'| \leq k = n + 2m$
  - V' must contain at least one vertex from each truth-setting gadget
  - V' must contain at least two vertices from each satisfaction testing gadget
  - Since this gives a total of at least k vertices already, we conclude V' contains exactly one vertex from each truth-setting gadget and two from each satisfaction testing gadget.

- The choice of vertex in each truth-setting gadget in V' induces a truth assignment

  set x_i = True if $x_i \in V'$ and x_i = False if $\bar{x}_i \in V'$

- This truth assignment satisfies each clause in C

- As observed exaclty two vertices a_{j1}, a_{j2}, a_{j3} are in V' so only two of the ~~the~~ edges E_{jk} can be covered by those vertices

- The third one must be covered by the vertex l_i. This means $l_i \in V'$ i.e. l_i evaluates to True hence C_j is satisfied

Conversly

- suppose we have a satisfying assignment for each clause in C
- let V' include each x_i such that x_i = True and each $\overline{x_i}$ such that x_i = False
- since each C_j is satisfied, this means at least one of the three E_{jk} is covered by vertex l_i
- To cover the other two edge E_{jk} as well as the ~~trai~~ triangle edges we can pick any two vertices a_{jk}
- The overall size of V' is $2m + n$