The Clayton Copula

The Clayton copula is

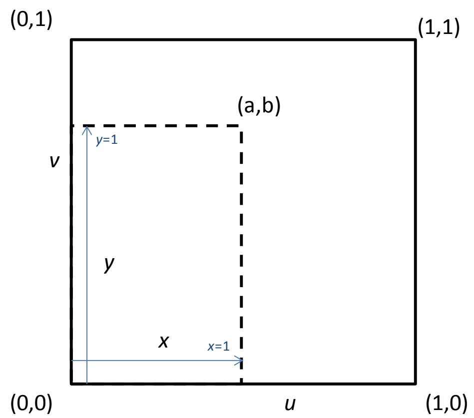
$$C(u,v) = \max\left[(u^{-\theta} + v^{-\theta} - 1), 0\right]^{-1/\theta} \qquad [-1,\infty) \setminus 0 \qquad (1)$$

For our applications $0 < \theta < \infty$ so this can be simplified to

$$C(u,v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta} \qquad (0,\infty) \qquad (2)$$

Truncation-Invariance

The Clayton copula has a remarkable invariance under truncation (Oakes, 2005[1]). To show this, suppose the copula in Eq. (2) is defined on the unit square u [0,1] and $v \in [0,1]$. Let's construct the copula on the sub-area $u \in [0,a]$ and $v \in [0,b]$. Define $x = u/a$ and $v = v/b$ so that $x \in [0,1]$ and $y \in [0,1]$ spans the sub-area.



The function

$$A(x,y) = \frac{[(xa)^{-\theta} + (yb)^{-\theta} - 1]^{-1/\theta}}{[a^{-\theta} + b^{-\theta} - 1]^{-1/\theta}} \qquad (3)$$

is the probability mass of the copula of Eq. (2) contained in the sub-regions of the sub-area, normalized by the total probability mass of the sub-area. Eq. (3) has all the properties of a copula on $[x, y] \in [0,1]^2$ (normalized, grounded, 2-increasing) except that the margins (for $x = 1$, and $y = 1$ separately) are not uniform. Setting $y = 1$, the marginal x distribution may be written

[1] David Oakes, On the Preservation of Copula Structure under Truncation,The Canadian Journal of Statistics / La Revue Canadienne de Statistique Vol. 33, No. 3, Dependence Modelling: Statistical Theory and Applications in Finance and Insurance (Sep., 2005), pp. 465-468. http://www.jstor.org/stable/25046191

$$p(x) = \frac{[(xa)^{-\theta} + b^{-\theta} - 1]^{-1/\theta}}{[a^{-\theta} + b^{-\theta} - 1]^{-1/\theta}} \tag{4}$$

and a similar expression may be written for the marginal *y* distribution, *q*, by setting $x = 1$. Note that $p(0) = 0$, and $p(1) = 1$, but $p(x) \neq x$. Partially solving Eq. (4),

$$\begin{aligned} (xa)^{-\theta} &= (a^{-\theta} + b^{-\theta} - 1)\,p^{-\theta} + 1 - b^{-\theta} \\ (yb)^{-\theta} &= (a^{-\theta} + b^{-\theta} - 1)\,q^{-\theta} + 1 - a^{-\theta} \end{aligned} \tag{5}$$

The copula on the sub-region is *A*, expressed in terms of uniform marginal distributions. That is, substituting Eqs. (5) into Eq. (3),

$$\begin{aligned} &C(p,q) = A\big(x(p), y(q)\big) \\ &= \frac{1}{(a^{-\theta} + b^{-\theta} - 1)^{-1/\theta}} \left[(a^{-\theta} + b^{-\theta} - 1)\,p^{-\theta} + 1 - b^{-\theta} + (a^{-\theta} + b^{-\theta} - 1)\,q^{-\theta} + 1 - a^{-\theta} - 1 \right]^{-1/\theta} \\ &= \left(p^{-\theta} + q^{-\theta} - 1 \right)^{-1/\theta} \end{aligned} \tag{6}$$

which is the same as the copula for the entire area!

Monte-Carlo Synthesis

The general prescription is to set $w(u,v) = \partial C(u,v)/\partial u$ and solve for $v(u,w)$. Then draw iid samples u_i and w_i from a uniform distribution on [0,1], and evaluate v_i. u_i and v_i are the desired pair. For the Clayton copula

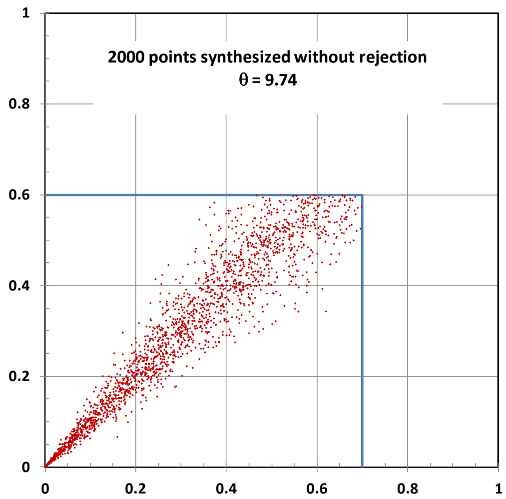
$$w = \frac{\partial C(u,v)}{\partial u} = u^{-(\theta+1)} \left(u^{-\theta} + v^{-\theta} + 1 \right)^{-\frac{\theta+1}{\theta}} \tag{7}$$

Solving Eq. (7) for *v*

$$v = [(w^{-\frac{\theta}{\theta+1}} - 1)u^{-\theta} - 1]^{-1/\theta} \tag{8}$$

The truncation-invariance property makes it possible to synthesize points in a sub-region sample of a Clayton copula, with one corner at (0,0), *without rejection*. If *p* and *q* are sampled for the copula of the sub-region (also a Clayton copula with parameter θ!) by the method of Eqs. (7) and (8) then, using Eq. (5), the corresponding values of *u* and *v* for the sampled copula are

$$\begin{aligned} u &= \left[(a^{-\theta} + b^{-\theta} - 1)\,p^{-\theta} + 1 - b^{-\theta} \right]^{-1/\theta} \\ v &= \left[(a^{-\theta} + b^{-\theta} - 1)\,q^{-\theta} + 1 - a^{-\theta} \right]^{-1/\theta} \end{aligned} \tag{9}$$



Probability Density

The probability density of the Clayton copula is

$$c(u,v)=\frac{\partial^2 C(u,v)}{\partial u \partial v}=(\theta+1)(uv)^{-(\theta+1)}(u^{-\theta}+v^{-\theta}-1)^{-\frac{2\theta+1}{\theta}} \tag{10}$$

Low-tail Dependence

$$LT=\lim_{u\to 0+}\frac{C(u,u)}{u}=\lim_{u\to 0+}\frac{\left(2u^{-\theta}-1\right)^{-1/\theta}}{u}=2^{-1/\theta} \tag{11}$$

because the second term in brackets can be ignored when *u* is small.

Tau

$$\tau=\frac{\theta}{\theta+2} \tag{12}$$

and

$$\theta=\frac{2\tau}{1-\tau} \tag{13}$$

Oakes showed that, because of the truncation invariance, this value of tau obtains for any truncation of the copula.