Oldroyd Viscoelastic Model
Lecture Notes

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ME 510: Non-Newtonian Fluids
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Putting the Oldroyd Model Into Context

Linear Models

Maxwell Model

\[ \tau + \lambda_1 \frac{\partial \tau}{\partial t} = -\eta_0 \dot{\gamma} \] (1)

Where

- \(\tau\) - stress tensor
- \(\lambda_1\) - relaxation time
- \(\eta_0\) - zero-strain rate viscosity
- \(\dot{\gamma}\) - shear-rate tensor

We can see a clear linear relationship between \(\tau\) and \(\dot{\gamma}\) in the equation 1. See Zach Wilson’s notes for details.

Jeffrey Model

If we add additional linear relationships, i.e. the time derivative of \(\dot{\gamma}\), then we arrive at the Jeffrey Model.

\[ \tau + \lambda_1 \frac{\partial \tau}{\partial t} = -\eta_0 \left( \dot{\gamma} + \lambda_2 \frac{\partial \dot{\gamma}}{\partial t} \right) \] (2)

Where

- \(\lambda_2\) - retardation time

Limitations

- For flows where \(\lambda_{\text{max}} \geq 1\) the linear models can not describe shear-rate dependence of viscosity.
- non-linear effects can not be described by linear model (normal stresses)
- if small-strain phenomena require large displacement gradients to manifest them the linear models will not work
- in steady shear free flows the model produces an infinite elongational viscosities

This motivates non-linear models.

Quasi-linear Models

Before we can introduce Non-linear models, there are a few building blocks we need to acquire.

Convected Time Derivative

For the convected time derivative of the stress tensor, \(\tau\), we will use the symbol \(\tau^{(1)}\).

\[ \tau^{(1)} = \frac{D\tau}{Dt} - \left\{ (\nabla \mathbf{v})^T \cdot \tau + \tau \cdot (\nabla \mathbf{v}) \right\} \] (3)

If \(\tau\) is symmetric, then equation 3 becomes:

\[ \tau^{(1)} = \frac{D\tau}{Dt} - \left\{ \tau \cdot \nabla \mathbf{v} \right\}^T - \left\{ \tau \cdot \nabla \mathbf{v} \right\} \] (4)

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Retarded-Motion Expansion

Asymptotic expansion of \( \tau \) used to describe small departures from Newtonian model. We could use the series expansion

\[
\tau = -\left[ b_1 \gamma(1) + b_2 \frac{\partial \gamma(1)}{\partial t} + b_{11} \left\{ \gamma(1) \cdot \gamma(1) \right\} \\
+ b_3 \frac{\partial^2 \gamma(1)}{\partial t^2} + b_{12} \left\{ \gamma(1) \cdot \frac{\partial \gamma(1)}{\partial t} \right\} + \ldots \right]
\]

However, this expansion does not lend itself to modeling certain flow very well. Also, we know from continuum mechanics, that the partial time derivative of \( \gamma(1) \) should not be in constitutive equations. Only convected time derivative of \( \gamma(1) \) should appear in constitutive equations. We, therefore, need a more suitable expansion. Assuming:

- incompressible fluid
- stress tensor is symmetric
- stress tensor can be expresses as polynomial in \( \gamma(n) \)

And:

- arranging terms of polynomial in increasing order
- collecting terms of equal order

We end up with the *Retarded-Motion Expansion*.

\[
\tau = -\left[ b_1 \gamma(1) + b_2 \gamma(2) + b_{11} \left\{ \gamma(1) \cdot \gamma(1) \right\} \\
+ b_3 \gamma(3) + b_{12} \left\{ \gamma(1) \cdot \gamma(2) + \gamma(2) \cdot \gamma(1) \right\} + b_{111} \left( \gamma(1) : \gamma(1) \right) \gamma(1) + \ldots \right]
\]

Where

- \( b_n \) - retarded-motion constant (material properties)
- these are dependent on the fluid, flow, and model

What is important here is understanding vocabulary. Ordered fluid model names come from truncating the Retarded-Motion Expansion. If we keep only the:

- first order terms, \( O(\gamma(1)) \), then we have the *Newtonian* model again with \( b_1 = \text{viscosity} \).
- second order terms, \( O(\gamma(2)) \), then we have an *incompressible second-order fluid*
- third order terms, \( O(\gamma(3)) \), then we have an *incompressible third-order fluid*

You will not often see a higher order than third-order fluids

Limitations of Retarded-Motion Expansion:

- Can predict a shear stress vs shear rate maximum
- Can predict a negative viscosity or elongational viscosity
- Can predict an unstable behavior for second-order fluids at rest
May require additional boundary conditions
- if perturbation expansion is used, this is not the cases
Can not qualitatively describe entire flows (i.e. stress relaxation experiment)
Plus more ...

The point is to reinforce the small De restriction. This comes out in the scaling (Not done in this lecture.).

**Convected Jeffrey Model (Oldroyd-B)**

If we plug equation 4 into the Jeffrey model (equation 2) we get
\[ \frac{\partial \tau}{\partial t} \rightarrow \tau^{(1)} \]
\[ \frac{\partial^n}{\partial t^n} (\dot{\gamma}) \rightarrow \gamma^{(n+1)} \]

Jeffrey Model \rightarrow Convected Jeffrey Model

Making the substitutions yields the Convected Jeffrey Model:
This is also known as Oldroyd-B Model

\[ \tau + \lambda_1 \tau^{(1)} = -\eta_0 \left( \gamma^{(1)} + \lambda_2 \gamma^{(2)} \right) \]  \hspace{1cm} (5)

Where
- \( \eta_0 \) - zero shear-rate viscosity
- \( \lambda_1 \) - relaxation time
- \( \lambda_2 \) - retardation time
- kinematic tensors defined previously in retarded-motion expansion.

One cool thing about the Convected Jeffrey Model is that it contains other models as special cases. So if

1. \( \lambda_2 = 0 \) then equation 5 transforms into the Maxwell Model (equation 1)
2. \( \lambda_1 = 0 \) then equation 5 transforms into a second order fluid with a normal stress coefficient \( \rightarrow 0 \)
3. \( \lambda_1 = \lambda_2 \) then equation 5 transforms into the Newtonian Model with viscosity, \( \eta_0 \)

**Non-Linear Differential Models**

**Recall** For a Newtonian fluid:

\[ \tau = \eta \dot{\gamma} \]  \hspace{1cm} (6)

Where
- \( \eta \) is a function of scalar \( \dot{\gamma} \)

The exact function is unknown but there are models. One famous model is the power law, \( \eta = m \dot{\gamma}^{n-1} \). While there are other models, the point is that for a Newtonian fluid, \( \eta = f(\dot{\gamma}) \). If we apply the same principle for non-Newtonian fluid, we can say:

\[ \dot{\gamma} = \sqrt{\frac{1}{2} \left( \gamma^{(1)} : \gamma^{(1)} \right)} \]  \hspace{1cm} (7)
If we plug equation 7 into 5 to get a new model called the \textit{White-Metzner Model}.

\[
\tau + \frac{\eta(\dot{\gamma})}{G} \tau_{(1)} = -\eta(\dot{\gamma})\gamma_{(1)}
\]  

(8)

Where

\( G \) - constant modulus

Benefits

- Simple
- Produces reasonable solutions for shear-rate dependent viscosity
- Can handle the first-order normal stress coefficient
- Can be used to model fast time dependent motions
- It is good at ballparking interaction of shear thinning and memory on flow fields

Shortcomings

- Poor model of fast time dependent motions
- In steady shear free flows the model produces an infinite elongational viscosities (same as in Maxwell Model)

It is also important to note that instead of using equation 7 we could have also used other models (ie \textit{Phan-Thien-Tanner} or \textit{FENE-P}) but those are not within the scope of this lecture.

** Oldroyd Model Fits Right Here. **

Non-Linear Integral Models

While there are many Integral Models in the literature, the details of each fall outside the scope of this lecture. That said, to name a few:

- Lodge
- K-BKZ
- Rivlin-Sawyers
- Wagner
- Doi-Edwards

Oldroyd Models

\[
\tau + \lambda_1 \tau_{(1)} + \frac{1}{2} \lambda_3 \left\{ \gamma_{(1)} : \tau + \tau : \gamma_{(1)} \right\} + \frac{1}{2} \lambda_5 (\text{tr} \ \tau) \gamma_{(1)} + \frac{1}{2} \lambda_6 \left( \tau : \gamma_{(1)} \right) \delta
\]

\[
= -\eta_0 \left[ \gamma_{(1)} + \lambda_2 \gamma_{(2)} + \lambda_4 \left\{ \gamma_{(1)} : \gamma_{(1)} \right\} + \frac{1}{2} \lambda_7 \left( \gamma_{(1)} : \gamma_{(1)} \right) \delta \right]
\]

(9)

Taking a closer look we may recognize that terms I and II are the terms of the convected Jeffreys model.
Restrictions on Oldroyd Model

1. $\lambda_1 > \lambda_2 > 0$ in order to ensure $\eta'$ decreases with increasing $\omega$.

2. $\sigma_1 > \sigma_2 > \ldots > \sigma_i > 0$ to ensure viscosity is generally a monotone decreasing function of $\dot{\gamma}$
   
   Where $\sigma_i = \lambda_1(\lambda_3 + \lambda_5) + \lambda_{i+2}(\lambda_1 - \lambda_3 - \lambda_5) + \lambda_{i+5}(\lambda_1 - \lambda_2 - \frac{3}{2}\lambda_5)$

3. for steady shear flow, $\sigma_2 \leq \frac{1}{2}\sigma_1$ to ensure $|\tau_{xy}|$ is a monotone increasing function of $\dot{\gamma}$

4. $\sigma_1 - \sigma_2 < \lambda_1(\lambda_1 - \lambda_2)$ to ensure that $\eta$ curve is above $\eta'$ curve where $\eta(\dot{\gamma})$ and $\eta''(\omega)$ are plotted together.

5. $\frac{2}{3}(\lambda_5 + \lambda_6) - \frac{1}{3}[4\lambda_6^2 - 11\lambda_5\lambda_6 + 4\lambda_6^2]^{1/2} < \lambda_1 - \lambda_3 < \frac{2}{3}(\lambda_5 + \lambda_6) + \frac{1}{3}[4\lambda_6^2 - 11\lambda_5\lambda_6 + 4\lambda_6^2]^{1/2}$ to ensure that elongational viscosity is bounded

Benefits of Oldroyd Model

- More variety in rheological response can be described for Oldroyd then Jeffreys.

- Wider range of of properties can be correctly described for Oldroyd then White-Wetzner.

- Easier to solve analytically then White-Wetzner because of it’s form.

- Like with the generalized Maxwell equation, more terms of $\lambda_i$ yield a more accurate result.

Limitations of Oldroyd Model

- Strong presence of singularities in $\tilde{\eta}_1$ and $\tilde{\eta}_2$ for $\dot{\epsilon} < 0$ and $0 < b < 1$.

- No reason why a constitutive equation must be linear in stress.

This motivates the Giesekus Model which is outside the scope of this lecture.
## Summary

<table>
<thead>
<tr>
<th>Name</th>
<th>Model</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Newtonian</strong></td>
<td>( \tau = \eta \dot{\gamma} )</td>
<td>Linear</td>
</tr>
<tr>
<td><strong>Maxwell</strong></td>
<td>( \tau + \lambda_1 \frac{\partial \tau}{\partial t} = -\eta_0 \dot{\gamma} )</td>
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<tr>
<td><strong>Jeffrey</strong></td>
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<td>Linear</td>
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<tr>
<td><strong>Convected Jeffrey (Oldroyd B)</strong></td>
<td>( \tau + \lambda_1 \tau(1) = -\eta_0 \left( \gamma(1) + \lambda_2 \gamma(2) \right) )</td>
<td>Quasi-linear</td>
</tr>
<tr>
<td><strong>White-Metzner</strong></td>
<td>( \tau + \frac{\eta(\dot{\gamma})}{G} \tau(1) = -\eta(\dot{\gamma}) \gamma(1) )</td>
<td>Non-Linear Differential</td>
</tr>
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<td>( \tau + \lambda_1 \tau(1) + \frac{1}{2} \lambda_3 \left{ \gamma(1) : \tau + \tau : \gamma(1) \right} + \frac{1}{2} \lambda_5 \left( \text{tr} \ \tau \right) \gamma(1) + \frac{1}{2} \lambda_6 \left( \tau : \gamma(1) \right) \delta = -\eta_0 \left[ \gamma(1) + \lambda_2 \gamma(2) + \lambda_4 \left{ \gamma(1) : \gamma(1) \right} + \frac{1}{2} \lambda_7 \left( \gamma(1) : \gamma(1) \right) \delta \right] )</td>
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</tr>
</tbody>
</table>

## Nomenclature

**Latin Letters**
- \( b_n \) - retarded-motion constant
- \( G \) - constant modulus

**Greek Letters**
- \( \dot{\gamma} \) - shear-rate tensor
- \( \dot{\epsilon} \) - extensional rate
- \( \bar{\eta}_1 \) - dynamic viscosity
- \( \bar{\eta}_2 \) - 
- \( \eta_0 \) - zero-strain rate viscosity
- \( \lambda_1 \) - relaxation time
- \( \lambda_2 \) - retardation time
- \( \tau \) - stress tensor