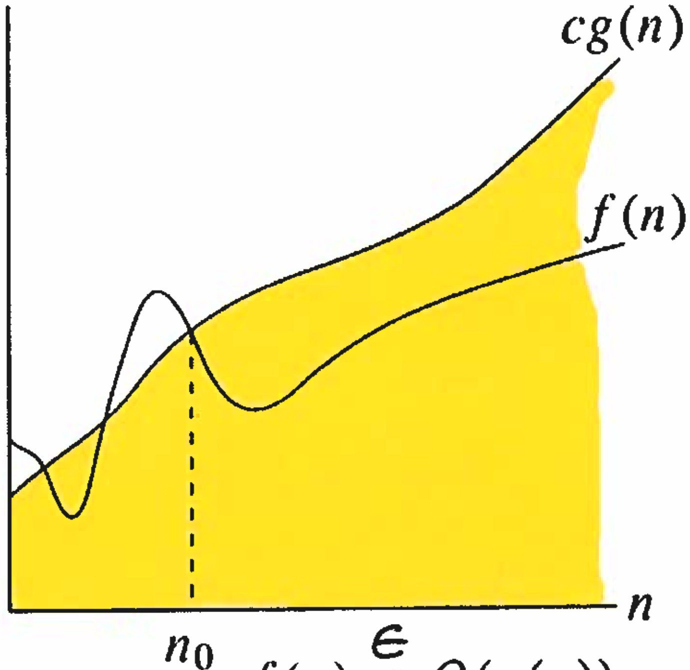
Asymtotic upper Bound <u>Big-O</u>

$O(g(n))$
- the set of all functions with a lower or the same order of growth as $g(n)$



$f(n) \overset{\in}{\cancel{=}} O(g(n))$

$$O(g(n)) = \{ f(n) : \exists\ c > 0 \text{ and } n_0 \text{ such that } \forall n \geq n_0\ \ f(n) \leq c\,g(n) \}$$

Big-O Examples

$n \in O(n^2)$

$n \leq \underbrace{c}\, n^2$

$c = 1$

$n_0 = 0$

$1000n + 45 \in O(n^2)$

$1000n + 45 \leq 1000n + 45n \quad \forall n \geq 1$

$\leq 1045n$

$\leq \underbrace{1045}\, n^2$

$c = 1045$

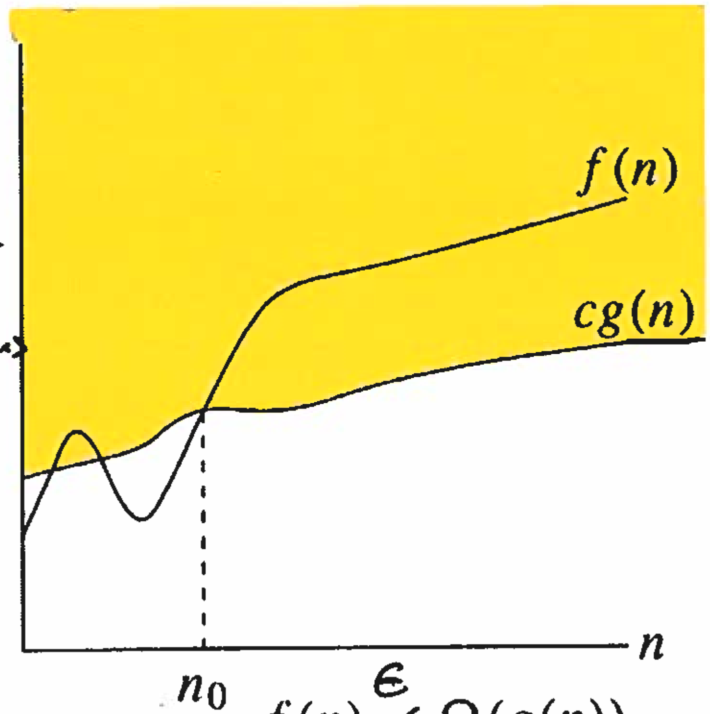
$n_0 = 1$

$\frac{n^3}{1000} \notin O(n^2)$

Big-Omega Ω

Asymtotic ~~&~~ Lower Bound

$\Omega(g(n))$ is the set of all functions with the same or higher order of growth as $g(n)$



$f(n) \in \Omega(g(n))$

$$\Omega(g(n)) = \{ f(n) : \exists\ c > 0 \text{ and } n_0 \text{ such that } \forall\ n \geq n_0 \quad c\,g(n) \leq f(n) \}$$

Big - Omega

$$\frac{n^3}{1000} \in \Omega(n^2)$$

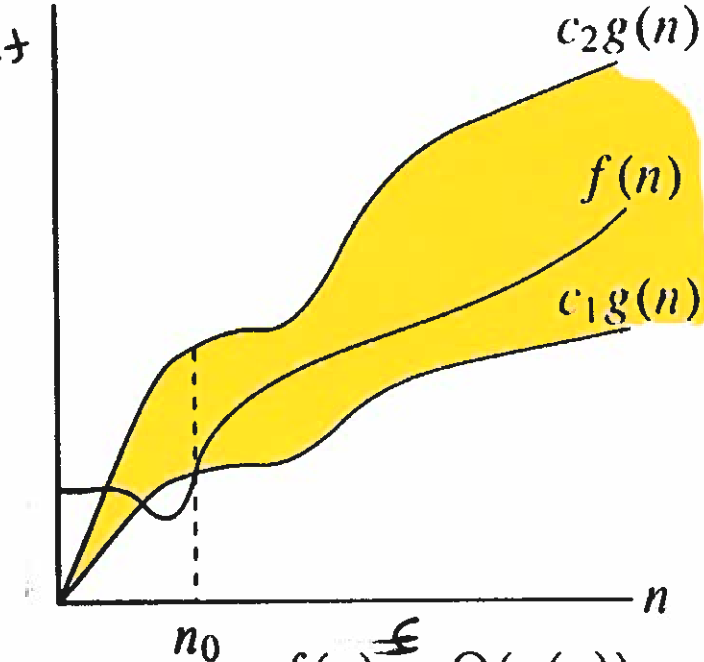
$$\frac{n^3}{1000} \geq \frac{n^2}{1000}$$

$c = \frac{1}{1000}$

$n_0 = 0$

Theta - notation

Asymtotic Tight Bound



$f(n) \overset{\in}{\cancel{=}} \Theta(g(n))$

$f(n) \in \Theta(g(n))$ iff $f(n) \in O(g(n))$ and
$f(n) \in \Omega(g(n))$

Theorem

$$\frac{n(n-1)}{2} \in \Theta(n^2)$$

Proof

- $\frac{n(n-1)}{2} \in O(n^2)$

$$\frac{n(n-1)}{2} = \frac{1}{2}n^2 - \frac{1}{2}n \qquad c = \frac{1}{2}$$
$$\leq \frac{1}{2}n^2 \qquad n_0 = 0$$

- $\frac{n(n-1)}{2} \in \Omega(n^2)$

$$\frac{n(n-1)}{2} = \frac{1}{2}n^2 - \frac{1}{2}n \qquad c = \frac{1}{4}$$
$$\geq \frac{1}{2}n^2 - \left(\frac{1}{2}n\right)^2 \qquad n_0 = 2$$
$$\geq \frac{1}{2}n^2 - \frac{1}{4}n^2$$
$$\geq \frac{1}{4}n^2$$

$$(x+y)^2 \in O(x^2+y^2) \qquad c(x^2+y^2)$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

without loss of generality

assume $x \geq y$

$$\leq x^2 + 2x^2 + y^2$$

$$\leq 3x^2 + y^2$$

$$\leq 3x^2 + 3y^2 \qquad \forall y \geq 0$$

$$\leq 3(x^2+y^2)$$

$c = 3 \qquad x_0 = 0$

$y_0 = 0$

$$f(n) = 1000n^2$$

$$g(n) = 0.001n^2$$

f(n) and g(n) have the same order of growth, so as far as asymtotic analysis is concerned f(n) and g(n) are equivalent.

Working with Big-O

Adding functions

$$O(f(n)) + O(g(n)) \in O(\max(f(n), g(n)))$$

$$\Omega(f(n)) + \Omega(g(n)) \in \Omega(\max(f(n), g(n)))$$

$$\Theta(f(n)) + \Theta(g(n)) \in \Theta(\max(f(n), g(n)))$$

Multiplying functions

$$O(c\, f(n)) \in O(f(n))$$

$$O(f(n)) \times O(g(n)) \in O(f(n)g(n))$$

$$\Omega(f(n)) \times \Omega(g(n)) \in \Omega(f(n)g(n))$$

$$\Theta(f(n)) \times \Theta(g(n)) \in \Theta(f(n)g(n))$$

- $n^3 + 50n^2 + 1000 \in O(n^3)$

$$n^3 + 50n^2 + 1000 \le 51n^3 + 1000$$
$$\le 51n^3 + 1000n^3 \quad \forall n \ge 1$$
$$\le 1051n^3$$

$c = 1051$

$n_0 = 1$

- $2^{n+3} + 15 \in O(2^n)$

$$2^{n+3} + 15 = 2^3\, 2^n + 15$$
$$= 8 \cdot 2^n + 15$$
$$\le 8 \cdot 2^n + 15 \cdot 2^n$$
$$\le 23 \cdot 2^n$$

$c = 23$

$n_0 = 0$