All Pass Network time constants using a method developed from Bedrosian [1] and Oppelt [2].
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We begin with a normalized network, where the geometric mean angular frequency \( \omega_{gm} = \sqrt{\omega_1 \omega_2} = 1 \) with \( \omega_1 \) the lower frequency and \( \omega_2 \) the upper frequency of the suppression stopband.

Having set the geometric mean frequency to 1, the remainder of the time constants are determined by the number of cascaded sections in the network and the ratio of \( \omega_1 \) and \( \omega_2 \). Note that the ratio of angular frequencies \( \omega \) is the same as the ratio of the lower frequency in Hz divided by the upper frequency in Hz. To illustrate, if the desired opposite sideband suppression band extends from 300 Hz to 3000 Hz, then the ratio is 0.1000. The calculations in this paper are carried out to 4 significant digits, and the final time constants to 3 significant digits. The ratio of \( \omega_1 \) to \( \omega_2 \) is designated \( \Omega \).

We begin calculating the time constants \( \nu \) for the above normalized all-pass network by obtaining a set of auxiliary variables from the bandwidth variable \( \Omega \).

First, obtain \( \varepsilon \) from:

\[
\varepsilon = \frac{1}{2} \frac{1 - \sqrt{\Omega}}{1 + \sqrt{\Omega}}
\]

and then \( q \) from:

\[
q = \varepsilon + 2\varepsilon^5 + 150\varepsilon^{13} + 1707\varepsilon^{17} + \ldots
\]

Rapid convergence of the series \( q \) is a useful check on the suitability of this approach. For example, when \( \Omega = 0.0333 \), as in a network with a suppression bandwidth from 200 Hz to 6000 Hz, \( \varepsilon = 0.3456 \), \( 1707\varepsilon^{17} = 0.00002445 \), and \( q = 0.3556 \). This network has a normalized bandwidth of 30.
Next, we develop a set of coefficients $k$ based on the number of individual all-pass sections. For the diagram on page 1, the number of sections $n$ is 3, and we need 3 values of $k_{in}$, obtained from:

$$k_{in} = \frac{4i + 1}{8n}$$

Plugging in the numbers and maintaining 4 significant figures:

$k_{03} = \frac{1}{24} = .04167$  \hspace{1cm} $k_{13} = \frac{5}{24} = .2083$  \hspace{1cm} $k_{23} = \frac{9}{24} = .3750$

Now use the $k_{in}$ values and sines and cosines to generate the following table of values:

<table>
<thead>
<tr>
<th>$k_{i3}$</th>
<th>$\sin \pi k_{i3}$</th>
<th>$\cos \pi k_{i3}$</th>
<th>$\sin 3\pi k_{i3}$</th>
<th>$\cos 3\pi k_{i3}$</th>
<th>$\sin 5\pi k_{i3}$</th>
<th>$\cos 5\pi k_{i3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k_{03}$</td>
<td>.1305</td>
<td>.9914</td>
<td>.3827</td>
<td>.9239</td>
<td>.6088</td>
<td>.7933</td>
</tr>
<tr>
<td>$k_{13}$</td>
<td>.6088</td>
<td>.7933</td>
<td>.9238</td>
<td>-.3828</td>
<td>-.1308</td>
<td>-.9914</td>
</tr>
<tr>
<td>$k_{23}$</td>
<td>.9239</td>
<td>.3826</td>
<td>-.3829</td>
<td>-.9238</td>
<td>-.3823</td>
<td>.9240</td>
</tr>
</tbody>
</table>

The set of $v_{i2}$ values for the time constants on page one may now be obtained from:

$$v_{i2} = \frac{\cos \pi k_{i3} + q^2 \cos 3\pi k_{i3} + q^6 \cos 5\pi k_{i3}}{\sin \pi k_{i3} - q^2 \sin 3\pi k_{i3} + q^6 \sin 5\pi k_{i3}}$$

And the set of $v_{i1}$ values from:

$$v_{i1} = \frac{1}{v_{i2}}$$

At this point it is possible to just proceed with a notebook and calculator, but in the slide rule era a designer would have noted the number of repeated values in the sine-cosine table, and perhaps done a little organization to simplify the calculations. These work with a memory calculator as well.

First, note that there are only 6 unique values in the table. Assign these the letters $a_1$ through $a_6$. The 6 values of $a$ are the same for all 3rd order networks, regardless of bandwidth. Bandwidth enters the equation when we multiply by $q$.

$a_1 = .9914$  \hspace{1cm} $q$ appears either as $q^2$ or $q^6$. Setting $q^2 = s$, we can now write a set of formulas:

$$v_{02} = \frac{a_1 + sa_2 + s^3a_3}{a_4 - sa_5 + s^3a_6}$$

$$v_{12} = \frac{a_3 - sa_5 - s^3a_1}{a_6 - sa_2 - s^3a_4}$$

$$v_{22} = \frac{a_5 - sa_2 + s^3a_2}{a_2 + sa_5 - s^3a_5}$$
The astute reader will note that the second term in each expression is either $sa_2$ or $sa_5$, which may be calculated and replaced with $b_2$ and $b_5$ to make the calculations even more compact:

\[
\begin{align*}
V_{02} &= \frac{a_1 + b_2 + s^3a_3}{a_4 - b_5 + s^3a_6} \\
V_{12} &= \frac{a_3 - b_5 - s^3a_1}{a_6 - b_2 - s^3a_4} \\
V_{22} &= \frac{a_5 - b_2 + s^3a_2}{a_2 + b_5 - s^3a_5}
\end{align*}
\]

As an example, using the previously calculated value of $q = .3556$ on page one for a network with a bandwidth of 30, we obtain:

\[
\begin{align*}
V_{02} &= 13.32 \\
V_{12} &= 1.511 \\
V_{22} &= 0.2755
\end{align*}
\]

These time constants are in seconds, and may be used to build a 3rd order all-pass network pair that has a bandwidth of 30 centered on geometric mean frequency $\omega = 1$.

To scale this network to some frequency range of interest, perhaps 200 Hz to 6000 Hz, we divide by the geometric mean angular frequency $(2\pi 200 \times 2\pi 6000)^{\frac{1}{2}} = 6.883 \times 10^{-3}$ to obtain:

\[
\begin{align*}
\tau_{02} &= 1.935 \times 10^{-3} \\
\tau_{12} &= 219.5 \times 10^{-6} \\
\tau_{22} &= 40.03 \times 10^{-6}
\end{align*}
\]

Finally, we can choose a capacitor value, for example 10.0 nF, and divide the time constants by $C$ to obtain resistor values:

\[
\begin{align*}
R_{02} &= 193.5k \\
R_{12} &= 21.95k \\
R_{22} &= 4.003k
\end{align*}
\]

Before checking a table of available 1% resistor values, it is useful at this point to check the network in a simulator. All values $R$ in the schematic may be 10.0k, all capacitors 10.0nF, and time constant $R$ values from the table above. Set the simulation to run from 10 Hz to 20 kHz to observe suppression. An LTspice simulation indicates that from 200 Hz to 6000 Hz, the suppressed opposite sideband is at least 46 dB below the desired sideband.