NP-Complete Problems

What to do?

- Approximate: get provably close to optimal
- Constrain the problem to an easier subset
- Only solve small instances
- Give up
- Huerisitic: follow some "good" logical rule.

Approximation

- an algorithm for a problem of size n has an approximation ratio $\rho(n)$ if for any input the algorithm produces a solution of cost C such that $\max\left(\frac{C}{C_{opt}}, \frac{C_{opt}}{C}\right) \leq \rho(n)$

Vertex Cover

- Given an undirected graph $G = (V, E)$, find a subset $V' \subseteq V$ such that if $(u,v) \in E$ then either $u \in V'$, $v \in V'$, or both. Find a V' of minimum size.

Greedy-VC

GreedyVC(G)

$V' \leftarrow \emptyset$

while E is not empty

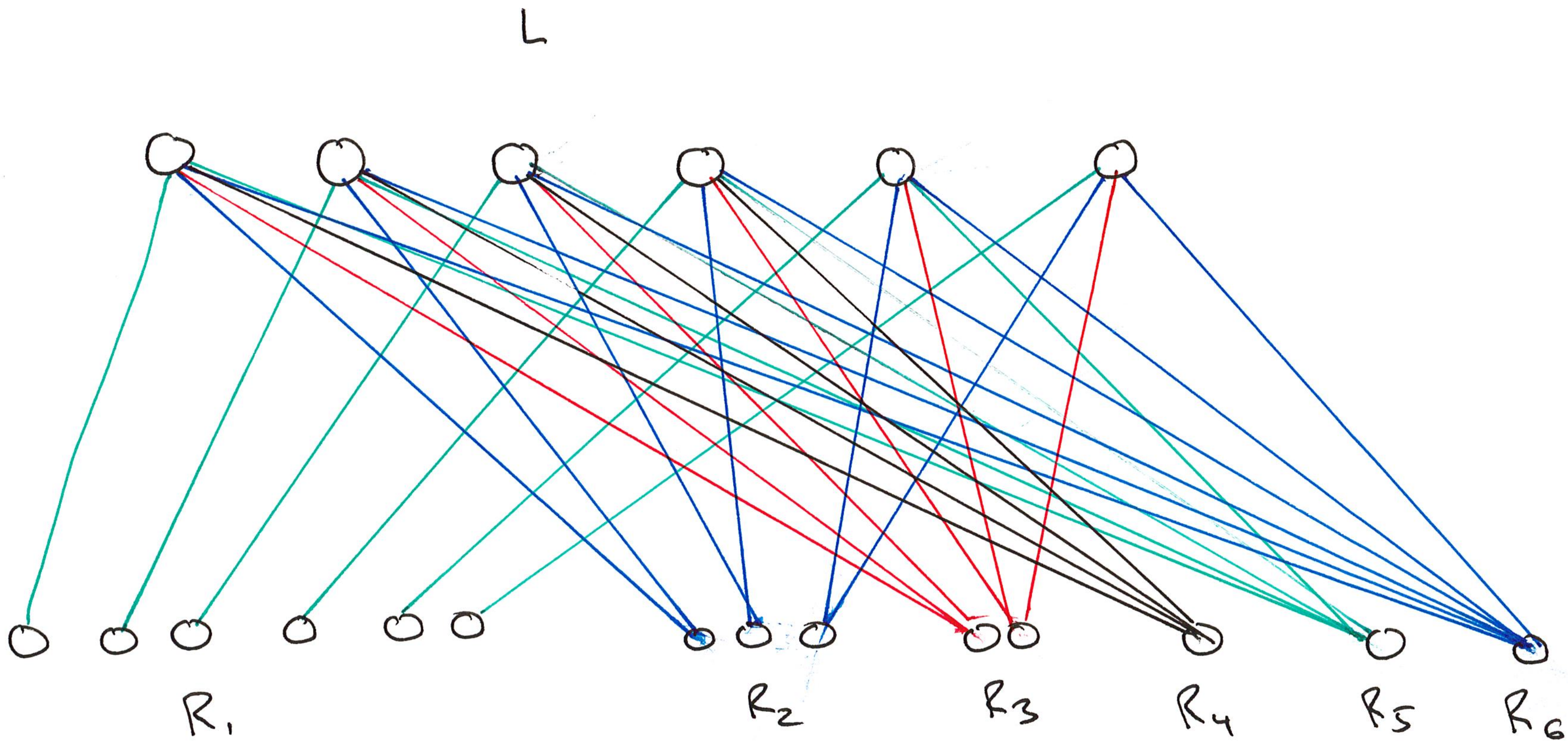
pick a vertex $v \in V$ of maximal degree

$V' \leftarrow V' \cup \{v\}$

$E \leftarrow E \setminus \{e \in E \mid v \in e\}$

return V'

- consider a bipartite graph with a set L of t nodes on the left and then a collection of sets R_1 R_2 $R_3 \ldots$ of nodes on the right.
- Each R_i has $\lfloor t/i \rfloor$ nodes
- This graph has $n \in \Theta(t \log t)$ nodes
- Connect each set R_i to L so that each $v \in R_i$ has i neighbors in L and no two vertices in R_i share any neighbors in common.



- The optimal VC is L of size t
- The greedy algorithm might first choose R_t then R_{t-1} and so on
- This achieves a $O(\log n)$ approximation ratio

Dumb Vertex Cover

DVC(G)

$V' \leftarrow \emptyset$

while E is not empty

$(u,v) \leftarrow$ any edge in E

$V' \leftarrow V' \cup \{u,v\}$

$E \leftarrow E \setminus \{$all edges incident to u or v$\}$

return V'

Theorem

DVC is a 2-approximation algorithm for VC.

Proof

- let A be the set of edges picked by DVC
- The optimal vertex cover V'_{opt} must include at least one endpoint of each edge in A
- No two edges in A share an endpoint
- $|A|$ is a lower bound for $|V'_{opt}|$
- the number of vertices in V' is $2|A|$
- therefore $|V'| \leq 2\,|V'_{opt}|$