Derivation of Formula for Synthesis of Frank Copula

The following derivation corrects an error in “Nelsen No. 5” the catalog of copula density maps by Armstrong.

Margaret Armstrong, "Copula Catalogue. Part 1 : Bivariate Archimedean Copulas," CERNA - Centre d'économie industrielle, Paris, France, 2003. [Online]. http://www.cerna.ensmp.fr/Documents/MA-CopulaCatalogue.pdf

The Frank copula is

$$C(u,v) = -\frac{1}{\theta}\ln\left(1+\frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right)$$

The conditional probability is

$$C(v\,|\,u) = \frac{\partial C(u,v)}{\partial u} = -\frac{1}{\theta}\frac{1}{F(u,v)}\frac{\partial F(u,v)}{\partial u}$$

where

$$F(u,v) = 1+\frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}$$

Now

$$\frac{\partial F(u,v)}{\partial u} = -\theta\frac{e^{-\theta u}(e^{-\theta v}-1)}{e^{-\theta}-1}$$

So

$$\begin{aligned} C(v\,|\,u) = \frac{\partial C(u,v)}{\partial u} &= w = -\frac{1}{\theta}\frac{1}{F(u,v)}\frac{\partial F(u,v)}{\partial u} \\ &= \frac{e^{-\theta}-1}{(e^{-\theta}-1)+(e^{-\theta u}-1)(e^{-\theta v}-1)}\frac{e^{-\theta u}(e^{-\theta v}-1)}{e^{-\theta}-1} \\ &= \frac{[(e^{-\theta u}-1)+1](e^{-\theta v}-1)}{(e^{-\theta}-1)+(e^{-\theta u}-1)(e^{-\theta v}-1)} = \frac{(U+1)V}{T+UV} \end{aligned}$$

where

$$U = e^{-\theta u}-1 \quad V = e^{-\theta v}-1 \quad T = e^{-\theta}-1$$

Solve for V

$$V = \frac{Tw}{1+U-Uw}$$

whence

$$v = -\frac{1}{\theta}\ln\left(1+\frac{w(e^{-\theta}-1)}{e^{-\theta u}-w(e^{-\theta u}-1)}\right)$$