The Clayton Copula

The Clayton copula is

\[
C(u, v) = \max\left[(u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}, 0\right] \quad [-1, \infty) \setminus 0
\]  

(1)

For our applications \(0 < \theta < \infty\) so this can be simplified to

\[
C(u, v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta} \quad (0, \infty)
\]  

(2)

Truncation-Invariance

The Clayton copula has a remarkable invariance under truncation (Oakes, 2005\(^1\)). To show this, suppose the copula in Eq. (2) is defined on the unit square \(u \in [0,1]\) and \(v \in [0,1]\). Let’s construct the copula on the sub-area \(u \in [0,a]\) and \(v \in [0,b]\). Define \(x = u/a\) and \(v = v/b\) so that \(x \in [0,1]\) and \(y \in [0,1]\) spans the sub-area.

\[
A(x, y) = \frac{[(xa)^{-\theta} + (yb)^{-\theta} - 1]^{-1/\theta}}{[a^{-\theta} + b^{-\theta} - 1]^{-1/\theta}}
\]  

(3)

is the probability mass of the copula of Eq. (2) contained in the sub-regions of the sub-area, normalized by the total probability mass of the sub-area. Eq. (3) has all the properties of a copula on \([x, y] \in [0,1]^2\) (normalized, grounded, 2-increasing) except that the margins (for \(x = 1\), and \(y = 1\) separately) are not uniform. Setting \(y = 1\), the marginal \(x\) distribution may be written

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\[ p(x) = \frac{[(xa)^\theta + b^\theta - 1]^{1/\theta}}{[a^\theta + b^\theta - 1]^{1/\theta}} \]  

(4)

and a similar expression may be written for the marginal \( y \) distribution, \( q \), by setting \( x = 1 \). Note that \( p(0) = 0 \), and \( p(1) = 1 \), but \( p(x) \neq x \). Partially solving Eq. (4),

\[
\begin{align*}
(xa)^\theta &= (a^\theta + b^\theta - 1)p^\theta + 1 - b^\theta \\
(yb)^\theta &= (a^\theta + b^\theta - 1)q^\theta + 1 - a^\theta
\end{align*}
\]

(5)

The copula on the sub-region is \( A \), expressed in terms of uniform marginal distributions. That is,

\[
C(p,q) = A(x(p), y(q))
\]

\[
= \frac{1}{(a^\theta + b^\theta - 1)^{1/\theta}} \left[ (a^\theta + b^\theta - 1)p^\theta + 1 - b^\theta + (a^\theta + b^\theta - 1)q^\theta + 1 - a^\theta - 1 \right]^{1/\theta}
\]

(6)

which is the same as the copula for the entire area!

**Monte-Carlo Synthesis**

The general prescription is to set \( w(u,v) = \partial C(u,v)/\partial u \) and solve for \( v(u,w) \). Then draw iid samples \( u_i \) and \( w_i \) from a uniform distribution on \([0,1]\), and evaluate \( v_i \). \( u_i \) and \( v_i \) are the desired pair. For the Clayton copula

\[
w = \frac{\partial C(u,v)}{\partial u} = u^{-(\theta+1)} \left( u^{-\theta} + v^{-\theta} + 1 \right)^{-\theta\over\theta+1}
\]

(7)

Solving Eq. (7) for \( v \)

\[ v = [(w^{\theta+1} - 1)u^{-\theta} - 1]^{-1/\theta} \]

(8)

The truncation-invariance property makes it possible to synthesize points in a sub-region sample of a Clayton copula, with one corner at \((0,0)\), without rejection. If \( p \) and \( q \) are sampled for the copula of the sub-region (also a Clayton copula with parameter \( \theta' \)) by the method of Eqs. (7) and (8) then, using Eq. (5), the corresponding values of \( u \) and \( v \) for the sampled copula are

\[
\begin{align*}
\hat{u} &= \left( (a^\theta + b^\theta - 1)p^\theta + 1 - b^\theta \right)^{1/\theta} \\
\hat{v} &= \left( (a^\theta + b^\theta - 1)q^\theta + 1 - a^\theta \right)^{1/\theta}
\end{align*}
\]

(9)
Probability Density

The probability density of the Clayton copula is

\[ c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v} = (\theta + 1)(uv)^{-\theta+1}(u^{-\theta} + v^{-\theta} - 1) \frac{2\theta+1}{\theta} \]  

(10)

Low-tail Dependence

\[ LT = \lim_{u \to 0^+} \frac{C(u,u)}{u} = \lim_{u \to 0^+} \frac{(2u^{-\theta} - 1)^{-1/\theta}}{u} = 2^{-1/\theta} \]  

(11)

because the second term in brackets can be ignored when \( u \) is small.

Tau

\[ \tau = \frac{\theta}{\theta + 2} \]  

(12)

and

\[ \theta = \frac{2\tau}{1 - \tau} \]  

(13)

Oakes showed that, because of the truncation invariance, this value of tau obtains for any truncation of the copula.