Supplemental Material for "Copula Models of Correlation"

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This document provides detailed derivation of results given in the paper "Copula Models of Correlation: A DRAM Case Study" by C. Glenn Shirley and W. Robert Daasch. References to sections in the paper and equations in the paper are enclosed in braces, {}.

A Model of Test and Use

Section {4.1} shows Eq. {15} which transforms the exchangeable copula, C, into the pseudo-copula, D, representing the Test/Use model defined in Eq. {14}. The references given[1] will guide the derivation, but a less general and more explicit demonstration will be easier to understand. The strategy is to do probability logical manipulations starting with Eq. {14} leading to expressions like $P(U_x \leq u,\ U_y \leq v)$ where U_x and U_y are independent uniform random variables on [0,1]. $P(U_x \leq u,\ U_y \leq v)$ is then recognized as a copula (or pseudo-copula). Some results from Nelsen's book are used on the way.

Preliminaries

Problem 2.16 on p29 of Nelsen[2]

Suppose random variables X and Y have copula C, and marginal distributions F and G.

Prove that

[1] Jorge Navarro and Fabio Spizzichino, "On the relationships between copulas of order statistics and marginal distributions," Statistics and Probability Letters, vol. 80, no. 5-6, pp. 473-479, March 2010. [Online]. http://dx.doi.org/10.1016/j.spl.2009.11.025

[2] Roger B. Nelsen, An Introduction to Copulas, 2nd ed. New York, New York, USA: Springer, 2010.

$$P\left(\max\left[X,Y\right]\le t\right)=C\left(F(t),G(t)\right) \tag{S1}$$

and

$$P\left(\min\left[X,Y\right]\le t\right)=F(t)+G(t)-C\left(F(t),G(t)\right). \tag{S2}$$

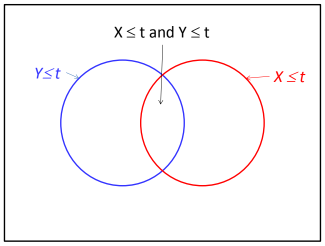
Proof of Eq.(S1).

The pairs with $X \le t$ and $Y \le t$ are the same as the pairs with $\max[X, Y] \le t$. So

$$\begin{aligned}P\left(\max\left[X,Y\right]\le t\right)&=P\left(X\le t,Y\le t\right)=P\left(F^{-1}(U_X)\le t,G^{-1}(U_Y)\le t\right)\\&=P\left(U_X\le F(t),U_Y\le G(t)\right)=C\left(F(t),G(t)\right)\end{aligned} \tag{S3}$$

Proof of Eq. (S2)

The event $\min[X,Y] \le t$ is the same as the event that one or both of X and Y are less or equal to t. These are the events covered by the two circles in the Venn diagram. So, from the inclusion/exclusion principle



$$\begin{aligned}P\left(\min\left[X,Y\right]\le t\right)&=P\left(X\le t\right)+P\left(Y\le t\right)-P\left(X\le t,Y\le t\right)\\&=F\left(t\right)+G\left(t\right)-C\left(F(t),G(t)\right)\end{aligned} \tag{S4}$$

Derivation of Eq. {15}

In Use the retention time for a "good" bit will always be the minimum retention time for any bit because a bit will be accessed repeatedly. Test, on the other hand, is a brief test in which either the maximum or minimum retention time will occur during the Test. If s is the proportion of the time that the Test retention time is the maximum for a bit, then the retention time model for retention times in Use and Test is

$$\begin{aligned}&Z_{Use}=\min[X,Y] && \text{All the time.}\\&Z_{Test}=\begin{cases}\max[X,Y] & \text{Proportion } s \text{ of the time.}\\ \min[X,Y] & \text{Proportion } 1-s \text{ of the time.}\end{cases}\end{aligned} \tag{S5}$$

where X and Y are retention times modeled by the fitted copula, C with equal margins, F

$$\begin{aligned}C\left(p,q\right)&\equiv P\left(X\le u,Y\le t)\right)\\&=P\left(F^{-1}(U_X)\le u,F^{-1}(U_Y)\le t\right)\\&=P\left(U_X\le F(u),U_Y\le F(t)\right)\\&=P\left(U_X\le p,U_Y\le q\right)\end{aligned} \tag{S6}$$

where $p = F(u)$ and $q = F(t)$.

The distribution function of interest in models of Test and Use is

$$
\begin{aligned}
D(p,q) &\equiv P\left(Z_{Use} \le u, Z_{Test} \le t\right) \\
&= sP\left(\min[X,Y] \le u, \max[X,Y] \le t\right) \\
&\qquad +(1-s)P\left(\min[X,Y] \le u, \min[X,Y] \le t\right) \\
&= sP\left(\min[F^{-1}(U_X),F^{-1}(U_Y)] \le u, \max[F^{-1}(U_X),F^{-1}(U_Y)] \le t\right) \\
&\qquad +(1-s)P\left(\min[F^{-1}(U_X),F^{-1}(U_Y)] \le u, \min[F^{-1}(U_X),F^{-1}(U_Y)] \le t\right) \\
&= sP\left(F^{-1}\left(\min[U_X,U_Y]\right) \le u, F^{-1}\left(\max[U_X,U_Y]\right) \le t\right) \\
&\qquad +(1-s)P\left(F^{-1}\left(\min[U_X,U_Y]\right) \le u, F^{-1}\left(\min[U_X,U_Y]\right) \le t\right) \\
&= sP\left(\min[U_X,U_Y] \le F(u), \max[U_X,U_Y] \le F(t)\right) \\
&\qquad +(1-s)P\left(\min[U_X,U_Y] \le F(u), \min[U_X,U_Y] \le F(t)\right) \\
&= sP\left(\min[U_X,U_Y] \le p, \max[U_X,U_Y] \le q\right) \\
&\qquad +(1-s)P\left(\min[U_X,U_Y] \le p, \min[U_X,U_Y] \le q\right)
\end{aligned} \tag{S7}
$$

and we seek D in terms of C. Notice that the manipulations in Eq. (S7) depend on exchanging the order of the inverse cumulative distribution, F^{-1}, with the min and max functions. This is only possible if the margins are equal. The margins of {Table 2} in the paper are equal by construction.

Consider the term with coefficient $(1 - s)$ in the final equality of Eq. (S7):

$$
\begin{aligned}
P\left(\min[U_X,U_Y] \le p, \min[U_X,U_Y] \le q\right) &= \begin{cases} P\left(\min[U_X,U_Y] \le p\right) & p \le q \\ P\left(\min[U_X,U_Y] \le q\right) & p \ge q \end{cases} \\
&= \begin{cases} 2p - C(p,p) & p \le q \\ 2q - C(q,q) & p \ge q \end{cases} \\
&= 2z - C(z,z) \qquad z = \min[p,q]
\end{aligned} \tag{S8}
$$

The first equality in Eq. (S8) is true because, for $p \le q$, both $\min[U_X, U_Y] \le p$ and $\min[U_X, U_Y] \le q$ are only true when $\min[U_X, U_Y] \le p$ since the events $\{\min[U_X, U_Y] \le p\}$ are a subset of the events $\{\min[U_X, U_Y] \le q\}$. A symmetrical argument is made when $q \le p$. The second equality in Eq. (S8) is a consequence of Eq. (S4) when $F = G = p$ or $F = G = q$.

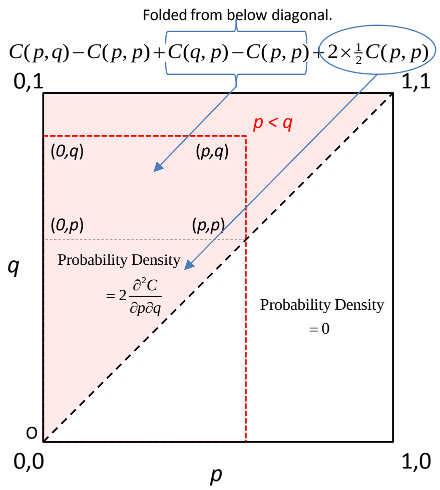
Now consider the term with coefficient s in the last equality of Eq. (S7). This is the pseudo-copula giving the dependence of the order statistics of samples of U_X and U_Y

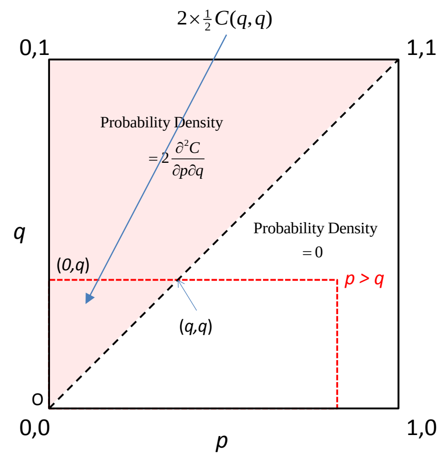
$$
OS(p,q) = P\left(\min[U_X,U_Y] \le p, \max[U_X,U_Y] \le q\right). \tag{S9}
$$

<u>Key Observation:</u> An expression for OS in terms of C is constructed by reflecting all probability points below the (0,0)/(1,1) diagonal in C to lie above the diagonal.

Referring to the figures, the probability density in OS may therefore be written as

$$
\begin{aligned}
OS(p,q) &= \begin{cases} C(p,q) + C(q,p) - C(p,p) & p \le q \\ C(q,q) & p \ge q \end{cases} \\
&= C(p,q) + C(q,z) - C(p,z) \qquad z = \min[p,q]
\end{aligned} \tag{S10}
$$





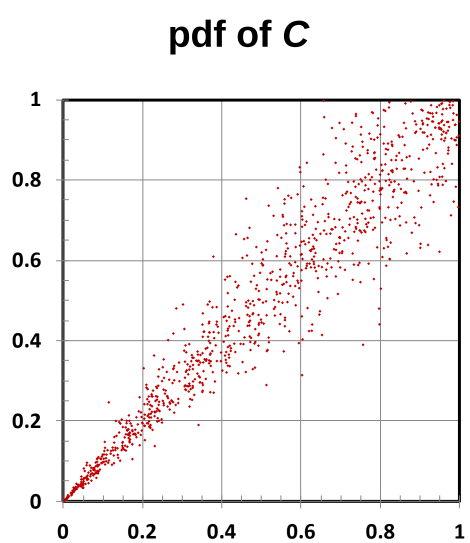
From Eqs. (S7), (S8), and (S10) the probability model for Test and Use is

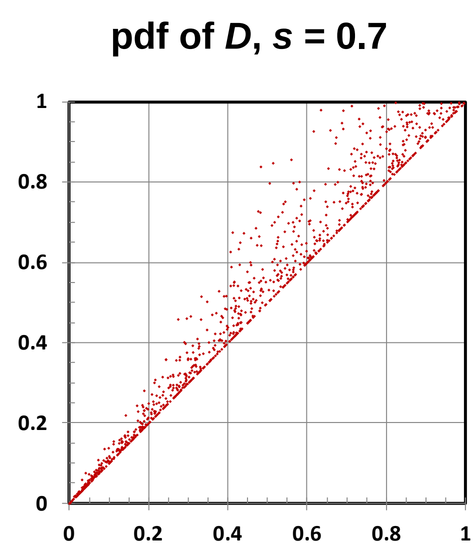
$$D(p,q)=s\left[C(p,q)+C(q,z)-C(p,z)\right]+(1-s)\left[2z-C(z,z)\right] \quad z=\min[p,q] \tag{S11}$$

where p is the Use condition and q is the Test condition. The most conservative model in which Test always "sees" the maximum retention time, and never the minimum, corresponds to $s = 1$.

Example

The Clayton copula C which fits the nominal DRAM data has $\theta = 9.75$. Samples synthesized from C and transformed according to Eq. (S5) into D are shown in the figure. The density of points is a measure of the probability density of C and of D. A value of $s = 0.7$ for D is shown. Notice that the term of Eq. (S11) with coefficient s corresponds to "folding" the part of C below the diagonal so that it lies above the diagonal of D, and the term with coefficient $(1 - s)$ gives a concentration of points lying on the diagonal of D.





Array Statistics from Bit Statistics

A statistical model of individual memory bits gives the probability that a bit is a member of one of the four mutually exclusive categories *pf*, *fp*, *ff*, *pp*. These bit-level probabilities are p_{pf}, p_{fp}, p_{ff}, and $p_{pp} = 1 - p_{pf} - p_{fp} - p_{ff}$. The bit-level statistical model gives the probabilities as functions of Test and Use conditions. In this notation the letter in the first subscript position indicates pass or fail in Use, and in the second position the letter indicates pass or fail in Test. For example, p_{pf} is the probability that a bit would pass Use but fail Test. It is important to keep this positional convention in mind in the following discussion. This note shows how to derive probabilities for *arrays* from the probabilities of *bits* in the four categories as a function of three factors:

- The number of bits in the array.
- Fault tolerance capacity in Test and in Use.
- Whether or not bits tolerated at Test are repaired/replaced at Test.

The objective is to derive expressions for three figures of merit (FOMs), yield loss (*YL*), overkill level (*OL*), and defect level (*DL*) for *arrays*. These FOMs are given by

$$YL = P(\text{Fails Test}) = 1 - P(\text{Passes Test}) \tag{S12}$$

$$\begin{aligned} OL = {} & P(\text{Good in Use}) \\ & - P(\text{Passes Test and Good in Use}) \end{aligned} \tag{S13}$$

$$\begin{aligned} DL &= P(\text{Fails in Use}|\text{Passes Test}) \\ &= 1 - P(\text{Good in Use}|\text{Passes Test}) \\ &= 1 - \frac{P(\text{Passes Test and Good in Use})}{P(\text{Passes Test})} \end{aligned} \tag{S14}$$

So, the objective becomes derivation of expressions for *P*(Passes Test), *P*(Good in Use), and *P*(Passes Test and Good in Use) for arrays.

Consider an array made from *n* of the bits characterized in this experiment. Assuming that the bits are statistically independent, the probability that the array has exactly n_{fp} bits in category *fp*, n_{pf} bits in category *pf*, and n_{ff} bits in category *ff* is given by the multinomial theorem,

$$P\left(N_{pf} = n_{pf}, N_{fp} = n_{fp}, N_{ff} = n_{ff}\right) = \binom{n}{n_{pf}, n_{fp}, n_{ff}} p_{pf}^{n_{pf}}\, p_{fp}^{n_{fp}}\, p_{ff}^{n_{ff}} (1 - p_{pf} - p_{fp} - p_{ff})^{n - n_{pf} - n_{fp} - n_{ff}}. \tag{S15}$$

For a memory array, conditions for the Poisson limit are well-justified. In this limit, $n \to \infty$, so $n_{pp} = n - n_{pf} - n_{fp} - n_{ff} \gg \max[n_{pf}, n_{fp}, n_{ff}]$, but λ's defined by

$$\lambda_{pf} = np_{pf} \qquad \lambda_{fp} = np_{fp} \qquad \lambda_{ff} = np_{ff} \tag{S16}$$

remain finite. In the Poisson limit Eq. (S15) becomes

$$P\left(N_{pf} = n_{pf}, N_{fp} = n_{fp}, N_{ff} = n_{ff}\right) \simeq \frac{\lambda_{ff}^{n_{ff}} \exp(-\lambda_{ff})}{n_{ff}\,!} \frac{\lambda_{pf}^{n_{pf}} \exp(-\lambda_{pf})}{n_{pf}\,!} \frac{\lambda_{fp}^{n_{fp}} \exp(-\lambda_{fp})}{n_{fp}\,!}. \tag{S17}$$

In the following manipulations expressions for probability functions will be derived. The probability functions are sums of "Poisson terms" such as Eq. (S17) taken over sets of integers $\{n_{pf}, n_{fp}, n_{ff}\}$ allowed by certain constraints. In some cases the sum of the Poisson terms gives a tidy analytical expression.

Arrays Without Fault Tolerance

We first derive expressions for FOMs assuming no fault tolerance at Test or in Use. In this case a single failing bit will cause the array to be classified as failing. The event that an array passes Test irrespective or whether it passes or fails in Use is defined by

$$n_{pf} + n_{ff} = 0 \tag{S18}$$

which expresses the condition that no bits fail in Test (note that the second subscript index is *f* in both bit count categories), and by

$$0 \le n_{fp} + n_{ff} < \infty \quad \Rightarrow \quad 0 \le n_{fp} < \infty \tag{S19}$$

which expresses the condition that any number of bits may fail in Use for arrays in this category (note that $n_{ff} = 0$ because of Eq. (S18)). So the probability of an array passing Test irrespective of whether the array passes or fails in Use is given by summing the probability in Eq. (S17) over the set of integers $\{n_{pf}, n_{fp}, n_{ff}\}$ allowed by the constraints of Eqs. (S18) and (S19)

$$P(\text{Passes Test}) = \sum_{\substack{n_{pf}=n_{ff}=0 \\ 0 \le n_{fp} < \infty}} \frac{\lambda_{ff}^{n_{ff}} \exp(-\lambda_{ff})}{n_{ff}!} \frac{\lambda_{pf}^{n_{pf}} \exp(-\lambda_{pf})}{n_{pf}!} \frac{\lambda_{fp}^{n_{fp}} \exp(-\lambda_{fp})}{n_{fp}!} = \exp\left[-(\lambda_{ff} + \lambda_{pf})\right]. \tag{S20}$$

Similarly, the probability that an array would be good in use irrespective of Test is given by summing Eq. (S17) over terms allowed by

$$n_{fp} + n_{ff} = 0, \qquad 0 \le n_{pf} < \infty \tag{S21}$$

so

$$P(\text{Good in Use}) = \exp\left[-(\lambda_{ff} + \lambda_{fp})\right] \tag{S22}$$

The event that an array passes Test and is good in Use is defined by $n_{fp} = n_{pf} = n_{ff} = 0$, so the probability is

$$P(\text{Passes Test and Good in Use}) = \exp\left[-(\lambda_{ff} + \lambda_{pf} + \lambda_{fp})\right] \tag{S23}$$

Substitution of the probabilities from Eqs. (S20), (S22), and (S23) into Eqs. (S12), (S13), (S14) gives figures of merit assuming no fault tolerance

$$\begin{aligned} YL &= 1 - \exp\left[-(\lambda_{ff} + \lambda_{pf})\right] \\ OL &= \exp\left[-(\lambda_{ff} + \lambda_{fp})\right]\left[1 - \exp(-\lambda_{pf})\right] \\ DL &= 1 - \exp(-\lambda_{fp}). \end{aligned} \tag{S24}$$

Arrays With Fault Tolerance

Fault tolerance is modeled by expanding the definition of a "good" array to include arrays with some "bad" bits. The effect of bad bits in arrays that are considered good is corrected by a fault tolerance scheme. Fault tolerance may be implemented by error correction of data in Test and/or in Use, or by replacing bad bits detected at Test with good bits. The maximum number of bad bits which can be tolerated is a measure of the capacity of the fault tolerance scheme. The following discussion supposes than an array can tolerate up to n_t bits bad in Test and up to n_u bits bad in Use. Two ways of implementing fault tolerance will be given. The first is fault tolerance with no repair, and the second is fault tolerance with repair at Test.

Fault Tolerance Without Repair

Consider a population of arrays each with the same number, n, of bits. The number of bits in the mutually exclusive bit categories *pf*, *fp*, *ff*, *pp* will vary from instance to instance of the array, with the constraint $n = n_{pp} + n_{pf} + n_{fp} + n_{ff}$. The condition for an array to "pass" Test is that the number of bits bad at Test is less than the fault tolerance capacity at Test: $0 \le n_{ff} + n_{pf} \le n_t$. And for an array to be "good" in Use: $0 \le n_{ff} + n_{fp} \le n_u$. These criteria may be used to categorize arrays as shown in Table 1.

Table 1 Array categories vs bit category counts and tolerance capacities for no-repair at Test.

fp: Passes Test and Bad in Use $0 \le n_{ff} + n_{pf} \le n_t$ $n_u < n_{ff} + n_{fp} < \infty$	***pp***: Passes Test and Good in Use $0 \le n_{ff} + n_{pf} \le n_t$ $0 \le n_{ff} + n_{fp} \le n_u$
ff: Fails Test and Bad in Use $n_t < n_{ff} + n_{pf} < \infty$ $n_u < n_{ff} + n_{fp} < \infty$	***pf***: Fails Test and Good in Use $n_t < n_{ff} + n_{pf} < \infty$ $0 \le n_{ff} + n_{fp} \le n_u$

Examples of four arrays of the same size categorized according to Table 1 are given in Figure 1. The number of bits in each bit category is shown by the vertical size of colored areas and varies from array to array. The diagrams in the figure do not accurately depict the Poisson limit because in that limit the number of *pp* bits will be much larger than the number of any other category. But this doesn't matter since the dimension of the *pp* region can be increased without limit without affecting the discussion. Keep in mind that Table 1 and Figure 1 describe the categorization of the entire population of bits and arrays irrespective of whether the test manufacturing flow physically removes failing arrays at Test. That is why the number of bits in various categories is unchanged between Test and Use in Figure 1. The definition of figures of merit in terms of conditional probabilities describes the effect of physically removing failures at Test.

The probability that an array falls into any of the four array categories is found by summing Poisson terms, Eq. (S17), over the set of integers $\{n_{pf}, n_{fp}, n_{ff}\}$ allowed by the criteria given in Table 1. The probabilities of interest may be expressed in terms of the bivariate correlated Poisson distribution introduced by Campbell[3]. If an array has *exactly* n_u bits which are bad in Use and *exactly* n_t bits which are bad at Test (that is, the arrays are at the limit of fault tolerance capacity), then the random variables N_{fp}, N_{pf} and N_{ff} may vary within the following constraints:

$$N_{ff} + N_{fp} = n_u \quad N_{ff} + N_{pf} = n_t \quad 0 \le N_{ff} \le \min[n_u, n_t] \tag{S25}$$

where the last inequality is a way of expressing the constraints $N_{pf} \ge 0$ and $N_{fp} \ge 0$. So the probability that an array has *exactly* n_u bits failing in Use and *exactly* n_t bits failing in Test is the sum of Poisson terms, Eq. (S17), over bit category counts allowed by Eq. (S25):

[3] J. T. Campbell (1934). The Poisson Correlation Function. Proceedings of the Edinburgh Mathematical Society (Series 2), 4 , pp 18-26 http://dx.doi.org/10.1017/S0013091500024135

$$
\begin{aligned}
&P\left(N_{ff}+N_{fp}=n_u,N_{ff}+N_{pf}=n_t\right)\\
&\quad=e^{-(\lambda_{fp}+\lambda_{pf}+\lambda_{ff})}\sum_{n_{ff}=0}^{\min[n_u,n_t]}\frac{\lambda_{fp}^{n_u-n_{ff}}\lambda_{pf}^{n_t-n_{ff}}\lambda_{ff}^{n_{ff}}}{(n_u-n_{ff})!(n_t-n_{ff})!n_{ff}!}\\
&\quad\equiv\text{pois}\left(n_u,n_t;\lambda_{fp},\lambda_{pf},\lambda_{ff}\right).
\end{aligned}
\qquad\text{(S26)}
$$

which is Campbell's bivariate Poisson distribution.

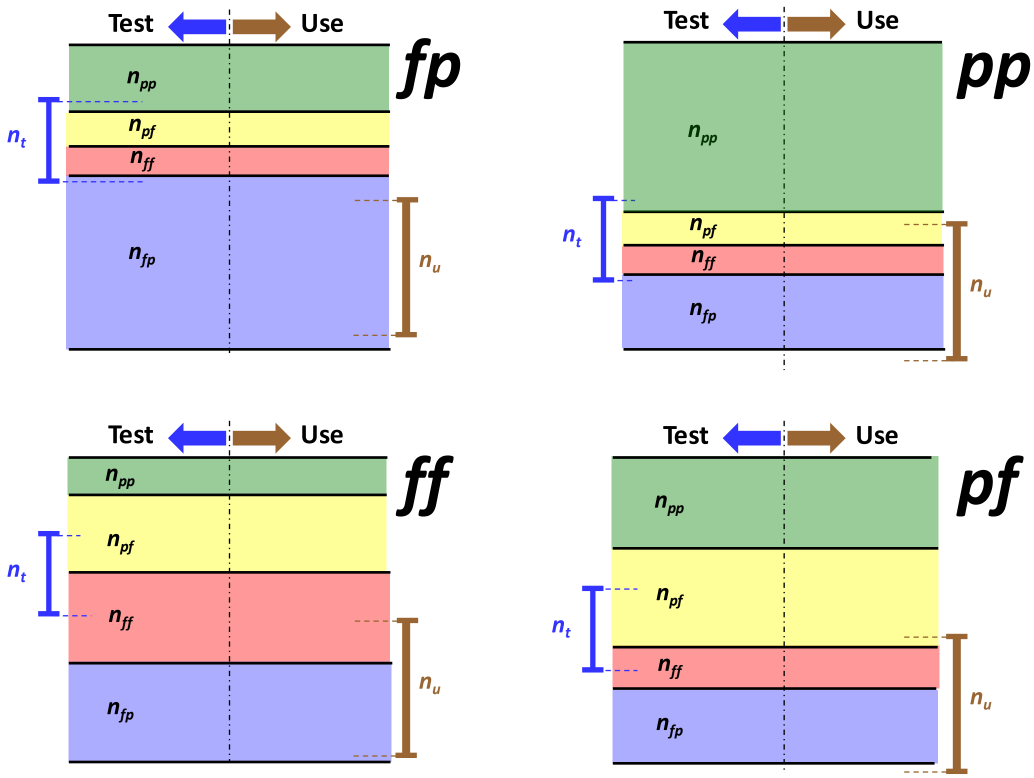


Figure 1 Examples of four arrays falling into each of the four array categories by the criteria of Table 1. The number of bits in each category is unchanged between Test and Use.

For the present application the cumulative form of Campbell's distribution is needed to sum Poisson terms, Eq. (S17), for all arrays within, not just at the limit of, the fault tolerance capacity:

$$
\begin{aligned}
&P\left(N_{ff}+N_{fp}\le n_u,N_{ff}+N_{pf}\le n_{tt}\right)\\
&\quad=\sum_{m=0}^{n_u}\sum_{n=0}^{n_t}\text{pois}\left(m,n;\lambda_{fp},\lambda_{pf},\lambda_{ff}\right)\\
&\quad=\sum_{i=0}^{\min[n_u,n_t]}\frac{\lambda_{ff}^{i}e^{-\lambda_{ff}}}{i!}R\left(\lambda_{fp},n_u-i\right)R\left(\lambda_{pf},n_t-i\right)\\
&\quad\equiv\text{Pois}\left(n_u,n_t;\lambda_{fp},\lambda_{pf},\lambda_{ff}\right)
\end{aligned}
\qquad\text{(S27)}
$$

where R is

$$
R\left(x,n\right)\equiv e^{-x}\sum_{0\le i\le n}\frac{x^i}{i!}. \qquad\text{(S28)}
$$

The second equality in Eq. (S27) is derived in an Appendix to these supplemental notes (Derivation of Eq. {22}). Eq. (S27) (aka Eq. {22}) provides a way to compute the cumulative bivariate Poisson distribution in terms of a sum over products of the function R given by Eq. (S28). R is a standard statistical function available, for example, in Excel. The sum in the second equality in Eq. (S27) is over a small number of terms since n_u and n_t are usually small integers.

From Table 1 and Eq. (S27) the probability that an array will be good in both Test and Use is

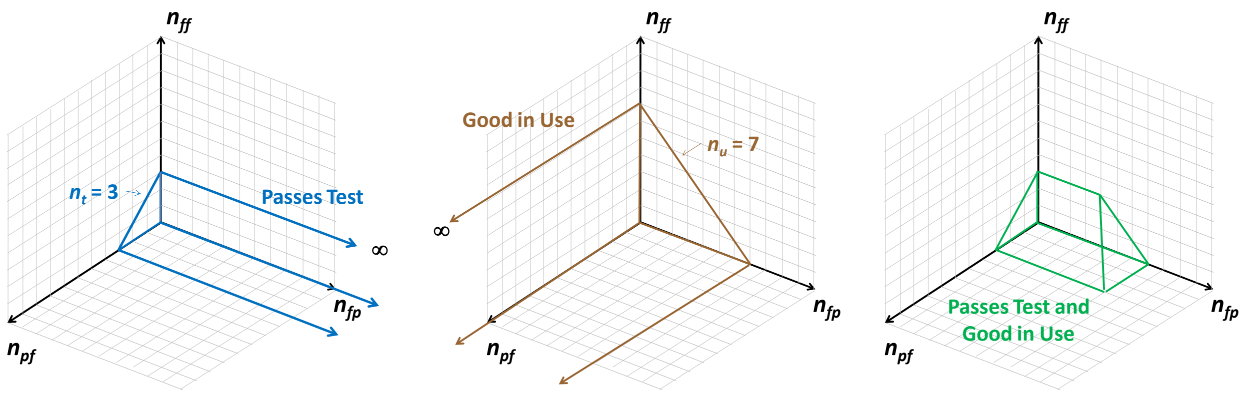
$$P(\text{Passes Test and Good in Use}) = \text{Pois}\left(n_u, n_t; \lambda_{fp}, \lambda_{pf}, \lambda_{ff}\right) \tag{S29}$$

The probability that an array is good in Use, irrespective of its Test category, is the union of the right-hand column of cells in Table 1 which amounts to setting $n_t = \infty$ in the cumulative Poisson distribution

$$P(\text{Good in Use}) = \text{Pois}\left(n_u, n_t \to \infty; \lambda_{fp}, \lambda_{pf}, \lambda_{ff}\right) = R\left(\lambda_{fp} + \lambda_{ff}, n_u\right) \tag{S30}$$

where we have used a property of Pois given in the Appendix (Eq. (S55). Similarly, the probability that an array is good in Test, irrespective of its Use category, is the union of the top row of cells in Table 1 which amounts to setting $n_u = \infty$ in the cumulative Poisson distribution

$$P(\text{Passes Test}) = \text{Pois}\left(n_u \to \infty, n_t; \lambda_{fp}, \lambda_{pf}, \lambda_{ff}\right) = R\left(\lambda_{pf} + \lambda_{ff}, n_t\right). \tag{S31}$$



$0 \le n_{ff} + n_{pf} \le n_t$
$0 \le n_{fp} < \infty$

$0 \le n_{ff} + n_{fp} \le n_u$
$0 \le n_{pf} < \infty$

$0 \le n_{ff} + n_{pf} \le n_t$
$0 \le n_{ff} + n_{fp} \le n_u$

Figure 2. Regions of {n_{pf}, n_{fp}, n_{ff}} space for sums over Poisson terms in no-repair probability functions. Shown for n_u = 7 and n_t = 3.

Each of the probabilities in Eqs. (S29), (S30), and (S31) may be regarded as sums of Poisson terms, Eq. (S17), over bit category indexes allowed by criteria given in Table 1. The allowed sets of integers may be visualized as regions in bit category count space shown in Figure 2. The three probabilities given in Eqs. (S29), (S30), and (S31) may be substituted into Eqs. (S12), (S13), and (S14) to give the figures of merit.

<u>Fault Tolerance With Repair</u>

Frequently a Test step actively repairs bits that it detects as bad by replacing a bad bit with a good one. The replacement may be accomplished by, for example, reconnecting decoder circuitry to a different spare column or block of circuitry. The repair re-categorizes bits in the *pf* and *ff* categories as *pp* bits. This has two consequences, first it increases the number of *pp* bits, and second it reduces the number of bad bits that fault tolerance mechanisms in Use must cover. The effect of repair at Test on bit categories in an array at Test and at Use is illustrated in Figure 3. In the figure, the example arrays have the same proportions of bits in categories before Test as those in Figure 1, but the proportions are modified in Use because of the effect of repair at Test.

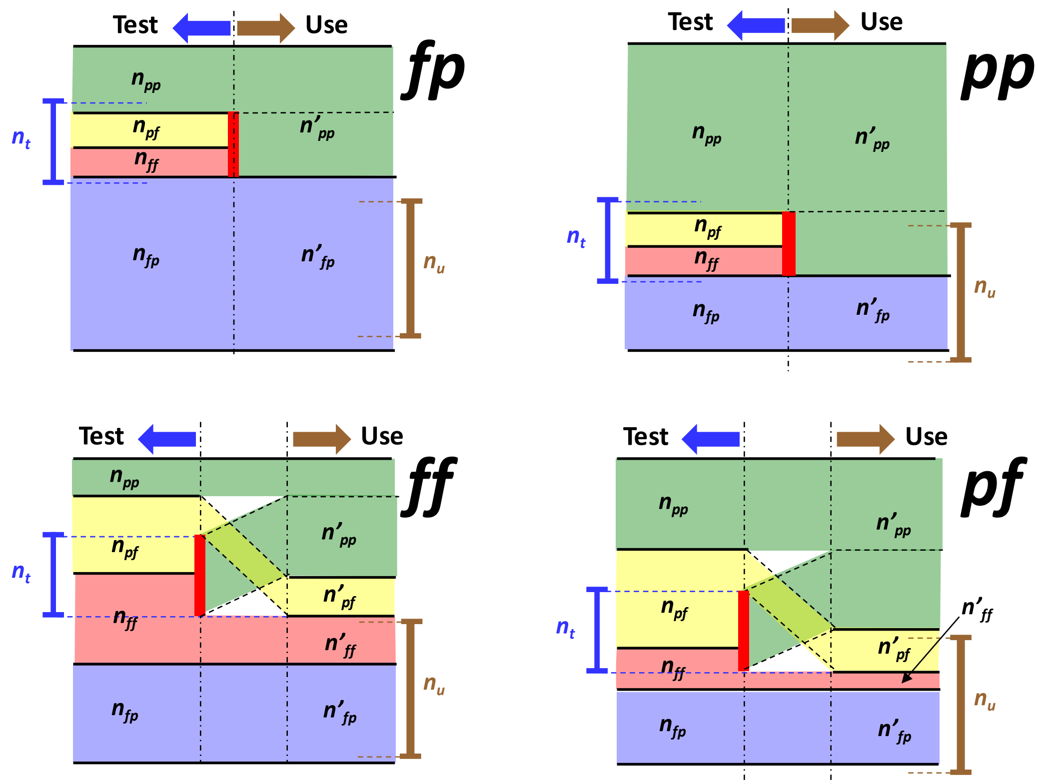


Figure 3. Examples of four arrays falling into each of the four array categories by the criteria of Table 2. The repair of bits failing at Test is represented by re-categorizing them as *pp* bits. The change in bit counts in categories describes the effect of active repair at Test. It is *not* due to screening (removal of arrays) at Test.

The model has three assumptions:

1. The effect of repair on a bit that fails in Test (a *pf* or *ff* bit) is to convert it to a *pp* bit. This assumes that the reservoir of spares has only perfect bits. Other models are possible. For example, the spare bits, and therefore the repaired bits, could be assumed to have the same category proportions as in the array before Test. In practice the proportion of *pp* bits is so much larger than the other categories that assuming that all repaired bits are in the *pp* category is an excellent approximation.

2. If the number of *pf* and *ff* bits in an array does not exceed the fault tolerance capacity of Test, then the array has no *pf* or *ff* bits in Use.

3. If the number of *pf* and *ff* bits in an array exceeds the fault tolerance capacity of Test, then the total number of *pf* and *ff* bits in the array in Use is $n_{pf} + n_{ff} - n_t$. The bits are *pf* and *ff* bits in the same proportion as in population of *pf* and *ff* bits before repair at Test because Test cannot "know" whether a bit that it repairs would have failed in Use or not.

In this model, for a given array the number of bits in the four categories after re-categorization by repair at Test is, in terms of the pre-Test categories (see Derivation of Eq. (S32) in the Appendix)

$$
\begin{aligned}
n'_{pp} &= n_{pp} + \min\left[n_{pf} + n_{ff}, n_t\right] \\
n'_{pf} &= \left\lfloor \max\left[n_{ff} + n_{pf} - n_t, 0\right] \frac{n_{pf}}{n_{ff} + n_{pf}} \right\rfloor \\
n'_{ff} &= \left\lceil \max\left[n_{ff} + n_{pf} - n_t, 0\right] \frac{n_{ff}}{n_{ff} + n_{pf}} \right\rceil \\
n'_{fp} &= n_{fp}
\end{aligned}
\tag{S32}
$$

where post-repair category counts are shown as primed, and the ceiling and floor functions force integer values in a way that is conservative to the end-user.

Figure 3 and Table 2 shows that unprimed pre-repair bit category counts are used to determine the Test pass/fail attribute of an array, whereas the primed bit category counts are used to determine the Use pass/fail category of an array.

Table 2. Array categories vs bit category counts, and tolerance capacities for repair at Test.

fp: Passes Test and Bad in Use $0 \le n_{ff} + n_{pf} \le n_t$ $n_u < n'_{ff} + n'_{fp} < \infty$	***pp***: Passes Test and Good in Use $0 \le n_{ff} + n_{pf} \le n_t$ $0 \le n'_{ff} + n'_{fp} \le n_u$
ff: Fails Test and Bad in Use $n_t < n_{ff} + n_{pf} < \infty$ $n_u < n'_{ff} + n'_{fp} < \infty$	***pf***: Fails Test and Good in Use $n_t < n_{ff} + n_{pf} < \infty$ $0 \le n'_{ff} + n'_{fp} \le n_u$

Passes Test. The union of the "Passes Test" criteria in the top row of Table 2 gives the indexes which must be summed over to give P(Passes Test), irrespective of Use:

$$0 \le n_{ff} + n_{pf} \le n_t,\ 0 \le n'_{ff} + n'_{fp} < \infty \tag{S33}$$

which is the same as

$$0 \le n_{ff} + n_{pf} \le n_t,\ 0 \le n_{fp} < \infty \tag{S34}$$

because $n'_{pf} = n_{fp}$ and because, from Eq. (S32), $n'_{ff} = 0$ when the first condition in Eq. (S33) is satisfied. Eq. (S34) is the same criterion for P(Passes Test) for the no-repair case, so

$$P(\text{Passes Test}) = R\left(\lambda_{pf} + \lambda_{ff}, n_t\right) \quad \text{(repair).} \tag{S35}$$

Passes Test and Good in Use. The same argument reduces the criterion in the top right cell of Table 2 to

$$0 \le n_{ff} + n_{pf} \le n_t,\ 0 \le n_{fp} \le n_u \tag{S36}$$

which differs from the corresponding cell in the no-repair case (Table 1). The corresponding probability when summed over Poisson terms is the tidy factorization (see Derivation of Eq. {29} in the Appendix)

$$P(\text{Passes Test and Good in Use}) = R\left(\lambda_{ff} + \lambda_{pf}, n_t\right) R\left(\lambda_{fp}, n_u\right) \quad \text{(repair)}\ . \tag{S37}$$

Good in Use. The union of the criteria in the right-hand column of Table 2 gives the gives the indexes which must be summed over to give P(Good in Use), irrespective of Test.

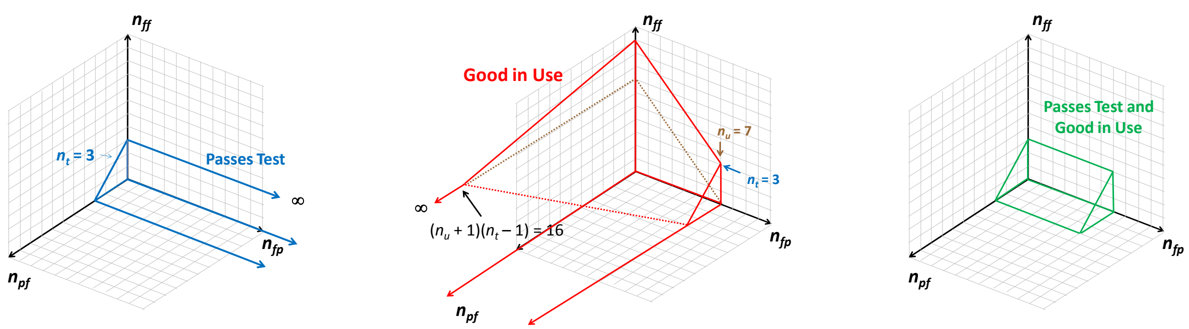
$$0 \le n_{ff} + n_{pf} < \infty, \;\; 0 \le n'_{ff} + n'_{fp} \le n_u \tag{S38}$$

which is the same as the following constraint on unprimed bit indexes to be summed over

$$0 \le n_{pf} < \infty, \quad 0 \le n'_{ff}(n_{ff}, n_{pf}, n_t) + n_{fp} \le n_u \tag{S39}$$

because $n'_{fp} = n_{fp}$ and the first condition in Eq. (S38) can be broken into $0 \le n_{ff} < \infty$ and $0 \le n_{pf} < \infty$, but the first of these is superfluous because of the tighter constraint on n_{ff} imposed by the second condition in Eq. (S39).

The volumes in $\{n_{pf}, n_{fp}, n_{ff}\}$ space allowed by the conditions of Eqs. (S34), (S36), and (S39) are shown in Figure 4 for example values of Test and Use tolerance criteria. The shape of the Good in Use volume suggests dividing it into two parts and then adding the probability functions for each part. The two parts are: 1) The semi-infinite prism which is the same as the no-repair Good in Use case leading to an analytical formula, Eq. (S30), for the probability, and 2) The extra "irregular volume" on top of the semi-infinite prism near the origin for which $\{n_{pf}, n_{fp}, n_{ff}\}$ points will be generated to sum the Poisson terms for this volume. The finite size of the irregular volume limits the amount of computation necessary.



$0 \le n_{ff} + n_{pf} \le n_t$
$0 \le n_{fp} < \infty$

$0 \le n_{pf} < \infty$
$0 \le n'_{ff}(n_{ff}, n_{pf}, n_t) + n_{fp} \le n_u$

$0 \le n_{ff} + n_{pf} \le n_t$
$0 \le n_{fp} \le n_u$

Figure 4 Regions of $\{n_{pf}, n_{fp}, n_{ff}\}$ space for sums over Poisson terms in the repair-at-Test probability functions. For $n_u = 7$ and $n_t = 3$.

The $\{n_{pf}, n_{fp}, n_{ff}\}$ points satisfying the Good in Use conditions for repair at Test, Eq. (S39) will satisfy

$$\left.\begin{array}{l} 0 \le n_{pf} < \infty \\ 0 \le n_{ff} + n_{fp} \le n_u \end{array}\right\} \text{Semi-infinite prism. OR } \left.\begin{array}{l} 0 \le n_{pf} < \infty \\ 0 \le n'_{ff}(n_{ff}, n_{pf}, n_t) + n_{fp} \le n_u \\ n_u < n_{ff} + n_{fp} \end{array}\right\} \text{Irregular volume.} \tag{S40}$$

The last condition for the irregular volume in Eq. (S40) ensures that the points in the semi-infinite prism and the irregular volume are mutually exclusive. Only points for which n_{pf} and n_t are such that $n'_{ff} < n_{ff}$ will be included in the irregular volume, because if $n'_{ff} = n_{ff}$ the second and last conditions for the irregular volume are mutually exclusive.

The irregular volume criteria in Eq. (S40) can be used directly to find $\{n_{pf}, n_{fp}, n_{ff}\}$ points and accumulate a sum of Poisson terms, Eq. (S17), to compute the probability function of the irregular volume. Upper bounds on each of n_{pf}, n_{fp}, and n_{ff} are needed to limit the number of points that need to be tested to find points satisfying the "Irregular volume" criteria in Eq. (S40). Inspection of the irregular volume criteria in Eq. (S40) and the function n'_{ff} in Eq.

(S32) shows that n_{ff} cannot exceed $n_u + n_t$ and that n_{fp} cannot exceed n_u. It is clear from the second of the Irregular volume criteria in Eq. (S40) that n_{pf} also has an upper bound but it is not obvious how to compute it. A useful upper bound on n_{pf} is given in the Appendix (Derivation of Eq. (S41)) to this supplemental document

$$n_{pf} \le (n_t - 1) n_{ff}. \tag{S41}$$

So now the irregular volume criteria of Eq. (S40) may be rewritten with additional limits which do not change the number of points in the irregular volume, but reduce the number of points that need to be tested to find them

$$\left.\begin{aligned} &0 \le n_{ff} \le n_u + n_t \\ &0 \le n_{fp} \le n_u \\ &0 \le n_{pf} \le n_{ff}(n_t - 1) \\ &0 \le n'_{ff}(n_{ff}, n_{pf}, n_t) + n_{fp} \le n_u \\ &n_u < n_{ff} + n_{fp} \end{aligned}\right\} \text{Irregular volume.} \tag{S42}$$

Notice that if $n_t = 0$, then no index points can satisfy the criteria and the irregular volume is empty of Poisson terms, as it should be. Also notice that the largest possible value of $n_{pf} = (n_t - 1)(n_u + 1)$ occurs when $n_{fp} = 0$ and $n_{ff} = n_u + 1$.

So the Good in Use probability function may be written

$$P(\text{Good in Use}) = L\left(\lambda_{ff}, \lambda_{fp}, \lambda_{pf}, n_u, n_t\right) + R\left(\lambda_{ff} + \lambda_{fp}, n_u\right) \tag{S43}$$

where L is the probability function corresponding to $\{n_{pf}, n_{fp}, n_{ff}\}$ points which satisfy Eq. (S42)

$$L\left(\lambda_{ff}, \lambda_{fp}, \lambda_{pf}, n_u, n_t\right) = \sum_{\text{Irregular Volume}} \frac{\lambda_{ff}^{n_{ff}} e^{-\lambda_{ff}}}{n_{ff}!} \frac{\lambda_{fp}^{n_{fp}} e^{-\lambda_{fp}}}{n_{fp}!} \frac{\lambda_{pf}^{n_{pf}} e^{-\lambda_{pf}}}{n_{pf}!}. \tag{S44}$$

Appendix

Properties of Campbell's Bivariate Correlated Poisson Distribution

Suppose that N_{fp}, N_{pf}, and N_{ff} are mutually independent Poisson random variables with means λ_{fp}, λ_{pf}, and λ_{ff}. Now construct two random variables N_u and N_t as follows:

$$\begin{aligned} N_u &= N_{pf} + N_{ff} \\ N_t &= N_{fp} + N_{ff} \end{aligned} \tag{S45}$$

The probability that N_u is exactly n_u and N_t is exactly n_t is the correlated Poisson distribution as Campbell expressed it

$$P\left(N_u = n_u, N_t = n_t\right) = e^{-(\lambda_{fp} + \lambda_{pf} + \lambda_{ff})} \sum_{n_{ff}=0}^{\min[n_u, n_t]} \frac{\lambda_{fp}^{n_u - n_{ff}} \lambda_{pf}^{n_t - n_{ff}} \lambda_{ff}^{n_{ff}}}{(n_u - n_{ff})!(n_t - n_{ff})! n_{ff}!} \equiv \text{pois}\left(n_u, n_t; \lambda_{fp}, \lambda_{pf}, \lambda_{ff}\right). \tag{S46}$$

which is the sum over all terms like

$$P(N_{fp} = n_{fp}, N_{pf} = n_{pf}, N_{ff} = n_{ff}) = \frac{\lambda_{fp}^{n_{fp}} \exp(-\lambda_{fp})}{n_{fp}!} \frac{\lambda_{pf}^{n_{pf}} \exp(-\lambda_{pf})}{n_{pf}!} \frac{\lambda_{ff}^{n_{ff}} \exp(-\lambda_{ff})}{n_{ff}!} \tag{S47}$$

allowed by the constraints

$$n_u = n_{fp} + n_{ff} \quad \text{and} \quad n_t = n_{pf} + n_{ff}. \tag{S48}$$

Of particular interest in the present application is the cumulative form of the distribution of Eq. (S46):

$$P\left(N_u \le n_u, N_t \le n_t\right) = \sum_{m=0}^{n_u}\sum_{n=0}^{n_t} \mathrm{pois}\left(m, n; \lambda_{fp}, \lambda_{pf}, \lambda_{ff}\right) \equiv \mathrm{Pois}\left(n_u, n_t; \lambda_{fp}, \lambda_{pf}, \lambda_{ff}\right). \tag{S49}$$

It is useful to express the cumulative form of Campbell's distribution in terms of the univariate cumulative Poisson distribution defined as:

$$R\left(x,n\right) \equiv e^{-x} \sum_{0\le i\le n} \frac{x^i}{i!} \ . \tag{S50}$$

for which $R(x,0) = \exp(-x)$, which vanishes for $n < 0$, and is unity for $n \to \infty$. The cumulative Poisson distribution is available in Excel as

$$R(x,n) = \mathrm{POISSON}(n, x, \mathrm{TRUE}). \tag{S51}$$

<u>Derivation of Eq. {22}</u>

Evaluation of the cumulative form of Campbell's bivariate correlated Poisson distribution is facilitated by expressing it as a sum over products of R because only a few terms are typically needed. We have not seen Eq. {22} in the litererature so a derivation is given here. Eq. {22} is

$$\mathrm{Pois}\left(n_u, n_t; \lambda_{fp}, \lambda_{pf}, \lambda_{ff}\right) = \sum_{i=0}^{\min[n_u, n_t]} \frac{e^{-\lambda_{ff}} \lambda_{ff}^i}{i!} R\left(\lambda_{fp}, n_u - i\right) R\left(\lambda_{pf}, n_t - i\right) \tag{S52}$$

Notice that the following properties of the cumulative bivariate Poisson distribution can be written by inspection of Eq. (S52):

$$\begin{aligned}
\mathrm{Pois}\left(n_u, n_t = 0; \lambda_{fp}, \lambda_{pf}, \lambda_{ff}\right) &= \exp\left[-(\lambda_{pf} + \lambda_{ff})\right] R\left(\lambda_{fp}, n_u\right) \\
\mathrm{Pois}\left(n_u = 0, n_t; \lambda_{fp}, \lambda_{pf}, \lambda_{ff}\right) &= \exp\left[-(\lambda_{fp} + \lambda_{ff})\right] R\left(\lambda_{pf}, n_u\right) \\
\mathrm{Pois}\left(n_u = 0, n_t = 0; \lambda_{fp}, \lambda_{pf}, \lambda_{ff}\right) &= \exp\left[-(\lambda_{fp} + \lambda_{pf} + \lambda_{ff})\right]
\end{aligned} \tag{S53}$$

Here is the derivation of Eq. (S52) (aka Eq. {22}):

$$
\begin{aligned}
\text{Pois}\left(n_u,n_t;\lambda_{fp},\lambda_{pf},\lambda_{ff}\right) &= \sum_{m=0}^{n_u}\sum_{n=0}^{n_t}\text{pois}\left(m,n;\lambda_{fp},\lambda_{pf},\lambda_{ff}\right) \\
&= \sum_{m=0}^{n_u}\sum_{n=0}^{n_t} e^{-(\lambda_{fp}+\lambda_{pf}+\lambda_{ff})} \sum_{n_{ff}=0}^{\min[m,n]} \frac{\lambda_{fp}^{m-n_{ff}}\lambda_{pf}^{n-n_{ff}}\lambda_{ff}^{n_{ff}}}{(m-n_{ff})!(n-n_{ff})!n_{ff}!} \\
&= \sum_{m=0}^{n_u}\sum_{n=0}^{n_t} \sum_{n_{ff}=0}^{\min[n_u,n_t]} \frac{e^{-\lambda_{fp}}\lambda_{fp}^{m-n_{ff}}}{(m-n_{ff})!} \frac{e^{-\lambda_{pf}}\lambda_{pf}^{n-n_{ff}}}{(n-n_{ff})!} \frac{e^{-\lambda_{ff}}\lambda_{ff}^{n_{ff}}}{n_{ff}!} && \text{(A)} \\
&= \sum_{n_{ff}=0}^{\min[n_u,n_t]} \frac{e^{-\lambda_{ff}}\lambda_{ff}^{n_{ff}}}{n_{ff}!} \sum_{m=0}^{n_u} \frac{e^{-\lambda_{fp}}\lambda_{fp}^{m-n_{ff}}}{(m-n_{ff})!} \sum_{n=0}^{n_t} \frac{e^{-\lambda_{pf}}\lambda_{pf}^{n-n_{ff}}}{(n-n_{ff})!} && \text{(B)} \qquad \text{(S54)} \\
&= \sum_{n_{ff}=0}^{\min[n_u,n_t]} \frac{e^{-\lambda_{ff}}\lambda_{ff}^{n_{ff}}}{n_{ff}!} \sum_{m=n_{ff}}^{n_u} \frac{e^{-\lambda_{fp}}\lambda_{fp}^{m-n_{ff}}}{(m-n_{ff})!} \sum_{n=n_{ff}}^{n_t} \frac{e^{-\lambda_{pf}}\lambda_{pf}^{n-n_{ff}}}{(n-n_{ff})!} && \text{(C)} \\
&= \sum_{n_{ff}=0}^{\min[n_u,n_t]} \frac{e^{-\lambda_{ff}}\lambda_{ff}^{n_{ff}}}{n_{ff}!} \sum_{j=0}^{n_u-n_{ff}} \frac{e^{-\lambda_{fp}}\lambda_{fp}^{j}}{j!} \sum_{k=0}^{n_t-n_{ff}} \frac{e^{-\lambda_{pf}}\lambda_{pf}^{k}}{k!} && \text{(D)} \\
&= \sum_{i=0}^{\min[n_u,n_t]} \frac{e^{-\lambda_{ff}}\lambda_{ff}^{i}}{i!} R\left(\lambda_{fp},n_u-i\right)R\left(\lambda_{pf},n_t-i\right) && \text{(E)}
\end{aligned}
$$

Notes for manipulations in Eq. (S54):

- Notice that $1/i!$ vanishes whenever i is a negative integer.
- Keeping in mind that $0 \le m \le n_u$ and $0 \le n \le n_t$, the upper limit in the sum over n_{ff} may be changed from $\min[m, n]$ to $\min[n_u, n_t]$ in (A) because all the additional terms will have the factorial of a negative integer in the denominator.
- The changed limit of the sum over n_{ff} in (A) permits the change of summation order in (B).
- The limits of sums over m and n in (C) are changed from (B) because the omitted terms all have the factorial of a negative integer in the denominator and so vanish.
- A change of summation index in (D) brings the sums over m and n in (C) into a form recognized as the definition of the univariate cumulative Poisson distribution.
- Recognition that n_{ff} is a dummy index leads to the final expression, (E).

Derivation of Eqs. {26} and {27}

Marginal distributions of Campbell's bivariate correlated Poisson distribution are given in Eqs. {26} and {27} of the paper:

$$
\begin{aligned}
&\text{Pois}\left(n_u\to\infty,n_t\to\infty;\lambda_{fp},\lambda_{pf},\lambda_{ff}\right)=1 \\
&\text{Pois}\left(n_u,n_t\to\infty;\lambda_{fp},\lambda_{pf},\lambda_{ff}\right)=R\left(\lambda_{fp}+\lambda_{ff},n_u\right) \qquad \text{(S55)} \\
&\text{Pois}\left(n_u\to\infty,n_t;\lambda_{fp},\lambda_{pf},\lambda_{ff}\right)=R\left(\lambda_{pf}+\lambda_{ff},n_t\right)
\end{aligned}
$$

Eq. (S55) is derived as follows:

$$
\begin{aligned}
\mathrm{Pois}\left(n_u,n_t\to\infty;\lambda_{fp},\lambda_{pf},\lambda_{ff}\right) &= \sum_{i=0}^{\min[n_u,n_t]}\frac{e^{-\lambda_{ff}}\lambda_{ff}^{i}}{i!}R\left(\lambda_{fp},n_u-i\right)R\left(\lambda_{pf},n_t-i\right)\\
&=\sum_{i=0}^{n_u}\frac{e^{-\lambda_{ff}}\lambda_{ff}^{i}}{i!}R\left(\lambda_{fp},n_u-i\right)\\
&=\sum_{i=0}^{n_u}\frac{e^{-\lambda_{ff}}\lambda_{ff}^{i}}{i!}\sum_{j=0}^{n_u-i}\frac{e^{-\lambda_{fp}}\lambda_{fp}^{j}}{j!}\\
&=\exp\left[-(\lambda_{ff}+\lambda_{fp})\right]\sum_{i=0}^{n_u}\sum_{j=0}^{n_u-i}\frac{\lambda_{ff}^{i}}{i!}\frac{\lambda_{fp}^{j}}{j!}\\
&=\exp\left[-(\lambda_{ff}+\lambda_{fp})\right]\sum_{i=0}^{n_u}\sum_{m=i}^{n_u}\frac{\lambda_{ff}^{i}}{i!}\frac{\lambda_{fp}^{m-i}}{(m-i)!} \qquad \text{(S56)}\\
&=\exp\left[-(\lambda_{ff}+\lambda_{fp})\right]\sum_{i=0}^{n_u}\sum_{m=0}^{n_u}\frac{\lambda_{ff}^{i}}{i!}\frac{\lambda_{fp}^{m-i}}{(m-i)!} \qquad \text{(F)}\\
&=\exp\left[-(\lambda_{ff}+\lambda_{fp})\right]\sum_{m=0}^{n_u}\frac{1}{m!}\sum_{i=0}^{m}\frac{m!}{i!(m-i)!}\lambda_{ff}^{i}\lambda_{fp}^{m-i} \qquad \text{(G)}\\
&=\exp\left[-(\lambda_{ff}+\lambda_{fp})\right]\sum_{m=0}^{n_u}\frac{(\lambda_{ff}+\lambda_{fp})^{m}}{m!}\\
&=R\left(\lambda_{ff}+\lambda_{fp},n_u\right)
\end{aligned}
$$

Notes for manipulations in Eq. (S56)

- The sum over m in (F) may be extended down to zero because terms involving $(m-i)!$ in the denominator will vanish for $0 \le m \le i-1$. (Factorial of a negative integer diverges). This enables the interchange of summation order in the next line.
- The sum over i in (G) may be truncated at m rather be than continued to n_u because terms involving $(m-i)!$ in the denominator with $m+1 \le i \le n_u$ will vanish. This enables direct invocation of the binomial theorem in the next line.

<u>Derivation of Eq. {29}</u>

A factorization of a sum over Poisson terms which is not expressible in terms of Campbell's function arose in the theory of Test with repair.

$$
P=\sum_{\substack{0\le n_{ff}+n_{pf}\le n_t\\ 0\le n_{fp}\le n_u}}\frac{\lambda_{ff}^{n_{ff}}\exp(-\lambda_{ff})}{n_{ff}!}\frac{\lambda_{pf}^{n_{pf}}\exp(-\lambda_{pf})}{n_{pf}!}\frac{\lambda_{fp}^{n_{fp}}\exp(-\lambda_{fp})}{n_{fp}!}=R\left(\lambda_{ff}+\lambda_{pf},n_t\right)R\left(\lambda_{fp},n_u\right) \qquad \text{(S57)}
$$

The proof proceeds by noticing that the sum over n_{fp} gives one of the R factors, leaving the rest as a sum recognizable as the binomial theorem which leads to the other R factor.

$$
\begin{aligned}
P&=\sum_{\substack{0\le n_{ff}+n_{pf}\le n_t\\ 0\le n_{fp}\le n_u}}\frac{\lambda_{ff}^{n_{ff}}\exp(-\lambda_{ff})}{n_{ff}!}\frac{\lambda_{pf}^{n_{pf}}\exp(-\lambda_{pf})}{n_{pf}!}\frac{\lambda_{fp}^{n_{fp}}\exp(-\lambda_{fp})}{n_{fp}!}=\exp\left[-(\lambda_{ff}+\lambda_{pf})\right]\sum_{j=0}^{n_t}\sum_{i=0}^{j}\frac{\lambda_{ff}^{j-i}}{(j-i)!}\frac{\lambda_{pf}^{i}}{i!}\exp(-\lambda_{fp})\sum_{k=0}^{n_u}\frac{\lambda_{fp}^{k}}{k!}\\
&=\exp\left[-(\lambda_{ff}+\lambda_{pf})\right]\sum_{j=0}^{n_t}\frac{1}{j!}\sum_{i=0}^{j}\frac{j!}{(j-i)!i!}\lambda_{ff}^{j-i}\lambda_{pf}^{i}R\left(\lambda_{fp},n_u\right)=\exp\left[-(\lambda_{ff}+\lambda_{pf})\right]\sum_{j=0}^{n_t}\frac{(\lambda_{ff}+\lambda_{pf})^{j}}{j!}R\left(\lambda_{fp},n_u\right)\\
&=R\left(\lambda_{ff}+\lambda_{pf},n_t\right)R\left(\lambda_{fp},n_u\right)
\end{aligned}
$$

(S58)

Derivation of Eq. (S32)

The number of bits in the *pp* category is increased by the action of repair converting bits in *ff* and *pf* categories to *pp* bits. Recall that *ff* and *pf* bits fail in Test, irrespective of whether the bits are good or bad in Use. That is, after repair, the number of bits in the *pp* category is

$$n'_{pp} = n_{pp} + \min\left[n_{ff} + n_{pf}, n_t\right] \tag{S59}$$

Eq. (S59) shows that if the number of bad bits at Test, $n_{ff} + n_{pf}$, exceeds the repair capacity, n_t, then the number of *pp* bits is increased by n_t. Otherwise, the number of *pp* bits is increased only by the number of available bits which are bad at Test, $n_{ff} + n_{pf}$.

Strictly, the model of Eq. (S59) is unrealistic because it assumes that Test can convert a failing bit into one that is certain to be good in Use. That is, that all replacement bits will be good in Use. A more realistic model would apportion $\min[n_{ff} + n_{pf}, n_t]$ across other categories using some rule. The rule might be that the repair bits are distributed across categories *pp*, *pf*, *fp*, *ff* in the same way as the bits in the unrepaired array. However, the very large preponderance of bits in the *pp* category makes Eq. (S59) a very good approximation.

The action of repair reduces the number of bits in the combined *ff* and *pf* categories by the same amount as the numbers in the *pp* category were increased:

$$\begin{aligned}
n'_{ff} + n'_{pf} &= n_{ff} + n_{pf} - \min\left[n_{ff} + n_{pf}, n_t\right] \\
&= n_{ff} + n_{pf} + \max\left[n_{ff} + n_{pf}, n_t\right] - \left(n_{ff} + n_{pf} + n_t\right) \\
&= \max\left[n_{ff} + n_{pf}, n_t\right] - n_t \\
&= \max\left[n_{ff} + n_{pf} - n_t, 0\right]
\end{aligned} \tag{S60}$$

where we have used $\min[x, y] + \max[x, y] = x + y$.

Because of causality, Test cannot discriminate between *ff* and *pf* bits when it repairs them, and when it does *not* repair them. So it is reasonable to have $n'_{ff} : n'_{pf} \simeq n_{ff} : n_{pf}$ as nearly as possible, consistent with category counts being integers. For integer a, x, and y the following is an identity for $x + y > 0$

$$a = \left\lceil a\frac{x}{x+y} \right\rceil + \left\lfloor a\frac{y}{x+y} \right\rfloor = \left\lfloor a\frac{x}{x+y} \right\rfloor + \left\lceil a\frac{y}{x+y} \right\rceil . \tag{S61}$$

Using this, the post-repair counts of *ff* and *fp* bits may be partitioned into integer counts as follows

$$\begin{aligned}
n'_{ff} + n'_{pf} &= \max\left[n_{ff} + n_{pf} - n_t, 0\right] \\
&= \left\lceil \max\left[n_{ff} + n_{pf} - n_t, 0\right]\frac{n_{ff}}{n_{ff} + n_{pf}} \right\rceil + \left\lfloor \max\left[n_{ff} + n_{pf} - n_t, 0\right]\frac{n_{pf}}{n_{ff} + n_{pf}} \right\rfloor .
\end{aligned} \tag{S62}$$

where the first term is taken to be n'_{ff} and the second is taken to be n'_{pf}. This choice of one of the two possible partitions will over-estimate Use failure rates and so is conservative from the User's point of view.

Finally, notice that the number of bits in the *fp* category in Use is not reduced by repair at Test (because they pass at Test), and the number doesn't increase in Use by action of repair at Test since it is assumed that repair at Test creates only *pp* bits. That is, $n'_{fp} = n_{fp}$.

Therefore the entire model for bit reclassification due to repair/replacement at Test is

$$n'_{fp} = n_{fp} \qquad n'_{pp} = n_{pp} + \min\left[n_{ff} + n_{pf}, n_t\right]$$
$$n'_{ff} = \left\lceil \max\left[n_{ff} + n_{pf} - n_t, 0\right]\frac{n_{ff}}{n_{ff} + n_{pf}} \right\rceil \qquad n'_{pf} = \left\lfloor \max\left[n_{ff} + n_{pf} - n_t, 0\right]\frac{n_{pf}}{n_{ff} + n_{pf}} \right\rfloor. \tag{S63}$$

Derivation of Eq. (S41)

Points in the irregular volume must satisfy all of the following conditions[4]

$$\left.\begin{array}{l} 0 \le n_{pf} < \infty \\ 0 \le n'_{ff}(n_{ff}, n_{pf}, n_t) + n_{fp} \le n_u \\ n_u < n_{ff} + n_{fp} \end{array}\right\} \text{Irregular volume.} \tag{S64}$$

where (note the "ceiling" function)

$$n'_{ff} = \left\lceil \max\left[n_{ff} + n_{pf} - n_t, 0\right]\frac{n_{ff}}{n_{ff} + n_{pf}} \right\rceil. \tag{S65}$$

Inspection of Eq. (S64) and Eq. (S65) shows that n_{pf} in the irregular volume must have an upper bound because as n_{pf} increases, eventually $n'_{ff} = n_{ff}$, making the second and last inequalities in Eq. (S64) mutually exclusive so that no points can satisfy all of the inequalities in Eq. (S64). It will be computationally useful to have an expression for the upper bound of n_{pf} to replace the first inequality in Eq. (S64).

For a point in the irregular volume it must be true that $n'_{ff} < n_{ff}$. Using the fact that $\lceil a \rceil < b$ is the same as $a \le b - 1$ where b is an integer and a is any real number (not necessarily an integer) we can write $n'_{ff} < n_{ff}$ as

$$\max\left[n_{ff} + n_{pf} - n_t, 0\right]\frac{n_{ff}}{n_{ff} + n_{pf}} \le n_{ff} - 1 \tag{S66}$$

When n_{pf} becomes sufficiently large, Eq. (S66) becomes

$$\frac{\left(n_{ff} + n_{pf} - n_t\right)n_{ff}}{n_{ff} + n_{pf}} \le n_{ff} - 1 \tag{S67}$$

which after some rearrangement is

$$n_{pf} \le (n_t - 1)n_{ff} \tag{S68}$$

which is the desired inequality for n_{pf}.

[4] Keep in mind that all of the variables are non-negative (including zero) integers.