True/False Questions

1. Answer whether the following statements are true or false and briefly explain your answer in the space provided.

   (a) [TRUE / FALSE] If a language A is Turing-decidable and $A \leq_m B$ then B is Turing-decidable. [5 pts]

   (b) [TRUE / FALSE] If a language L is Turing-decidable then all subsets of L are Turing-decidable. [5 pts]

   (c) [TRUE / FALSE] If $\overline{L}$ is Turing-recognizable then L is also Turing-recognizable. [5 pts]

   (d) [TRUE / FALSE] The Turing-recognizable languages are closed under the complement operation. [5 pts]

   (e) [TRUE / FALSE] If $A \leq_m B$ and B is Turing-decidable then A is also Turing-decidable. [5 pts]

   (f) [TRUE / FALSE] The set of all Turing-undecidable language is countable. [5 pts]

   (g) [TRUE / FALSE] If A and B are Turing-recognizable then $A - B$ is also Turing-recognizable. [5 pts]

   (h) [TRUE / FALSE] The language ALL_{DFA} is $\in NP$. Where [5 pts]

   $$ALL_{DFA} = \{\langle D \rangle \mid D \text{ is a DFA and } L(D) = \Sigma^*\}$$

   (i) [TRUE / FALSE] A PDA with two stacks is equivalent to a Multi-tape Turing machine. [5 pts]

Decidable and Recognizable Languages

2. Give an informal description of a Turing machine that decides the following language: [10 pts]

$$A = \{w \in \{0,1\}^* \mid w \text{ contains twice as many 0's as 1's}\}$$

3. Write a high-level description of a Turing Machine that decides the following language: [10 pts]

$$ALL_{DFA} = \{\langle D \rangle \mid D \text{ is a DFA and } L(D) = \Sigma^*\}$$

4. Let $\Sigma = \{0, 1\}$. Show that the problem of determining whether a CFG generates at least one string in 1^* is decidable. This can be written as the following language: [10 pts]

$$\{\langle G \rangle \mid G \text{ is a CFG and } 1^* \cap L(G) \neq \emptyset\}$$

5. Let $A = \{\langle D \rangle \mid D \text{ is a DFA that doesn't accept any string containing an odd number of } 1s\}$. Show that A is decidable. [10 pts]

Reductions

6. Show that the following language is undecidable using a reduction. [10 pts]

$$HaltSome_{TM} = \{\langle M \rangle \mid M \text{ is a TM and there exists a } w \in \Sigma^* \text{ such that } M \text{ halts on } w\}$$

7. Prove that the following language is not recognizable [10 pts]

$$LOOP_{TM} = \{\langle M \rangle \mid M \text{ is a TM that loops on all inputs}\}$$

8. Show that the following language is not Turing-decidable: [10 pts]

$$L2_{TM} = \{\langle M \rangle \mid M \text{ is a TM that accepts all strings of length 2 or less}\}$$

9. Show that the following language is not Turing-recognizable [10 pts]

$$LoopSingle_{TM} = \{\langle M \rangle \mid M \text{ is a TM that loops on the string 0011}\}$$

Closure Properties of Turing Machines

10. Prove that the Turing-Decidable languages are closed under concatenation. [5 pts]

11. Prove that the Turing-Decidable languages are closed under the Kleene star operation. [5 pts]

12. Prove that the Turing-Recognizable languages are closed under the Kleene star operation. [5 pts]

13. Prove that the Turing-Recognizable languages are not closed under complement. [5 pts]

P vs. NP

14. Briefly explain what it means for a language to be NP-Complete and why such languages are important. [5 pts]

15. Let the Graph coloring problem be defined as: [10 pts]

$$\{\langle G, k \rangle \mid G \text{ is a graph that can be colored with } k \text{ colors}\}$$

Prove that the graph coloring problem is an element of NP. (Note that colored means that no two adjacent nodes have the same color)

16. Show that the Knapsack problem is an element of NP, where the Knapsack problem is defined as the following: [10 pts]

$$\{\langle L, W, V \rangle \mid L \text{ is a list of tuples } (w_i, v_i) \text{ and } \exists \text{ a subset of } L \text{ where } \sum w_i \leq W \text{ and } \sum v_i \geq V\}$$