Copula Manipulations Using Generalized Functions[1].

Analytical methods to compute conditional probabilities and probability densities from the copula involve taking derivatives. Conversely, methods to compute the copula from probability densities involve integration. Methods for performing these manipulations for continuous copulae are given in the standard texts[2]. However, many copulae have singular parts. The most elementary copulae describing perfect dependence and perfect anti-dependence are singular, albeit with no continuous parts. The Marshall-Olkin copula is a well-known copula with both continuous and singular parts. The usual way[2] of handling a copula with continuous and singular parts is to deal with each part separately. Kreps[3] showed how to handle singularities in an elegant way by using the Dirac delta function. This note follows Kreps to show how the same methods used to manipulate continous copulae can also be used to handle all parts, continous and singular, of a general copula if certain "generalized functions" are used to describe the singular parts. The practical use of generalized functions is described by Bracewell[4], but for practical copula manipulations only the "Useful Identities" shown in this note are needed.

After providing the "Useful Identities" some examples show how to apply them.

Useful Identities

The following identities involving the Heaviside, Θ, and Dirac, δ, generalized functions are convenient for copula manipulations.

$$\Theta(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases} \tag{1}$$

$$\delta(x) = \begin{cases} 0 & x < 0 \\ \infty & x = 0 \\ 0 & x > 0 \end{cases} \tag{2}$$

$$\Theta(x) = \int_0^x \delta(x')\,dx' \tag{3}$$

$$\frac{d\Theta(x)}{dx} = \delta(x) \tag{4}$$

$$\int_{-b}^{a} \Theta(x)\delta(x)\,dx = \frac{1}{2} \quad a,b > 0 \tag{5}$$

Proof of Eq. (5)

[1] Glenn Shirley, December 2012
[2] Roger B. Nelsen, "An Introduction to Copulas" Springer, NY, 2^{nd} edition (2009) .
[3] Rodney E. Kreps, "A Partially Comonotonic Algorithm for Loss Generation," ASTIN Colloquium Papers 2000, p165.
[4] Ron Bracewell, "The Fourier Transform and its Applications," McGraw-Hill, NY (1965).

$$\int_{-b}^{a}\Theta(x)\delta(x)dx=\int_{-b}^{a}\Theta\frac{d\Theta}{dx}dx=\frac{1}{2}\int_{-b}^{a}\frac{d\Theta^2}{dx}dx$$
$$=\frac{1}{2}\int_{\Theta(-b)}^{\Theta(a)}d\Theta^2=\frac{1}{2}\Theta^2(x)\Big|_{-b}^{a}=\frac{1}{2}\quad a,b>0 \tag{6}$$

Also..

$$\frac{\partial\min[x,y]}{\partial x}=\frac{\partial\max[x,y]}{\partial y}=\Theta(y-x) \tag{7}$$

$$\frac{\partial\min[x,y]}{\partial y}=\frac{\partial\max[x,y]}{\partial x}=\Theta(x-y) \tag{8}$$

$$\frac{\partial\Theta(y-x)}{\partial y}=\delta(y-x) \tag{9}$$

$$\frac{\partial\Theta(x-y)}{\partial y}=-\delta(y-x) \tag{10}$$

$$\Theta(-x)=1-\Theta(x) \tag{11}$$

$$\delta(-x)=\delta(x) \tag{12}$$

$$\frac{\partial^2\min[x,y]}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial\min[x,y]}{\partial y}\right)=\frac{\partial}{\partial x}\Theta(x-y)=\delta(x-y) \tag{13}$$

and equivalently

$$\min[x,y]=\int_{-\infty}^{x}\int_{-\infty}^{y}\delta(s-t)dsdt \tag{14}$$

$$\max[x+y-1,0]=\int_{-\infty}^{x}\int_{-\infty}^{y}\delta(s+t-1)dsdt \tag{15}$$

$$\min[f(x),f(y)]=f(\min[x,y])\qquad f(x)\text{ is a monotonically increasing function} \tag{16}$$

$$\max[f(x),f(y)]=f(\max[x,y])\qquad f(x)\text{ is a monotonically increasing function} \tag{17}$$

$$\min[g(x),g(y)]=g(\max[x,y])\qquad g(x)\text{ is a monotonically decreasing function} \tag{18}$$

$$\max[g(x),g(y)]=g(\min[x,y])\qquad g(x)\text{ is a monotonically decreasing function} \tag{19}$$

$$\max[x,y]+\min[x,y]=x+y \tag{20}$$

$$a-\min[x,y]=\max[a-x,a-y] \tag{21}$$

$$a - \max[x, y] = \min[a - x, a - y] \tag{22}$$

$$\begin{aligned} &\frac{\partial \min[x, y]}{\partial x} + \frac{\partial \max[x, y]}{\partial x} = 1 \\ &\Theta(y - x) + \Theta(x - y) = 1 \end{aligned} \tag{23}$$

Examples

Perfect Correlation

$$\begin{aligned} &C(x, y) = \min[x, y] \\ &\frac{\partial C}{\partial x} = \Theta(y - x) \\ &\frac{\partial^2 C}{\partial y \partial x} = \delta(y - x) \end{aligned} \tag{24}$$

So, for perfect correlation,

$$\begin{aligned} \tau &= 4\int_0^1 \int_0^1 C \frac{\partial^2 C}{\partial x \partial y} dxdy - 1 \\ &= 4\int_0^1 \int_0^1 \min[x, y]\delta(y - x) dxdy - 1 \\ &= 4\int_0^1 \min[x, x] dx - 1 = 4\int_0^1 xdx - 1 = 4 \times \tfrac{1}{2} x^2 \Big|_0^1 - 1 = 1 \end{aligned} \tag{25}$$

Independence

$$C(x, y) = xy \tag{26}$$

The mixed second derivative is unity so

$$\begin{aligned} \tau &= 4\int_0^1 \int_0^1 CdC - 1 \\ &= 4\int_0^1 \int_0^1 C \frac{\partial^2 C}{\partial x \partial y} dxdy - 1 \\ &= 4\int_0^1 \int_0^1 xydxdy - 1 \\ &= 4\left(\tfrac{1}{2} x^2 \Big|_0^1\right)^2 - 1 = 0 \end{aligned} \tag{27}$$

Perfect Anticorrelation

$$C(x, y) = \max[x + y - 1, 0] \tag{28}$$

$$
\begin{aligned}
&f = x + y - 1 \\
&C(f) = \max[f, 0] \\
&\frac{\partial C}{\partial f} = \Theta(f) = \Theta(x + y - 1) \\
&\frac{\partial f}{\partial x} = 1 \\
&\frac{\partial C}{\partial x} = \frac{\partial C}{\partial f}\frac{\partial f}{\partial x} \\
&\frac{\partial C}{\partial x} = \Theta(x + y - 1)
\end{aligned} \tag{29}
$$

and, by symmetry,

$$
\frac{\partial C}{\partial y} = \frac{\partial C}{\partial x} = \Theta(x + y - 1) \tag{30}
$$

and

$$
\frac{\partial^2 C}{\partial x \partial y} = \frac{\partial}{\partial x}\Theta(x + y - 1) = \frac{\partial \Theta(f)}{\partial f}\frac{\partial f}{\partial x} = \delta(f) \times 1 = \delta(x + y - 1) \tag{31}
$$

Whence

$$
\begin{aligned}
\tau &= 4\int_0^1\int_0^1 \max[x + y - 1, 0]\delta(x + y - 1)\,dxdy - 1 \\
&= 4\int_0^1 \max\left[x + (1 - x) - 1, 0\right]dx - 1 \\
&= 4\int_0^1 \max[0, 0]\,dx - 1 \\
&= -1
\end{aligned} \tag{32}
$$

PDF for Marshall-Olkin Copula

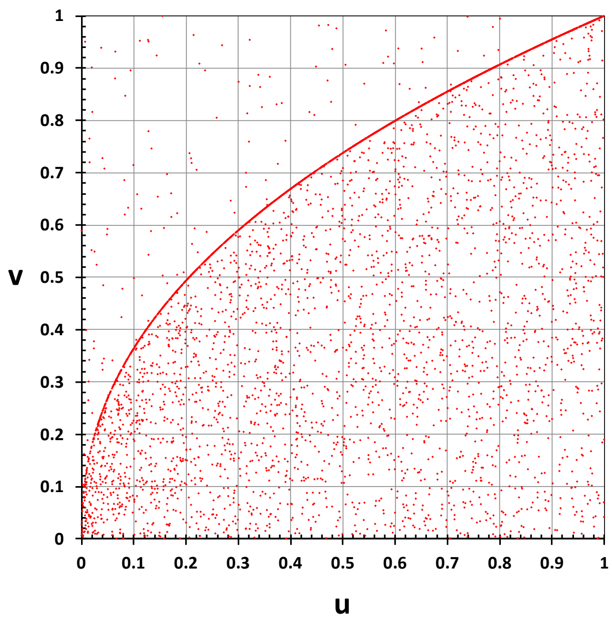
The objective here is to compute the pdf of the Marshall-Olkin copula from the cdf. That is, to compute

$$
c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v} \tag{33}
$$

where C is the Marshall-Olkin copula

$$
C(u,v) = \min\left[u^{1-\alpha}v, uv^{1-\beta}\right] = \begin{cases} u^{1-\alpha}v & u^{\alpha} \geq v^{\beta} \\ uv^{1-\beta} & u^{\alpha} \leq v^{\beta} \end{cases}. \tag{34}
$$

The Marshall-Olkin copula is a good way to show the benefit of using the method of generalized functions because it has a singularity as well as a region in which the copula is non-zero and continuous.



Numpoints	5000
Tau [0 < tau < 1]	0.4
s [-1 < s < 1]	0.9
Alpha	0.411
Beta	0.937

The Marshall-Olkin copula may be parameterized either by α and β (the usual definition), or by τ and *s*:

$$\alpha = \frac{(1+\tau^{s})\tau}{1+\tau} \quad \beta = \frac{(1+\tau^{-s})\tau}{1+\tau}$$

where $-1 \le s \le 1$ controls the symmetry of the copula.

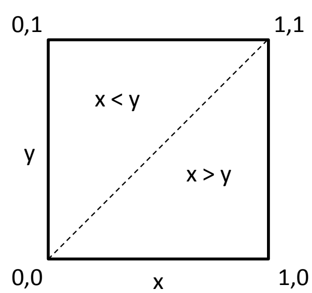
The method used by Nelsen[5] to study this copula treats the continuous and singular parts separately. But the method using generalized functions treats the entire copula as one entity, and the results for singular and continuous parts fall out naturally.

<u>First Way</u>

Let's write

$$x = u^{\alpha}, \quad y = v^{\alpha} \tag{35}$$

This transformation separates the two parts of (34) into above and below the diagonal in the *x,y* unit square,



so the copula becomes the function[6]

[5] Roger B. Nelsen, "An Introduction to Copulas" Springer, NY, 2nd edition (2009), Section 3.1.1, p52.

[6] A pseudo-copula because, for example, $C(x,1) = x^{1/\alpha} \ne x$

$$C(x,y)=\begin{Bmatrix} x^{\frac{1}{\alpha}-1}y^{\frac{1}{\beta}} & x\ge y \\ x^{\frac{1}{\alpha}}y^{\frac{1}{\beta}-1} & x\le y \end{Bmatrix}=\min\left[x^{\frac{1}{\alpha}-1}y^{\frac{1}{\beta}},x^{\frac{1}{\alpha}}y^{\frac{1}{\beta}-1}\right]\equiv\min[f,g] \tag{36}$$

where

$$f=x^{\frac{1}{\alpha}}y^{\frac{1}{\beta}}x^{-1} \quad \text{and} \quad g=x^{\frac{1}{\alpha}}y^{\frac{1}{\beta}}y^{-1} \tag{37}$$

Notice that

$$f\le g \quad \text{when} \quad x\ge y \tag{38}$$

We need the probability density in (x,y) space:

$$\begin{aligned}
\frac{\partial^2 C}{\partial x\partial y}&=\frac{\partial}{\partial x}\left(\frac{\partial\min[f,g]}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\frac{\partial\min[f,g]}{\partial f}+\frac{\partial g}{\partial y}\frac{\partial\min[f,g]}{\partial g}\right)\\
&=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\Theta(g-f)+\frac{\partial g}{\partial y}\Theta(f-g)\right)\\
&=\frac{\partial}{\partial x}\left(\frac{1}{\beta}x^{\frac{1}{\alpha}-1}y^{\frac{1}{\beta}-1}\Theta(g-f)+\frac{(1-\beta)}{\beta}x^{\frac{1}{\alpha}}y^{\frac{1}{\beta}-2}\Theta(f-g)\right)\\
&=\frac{1-\alpha}{\alpha\beta}x^{\frac{1}{\alpha}-2}y^{\frac{1}{\beta}-1}\Theta(g-f)+\frac{1-\beta}{\alpha\beta}x^{\frac{1}{\alpha}-1}y^{\frac{1}{\beta}-2}\Theta(f-g)+\left(\frac{1}{\beta}x^{\frac{1}{\alpha}-1}y^{\frac{1}{\beta}-1}-\frac{1-\beta}{\beta}x^{\frac{1}{\alpha}}y^{\frac{1}{\beta}-2}\right)\frac{\partial\Theta(g-f)}{\partial x}\\
&=\frac{1-\alpha}{\alpha\beta}x^{\frac{1}{\alpha}-2}y^{\frac{1}{\beta}-1}\Theta(x-y)+\frac{1-\beta}{\alpha\beta}x^{\frac{1}{\alpha}-1}y^{\frac{1}{\beta}-2}\Theta(y-x)+\left(\frac{1}{\beta}x^{\frac{1}{\alpha}-1}y^{\frac{1}{\beta}-1}-\frac{1-\beta}{\beta}x^{\frac{1}{\alpha}}y^{\frac{1}{\beta}-2}\right)\frac{\partial\Theta(x-y)}{\partial x}
\end{aligned} \tag{39}$$

where Θ is the Heaviside function

$$\Theta(x) \quad =\begin{cases} 0 & x<0 \\ \frac{1}{2} & x=0 \\ 1 & x>0 \end{cases} \tag{40}$$

and where, from (38)[7]

$$\Theta(g-f)=\Theta(x-y) \tag{41}$$

Now, since

$$\frac{\partial\Theta(x-y)}{\partial x}=\delta(x-y) \tag{42}$$

we have

[7] I thought this was pretty neat! G.

$$\frac{\partial^2 C}{\partial x \partial y} = \frac{1-\alpha}{\alpha\beta} x^{\frac{1}{\alpha}-2} y^{\frac{1}{\beta}-1} \Theta(x-y) + \frac{1-\beta}{\alpha\beta} x^{\frac{1}{\alpha}-1} y^{\frac{1}{\beta}-2} \Theta(y-x) + x^{\frac{1}{\alpha}+\frac{1}{\beta}-2} \delta(x-y) \quad (43)$$

since, because of the delta function, we may set *x* = *y* in the final term of (43).

In terms of the uniform marginal distributions:

$$\frac{\partial C(u,v)}{\partial v} = u^{1-\alpha}\Theta(u^\alpha - v^\beta) + (1-\beta)uv^{-\beta}\Theta(v^\beta - u^\alpha) \quad (44)$$

$$\frac{\partial^2 C(u,v)}{\partial u \partial v} = (1-\alpha)u^{-\alpha}\Theta(u^\alpha - v^\beta) + (1-\beta)v^{-\beta}\Theta(v^\beta - u^\alpha) + \alpha\beta\delta(u^\alpha - v^\beta) \quad (45)$$

<u>Second Way</u>

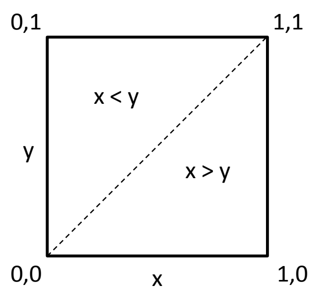
The Marshall-Olkin copula is

$$C(u,v) = \min\left[u^{1-\alpha}v, uv^{1-\beta}\right] = \begin{cases} u^{1-\alpha}v & u^\alpha \ge v^\beta \\ uv^{1-\beta} & u^\alpha \le v^\beta \end{cases} \quad (46)$$

Let's write

$$x = u^\alpha, \quad y = v^\alpha \quad (47)$$

This transformation separates the two parts of (34) into above and below the diagonal in the *x*,*y* unit square,



so the copula becomes the pseudo-copula

$$C(x,y) = \begin{Bmatrix} x^{\frac{1}{\alpha}-1} y^{\frac{1}{\beta}} & x \ge y \\ x^{\frac{1}{\alpha}} y^{\frac{1}{\beta}-1} & x \le y \end{Bmatrix} = x^{\frac{1}{\alpha}-1} y^{\frac{1}{\beta}} \Theta(x-y) + x^{\frac{1}{\alpha}} y^{\frac{1}{\beta}-1} \Theta(y-x) \quad (48)$$

where Θ is the Heaviside function

$$\Theta(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases} \quad (49)$$

We need the probability density in (*x*,*y*) space:

$$\begin{aligned}\frac{\partial^2 C}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{1}{\beta} x^{\frac{1}{\alpha}-1} y^{\frac{1}{\beta}-1}\Theta(x-y) + \frac{1-\beta}{\beta} x^{\frac{1}{\alpha}} y^{\frac{1}{\beta}-2}\Theta(y-x) + \left\{ x^{\frac{1}{\alpha}-1} y^{\frac{1}{\beta}} \frac{\partial\Theta(x-y)}{\partial y} + x^{\frac{1}{\alpha}} y^{\frac{1}{\beta}-1} \frac{\partial\Theta(y-x)}{\partial y} \right\}\right) \\ &= \frac{1-\alpha}{\alpha\beta} x^{\frac{1}{\alpha}-2} y^{\frac{1}{\beta}-1}\Theta(x-y) + \frac{1-\beta}{\alpha\beta} x^{\frac{1}{\alpha}-1} y^{\frac{1}{\beta}-2}\Theta(y-x) + \left(\frac{1}{\beta} x^{\frac{1}{\alpha}-1} y^{\frac{1}{\beta}-1} - \frac{1-\beta}{\beta} x^{\frac{1}{\alpha}} y^{\frac{1}{\beta}-2}\right)\frac{\partial\Theta(x-y)}{\partial x}\end{aligned} \quad (50)$$

where we have noticed that the term in braces vanishes because

$$\frac{\partial\Theta(x-y)}{\partial x} = \frac{\partial\Theta(y-x)}{\partial y} = -\frac{\partial\Theta(x-y)}{\partial y} = \delta(x-y) \quad (51)$$

and we can set $x = y$ in that term. Moreover, we can set $x = y$ in the final term of Eq. (50) and replace the derivative of the Heaviside function by the Dirac delta function to get

$$\frac{\partial^2 C}{\partial x \partial y} = \frac{1-\alpha}{\alpha\beta} x^{\frac{1}{\alpha}-2} y^{\frac{1}{\beta}-1}\Theta(x-y) + \frac{1-\beta}{\alpha\beta} x^{\frac{1}{\alpha}-1} y^{\frac{1}{\beta}-2}\Theta(y-x) + x^{\frac{1}{\alpha}+\frac{1}{\beta}-2}\delta(x-y) \quad (52)$$

or

$$\frac{\partial^2 C}{\partial x \partial y} = \begin{cases} \frac{1-\alpha}{\alpha\beta} x^{\frac{1}{\alpha}-2} y^{\frac{1}{\beta}-1} & x > y \\ \frac{1-\beta}{\alpha\beta} x^{\frac{1}{\alpha}-1} y^{\frac{1}{\beta}-2} & x < y\ . \\ x^{\frac{1}{\alpha}+\frac{1}{\beta}-2} & x = y \end{cases} \quad (53)$$