It is recommended that you read through the exam before you begin. Answer all questions in the space provided.

Name: ____________________

Answer whether the following statements are true or false and support your answer with a proof sketch or counter-example in the space provided.

1. [TRUE / FALSE] Let G be an undirected connected graph. If G has a unique minimum spanning tree, then all the edge weights in G are distinct. [5 pts]

2. [TRUE / FALSE] If an operation takes $O(1)$ expected time, then is takes $O(1)$ amortized time. [5 pts]

3. [TRUE / FALSE] When using the path compression and the smaller-into-larger optimizations the Union Find data structure has $O(1)$-time performance for all operations. [5 pts]

4. [TRUE / FALSE] All problems in NP are known to be reducible to one another. [5 pts]

5. [TRUE / FALSE] If a problem A is polynomial time reducible to a problem B, and B has a polynomial time algorithm, then A has a polynomial time algorithm. [5 pts]

6. If some data structure supports some operation Foo such that a sequence of n calls to Foo takes $\Theta(n \log n)$ time to perform in the worst case.
   (a) What is the amortized cost of a single call to Foo? Briefly explain your answer. [5 pts]

   (b) What is the worst-case cost of a single call to Foo? Briefly explain your answer. [5 pts]

7. The problem of finding the prime factors of an integer has no known poly-time algorithm, but the problem has not been shown to be NP-Complete. Why is this fact significant? What would a poly-time solution to this mean with regards to P vs NP? [10 pts]

8. In class we analyzed Dynamic Table Expansion under the assumption that if we want to insert a new element in a table T that is full, we first copy all the elements of T into a new table T' of size $|T'| = 2|T|$, and then enter the new element in T'. In this question, we consider cases where the size of T' is not double the size of T. Assume (as in class) that entering an element in an empty slot of a table costs 1, and copying an element from a table into a new table also costs 1. Suppose that $|T'| = |T| + 1000$, i.e, each new table has 1000 more slots than the previous one. [10 pts]

   Starting with an empty table T with 1000 slots, we insert a sequence of n elements. What is the amortized cost per insertion? Explain your answer. (Hint: aggregate analysis might help)

9. For a hash table of size m consider the following obvious hash function $h(x) = x \mod m$.

   (a) Assuming that each entry in the table stores an unordered linked list, and that when a new element is added it is inserted at the beginning of the list. In the worst case, what is the time complexity to insert n keys into the table if chaining is used to resolve collisions? Briefly explain your answer. [5 pts]

   (b) Assuming that n items have already been inserted into the hash table, what is the worst case time complexity to search for an item? Briefly explain your answer. [5 pts]

   (c) Describe a set of n unique items that would result in the worst case time complexity to search for an item. [5 pts]

10. You've been asked to help a friend move. For simplicity, let us assume that your car has size 1, and your friend's n possessions $x_1, x_2, ..., x_n$ have real number sizes between 0 and 1. The goal is to divide those items into as few carloads as possible, such that all items arrive at the new location, and the car is never overpacked. Consider the following **First-Fit** approximation algorithm. Put item x_1 in the first carload. Then, for $i = 2, 3, ..., n$, put x_i in the first carload that has room for it, starting a new carload if necessary. For example: if $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, and $x_4 = 0.3$, the **First-Fit** algorithm would place x_1 and x_2 into the first carload, x_3 in the second carload, and then x_4 in the first carload where there is still room. Note that all decision are made offline; we divide the n possessions into carloads before any trips are actually made.

    (a) Give an example input where this **First-Fit** algorithm with fail to minimize the number of carloads. [5 pts]

(b) Prove that the **First-Fit** algorithm is a 2-approximation algorithm. (Hint: How many carloads can be less than half full) [10 pts]

11. An event organizer has been tasked with planning the invitations for a conference, but several of the potential attendees have started to develop conflicts. With careful research the organizer has managed to create a list of all possible arguing pairs. To keep the conference on schedule they can't invite both members of an arguing pair. Being a perfectionist, the organizer wants to find the smallest number of people that they can remove from the invitation list such that the final list will contain no arguing pairs. After much work they have been unable to find an optimal solution using any efficient algorithm.

    (a) Explain why? (i.e. What problem is the organizer trying to solve and why are they unable to do it?) [7 pts]

    (b) How could they weaken their goals to find a reasonable solution? [8 pts]