TSP

Given a complete weighted graph find the minimum weight Hamiltonian Cycle.

Brute Force $O(n!)$

Dynamic Programming $O(n 2^n)$

$\frac{C}{C_{opt}}$

Metric TSP

$d(x,y) = d(y,x)$ undirected

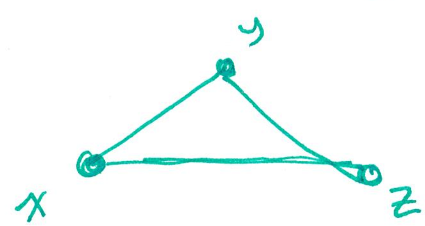
$d(x,y) \geq 0$

$d(x,y) + d(y,z) \geq d(x,z)$

2-approx

triangle inequality

$\frac{3}{2}$ - approx



Define

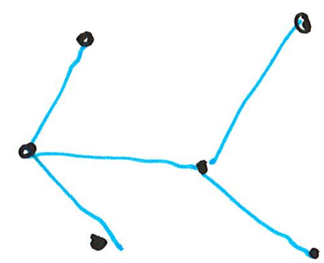
$c(S)$ - total cost of a set of edges

H_G^* - minimum weight Hamiltonian Cycle

$c(H_G^*)$ - length of optimal solution

2-approx

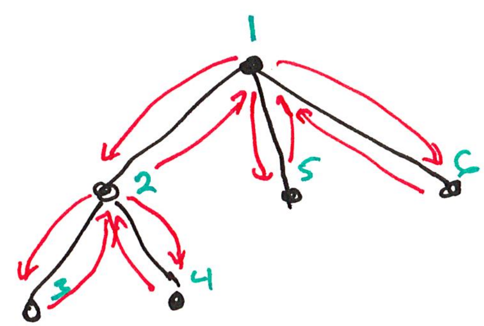
- using MST as our start
- not a cycle
- DFS



1, 2, 3, 2, 4, 2, 1, 5, 1, 6, 1

visits all nodes

some more than once



- Bypass duplicate vertices

1, 2, 3, ~~2~~, 4, ~~2~~, ~~1~~, 5, ~~1~~, 6, 1

Given the triangle inequality this will not increase path length

$1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 1$

Steps

- compute MST T
- construct path P using DFS

$c(P') \leq c(P)$

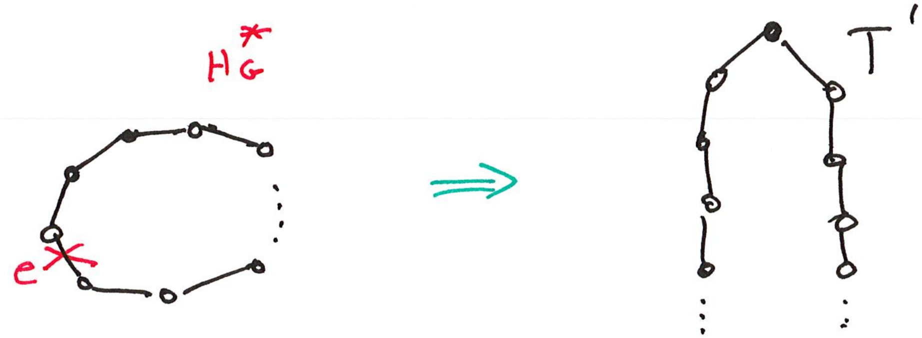
- strip duplicates to make P'

what is $c(P)$

$c(P') \leq 2c(T)$

$$c(P) = 2c(T)$$

- How does $c(T)$ relate to $c(H_G^*)$?



Delete an edge e from H_G^* creates a spanning tree T'

$$c(T) \leq c(T')$$

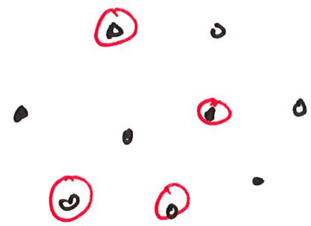
$$c(H_G^*) \geq c(T') \geq c(T)$$

$$c(P') \leq 2c(H_G^*)$$

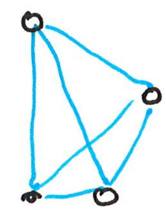
Christofides Algorithm

$\frac{3}{2}$ - approximation

$S \subseteq V$

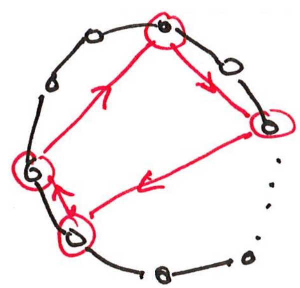


S

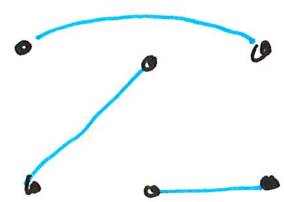


S has a Hamiltonian Cycle

$$c(H_S^*) \leq c(H_G^*)$$



Perfect - Matching

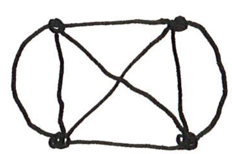


every node has ~~exact~~ exactly one edge

minimum cost perfect matching can be found in poly-time

Euler Circuit

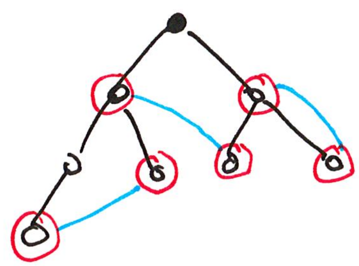
A circuit that traverses every edge exactly once



- the degree of every node must be even

$\frac{3}{2}$ - approximation

- add edges to the MST so that the graph has an Euler Circuit



S - set of odd degree vertices

- use perfect matching on S to increase the degree of all vertices in S by 1

- Can the number of odd degree vertices be odd? No

$$\sum d_i = 2|E|$$

- compute MST T
- $S \leftarrow$ odd degree vertices
- Find minimum perfect matching M on S
- Add the edges from M to T
- compute Euler Circuit E
- strip duplicates from E to set E'

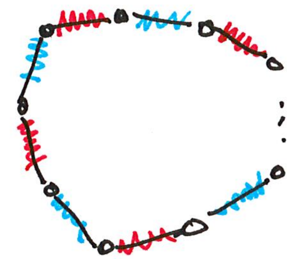
$$c(E) = c(T) + c(M)$$

$$c(E') \le c(\cancel{H}\, E)$$

$$c(T) \le c(H_G^*)$$

$c(M)$

H_s^*



- Let M_1 be the red perfect matching
- Let M_2 be the blue perfect matching

$$c(H_s^*) = c(M_1) + c(M_2)$$

$$c(M) \leq c(M_1) \qquad c(M) \leq c(M_2)$$

$$\begin{aligned} c(M) &\leq \frac{1}{2}\left(c(M_1) + c(M_2)\right) \\ &\leq \frac{1}{2}\left(c(H_s^*)\right) \\ &\leq \frac{1}{2}\left(c(H_G^*)\right) \end{aligned}$$

~~c(H~~

$$
\begin{aligned}
c(E') &\le c(T) + c(M) \\
&\le c(H_G^*) + \frac{1}{2} c(H_G^*) \\
&\le \frac{3}{2} c(H_G^*)
\end{aligned}
$$