CS 350 Algorithms and Complexity

Winter 2019

Lecture 14: Greedy Algorithms

(slides based on those of Mark Jones)

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Greedy Algorithms

- Solves an optimization problem by breaking it into a sequence of steps, and making the best choice at each step.
- Key idea: a series of locally-optimal choices yields a globally-optimal choice.
- Not all problems can be solved by Greedy Algorithms; if the problem forms a matroid, then it can be so solved.

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A. Yes

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- Now suppose that the US had a 20¢ coin (as does the UK, for example). Can you still solve the problem using a Greedy Algorithm?

A. Yes

B. No

Example: Knapsack problem

item	weight	value
1	3	\$25
2	2	\$20
3	1	\$15
4	4	\$40
5	5	\$50

, capacity $W = 6$.

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A. Item 5
B. Items 3 & 5
C. Items 2 & 4
D. Items 1 & 5
E. None of the above

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Huffman Coding

The Coding Problem:

- A data file contains 100,000 "characters" each of which is either an a, b, c, d, e, or f

- Using three bits for each character takes:

  3 x 100,000 = 300,000 bits

- How could we do better?

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Letter	Code
a	000
b	001
c	010
d	011
e	100
f	101

Using Frequency Information:

- *Variable length coding* gives shorter codes to more frequent letters.
- Encoded size:
  (45 * 1
  + (13+12+16+9) * 2
  + 5 * 3) * 1,000
  = 160,000
- A saving of of over 46%
- Is there a flaw?

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Unique Decoding:

- What string does the code 10000011010 represent?

- One reading:
  100 0 00 11 01 0
  f a d e b a

- Another reading:
  10 00 0 01 10 10
  c d a b c c

- Oh dear: we've lost too much of the information that was in the original!

Letter	Frequency	Code
a	45,000	0
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c	12,000	10
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Use a Prefix-free Code

- Prefix(-free) property:
  no codeword is a prefix of
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- Encoded size:
  (45 * 1
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- Still reduce size by ~25%

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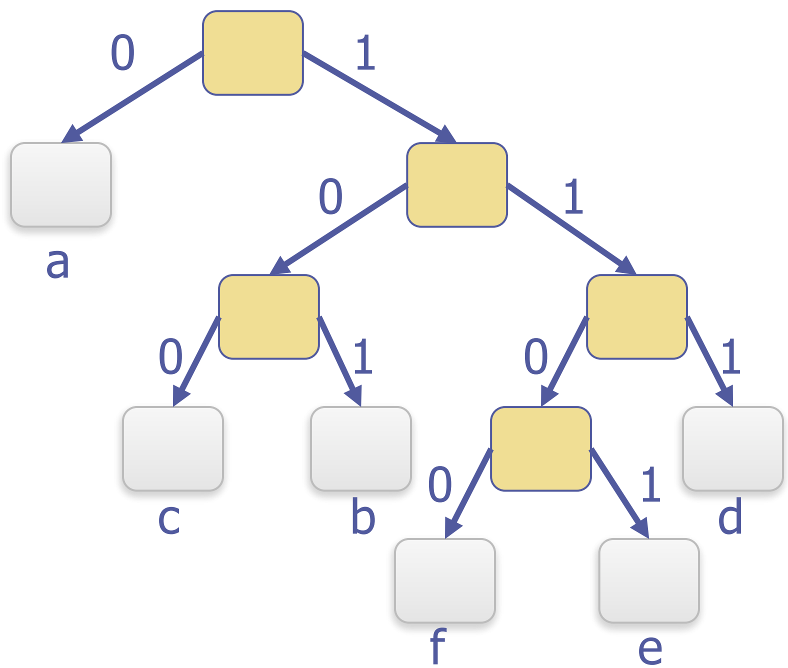

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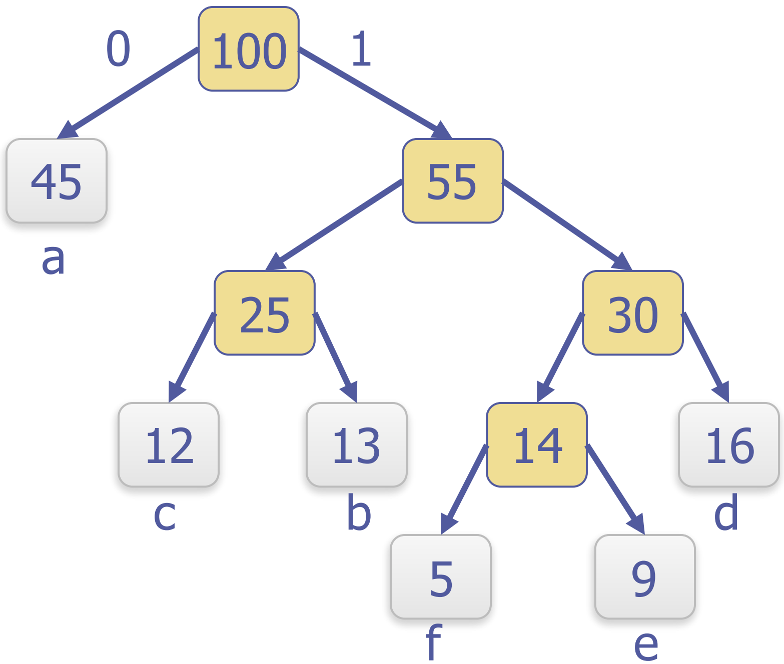
Frequencies & Costs:

- For any given coding tree T, the number of bits required to code a message is:

$$cost(T) = \sum_{c \in C} freq(c) \cdot depth_T(c)$$



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Building a Huffman Coding Tree

- We can use a table to avoid doing a calculation more than once:

```
initialize a empty priority queue, Q
add a leaf node to Q for each character
while (|Q|>1) do
   l = extractMin(Q)
   r = extractMin(Q)
   t = new tree node
       with left=l, right=r, freq=l.freq+r.freq
   insert t into Q
return extractMin(Q)
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Using frequency as key

- Complexity?
- Complexity for computing frequencies?

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Using frequency as key

Greedy choices!

Last element in the queue

- Complexity?
- Complexity for computing frequencies?

Example:

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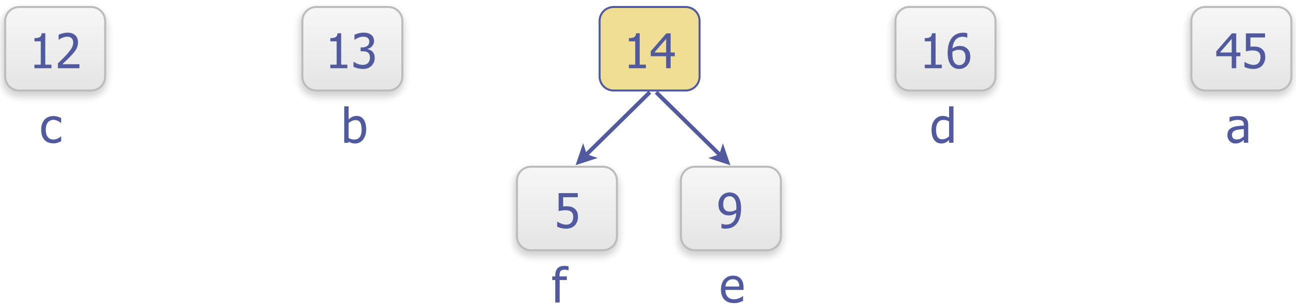

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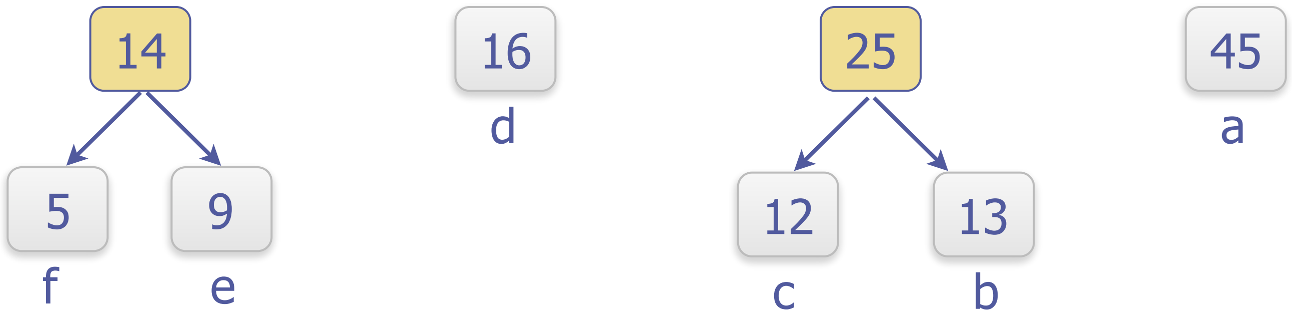

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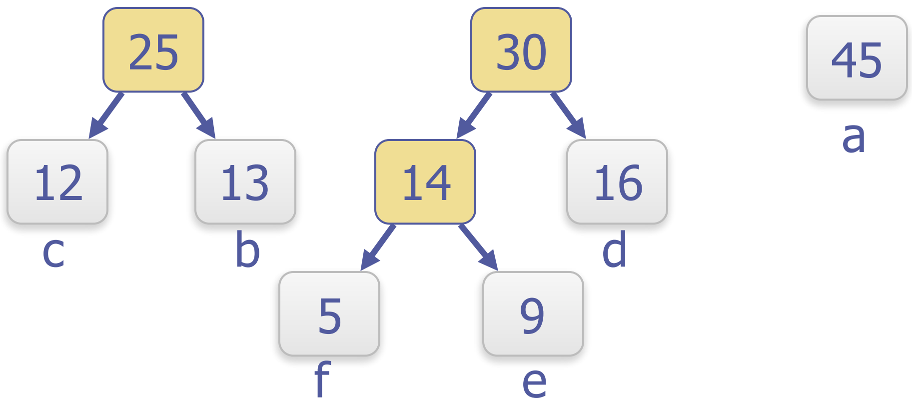

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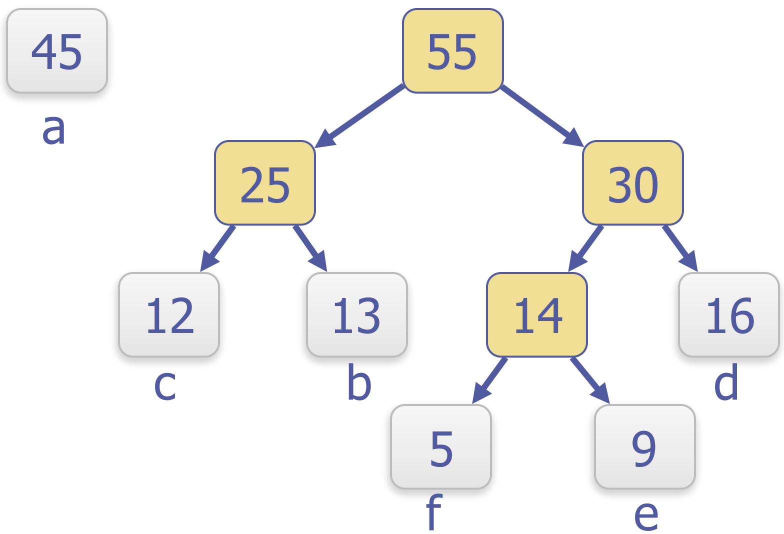

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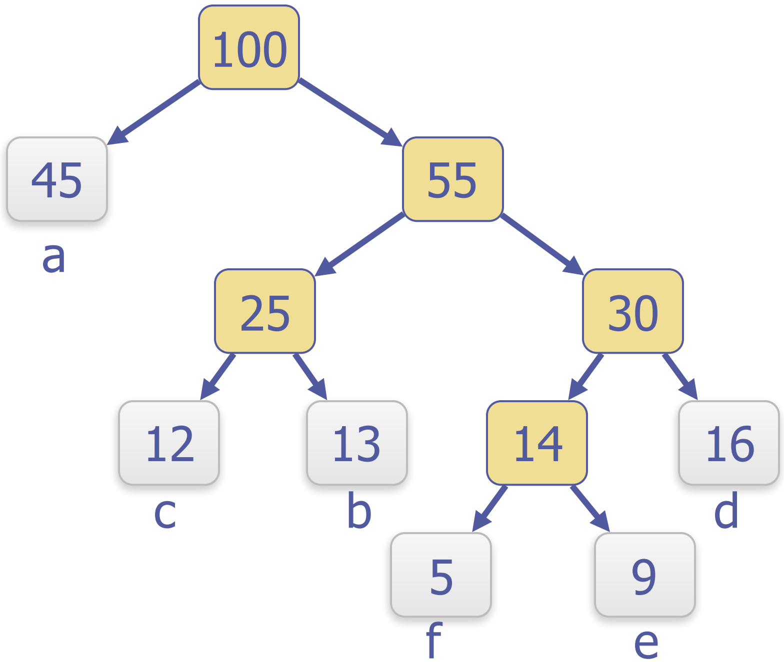


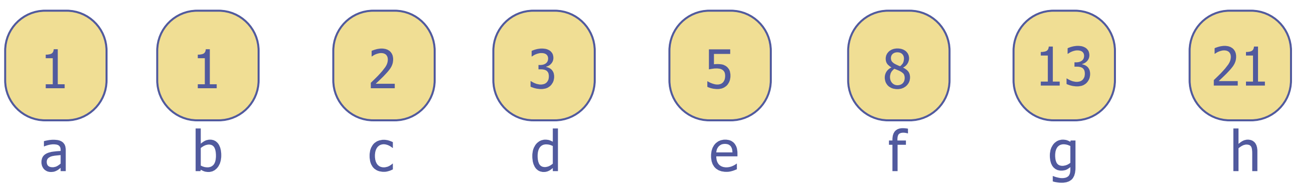
“Optimal Subproblems”

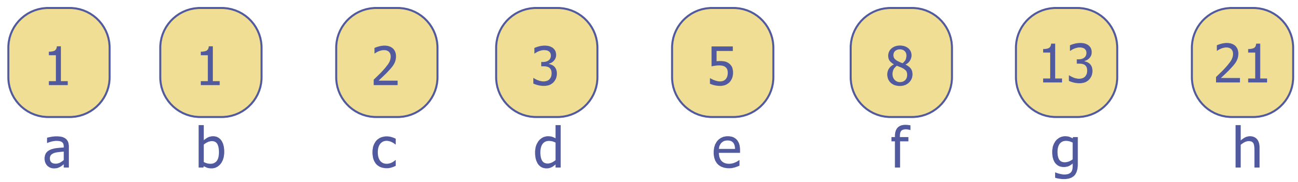
- At each iteration, our task is to find an optimal code for |Q| items
- We pick the pair of characters that have the lowest frequencies
- We reduce the original problem to the task of finding an optimal code for |Q|-1 items
- We can prove that the resulting coding scheme is indeed optimal

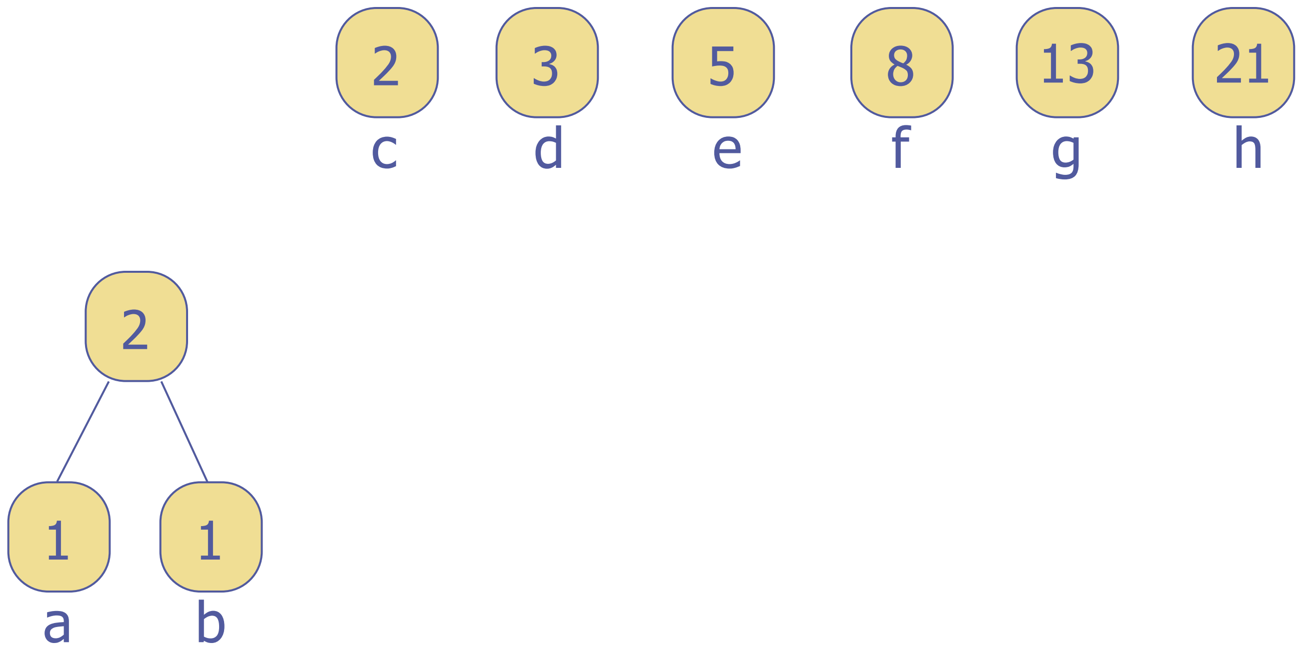
Huffman Trees (2nd Example)

- Build the optimal Huffman code for the following set of frequencies

  a:1 b:1 c:2 d:3 e:5 f:8 g:13 h:21




1
1
2
3
5
8
13
21
a
b
c
d
e
f
g
h


2
3
5
8
13
21
c
d
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f
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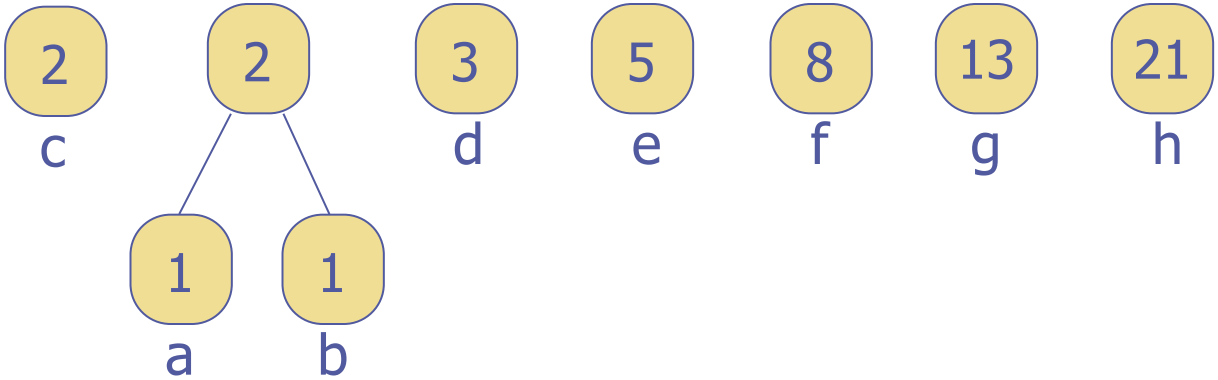
h
2
1
1
a
b


2
c
2
1
a
1
b
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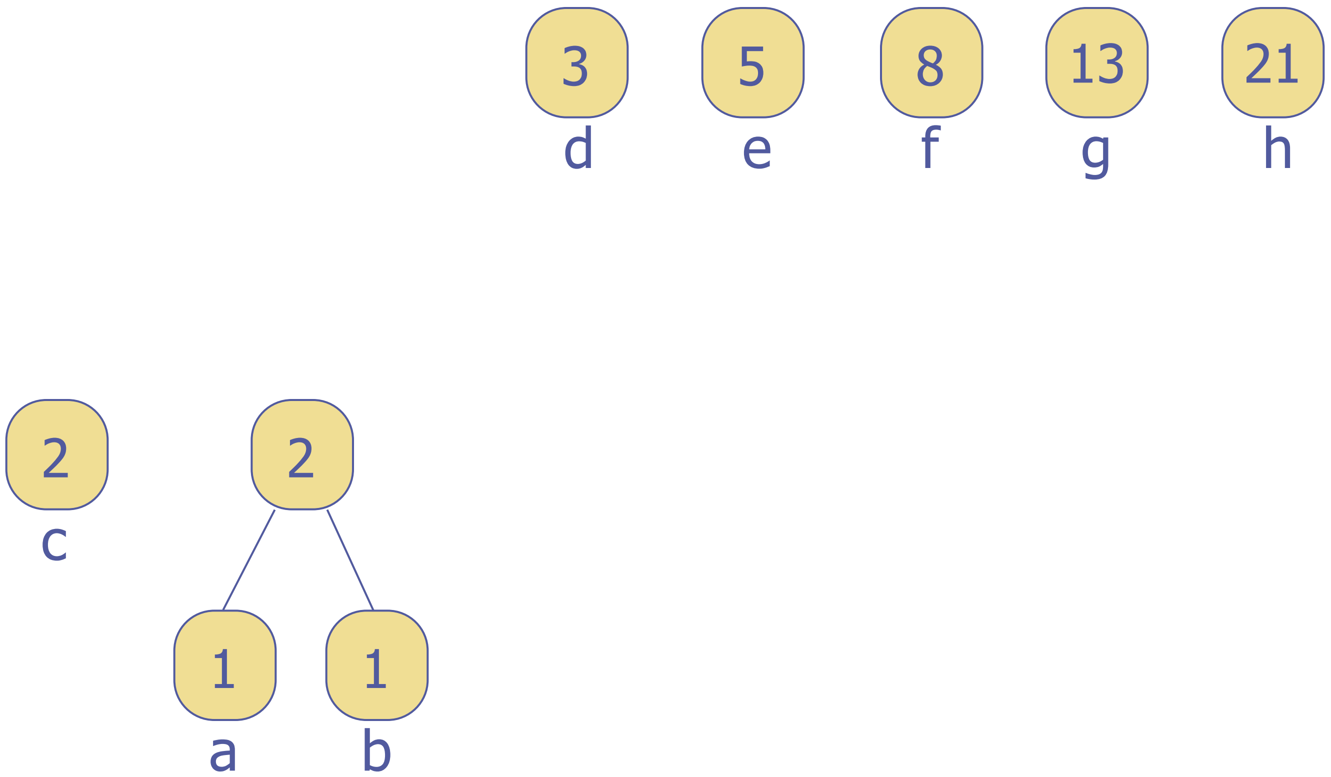
e
8
f
13
g
21
h


3
d
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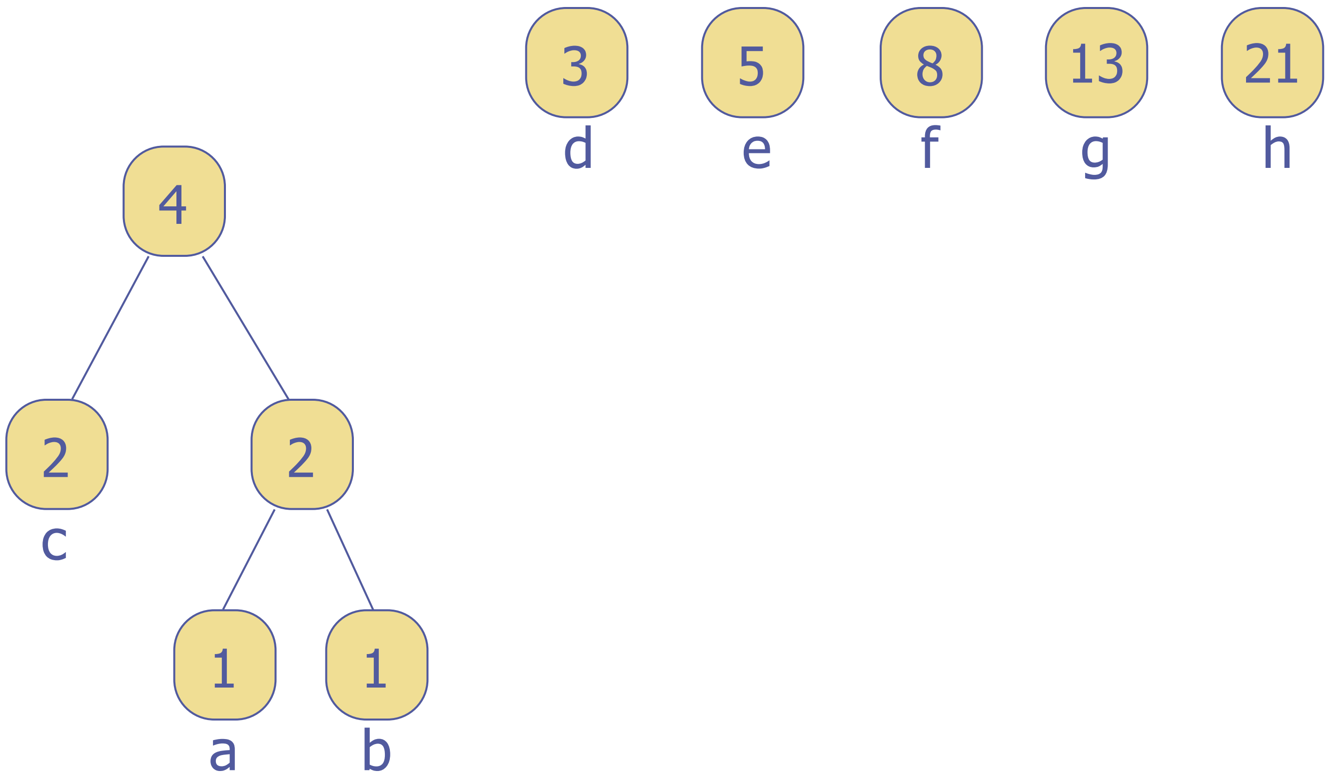
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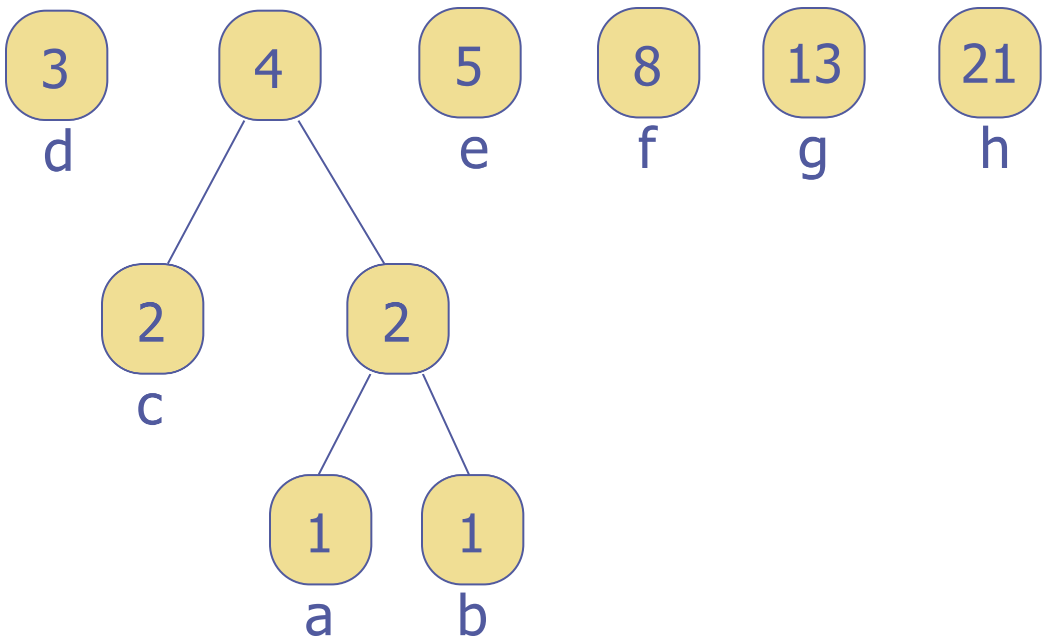
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Correctness of Huffman Code

Proof Idea

- Step 1: Show that this problem satisfies the greedy choice property, that is, if a greedy choice is made by Huffman's algorithm, an optimal solution remains possible.
- Step 2: Show that this problem has an optimal substructure property, that is, an optimal solution to Huffman's algorithm contains optimal solutions to subproblems.
- Step 3: Conclude correctness of Huffman's algorithm using step 1 and step 2.

Lemma: Greedy Choice Property

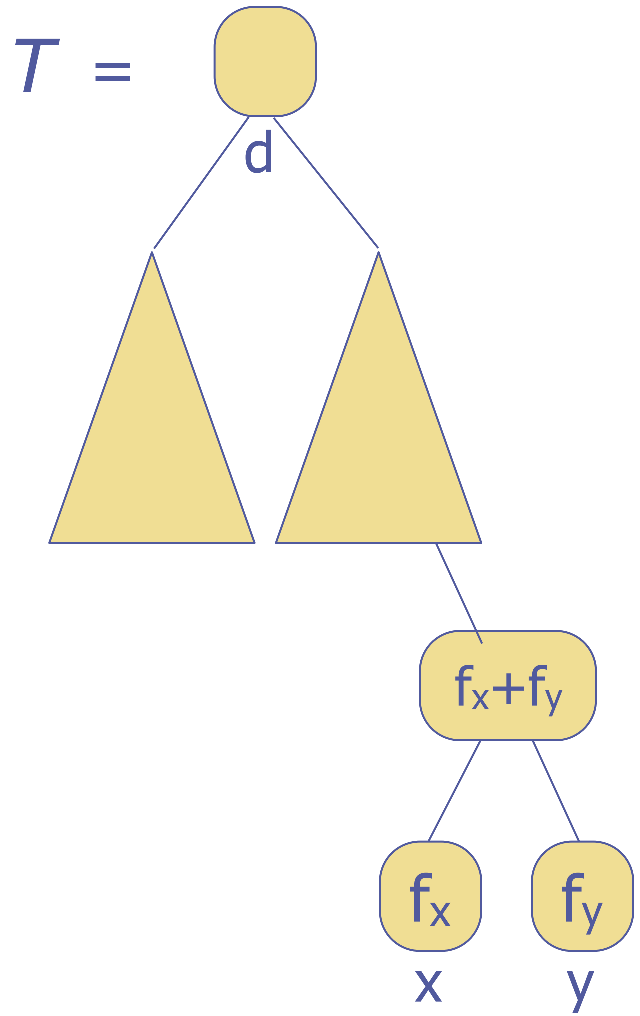
Let c be an alphabet in which each character c has frequency f[c]. Let x and y be two characters in C having the lowest frequencies. Then there exists an optimal prefix code for C in which the codewords for x and y have the same length and differ only in the last bit.

Lemma: Optimal Substructure Property

- Let T be a full binary tree representing an optimal prefix code over an alphabet C, where each $c \in C$ has frequency f_c.
- Consider any two characters x and y that appear as sibling leaves in the tree T.
- Consider alphabet $C' = C - \{x, y\} \cup \{z\}$ with frequency $f_z = f_x + f_y$, and label with z the parent of x and y
- Then $T' = T - \{x, y\}$ represents an optimal code for alphabet C'

T represents an optimal prefix code for alphabet *C*

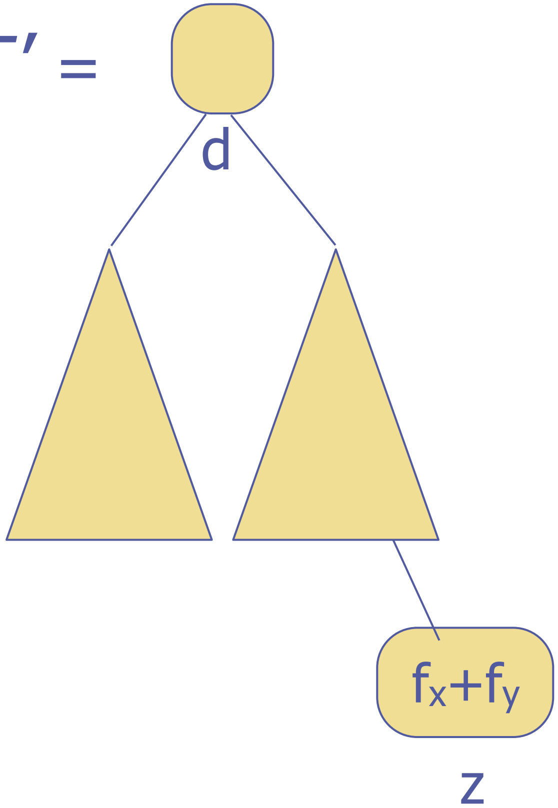
x and *y* appear as sibling leaves



T′ represents an optimal prefix code for alphabet *C′*

x and *y* replaced by *z*

$T' =$



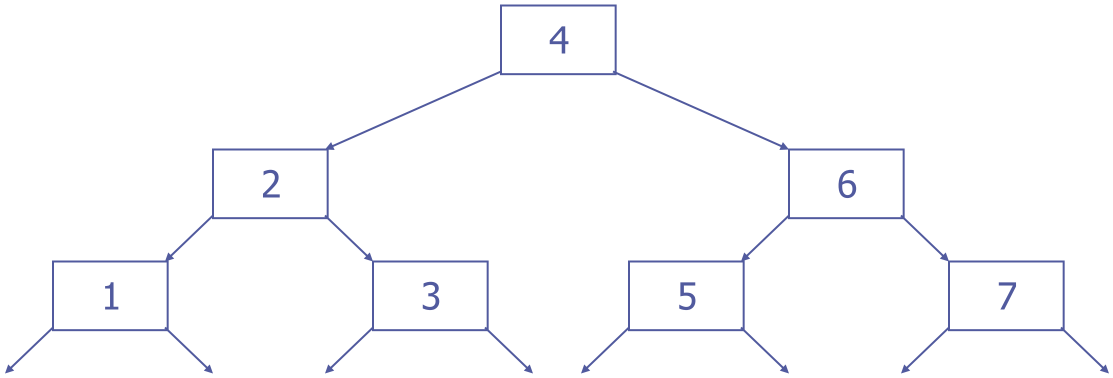
Priority Queues

Priority Queues

- A Priority Queue is a data structure optimized for finding and removing the element with the max (or min) key. It has operations to:
  - find the highest priority element (with max key)
  - delete the highest priority element
  - add a new item
- We want to avoid insertion sort at each step
  - Complexity of insertion would be O(n)
- We use a *Heap* (Levitin §6.4) — a particular kind of balanced tree.

The ideal:

- O(log n) complexity
- Everybody happy



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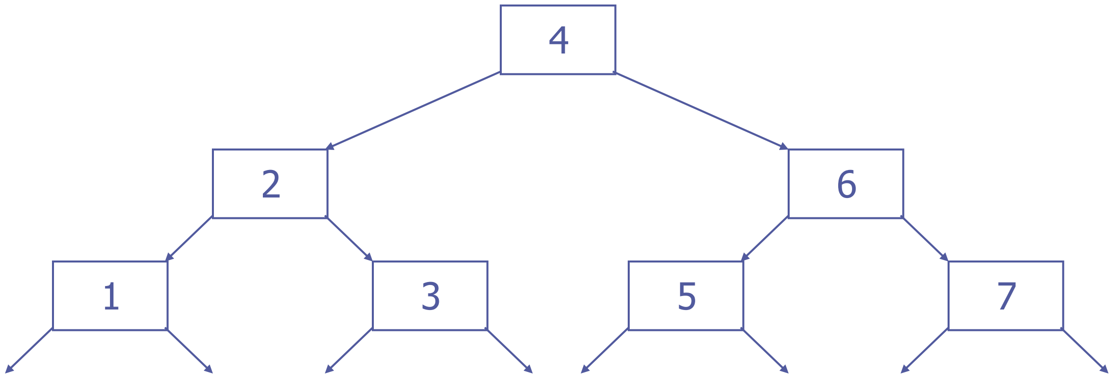

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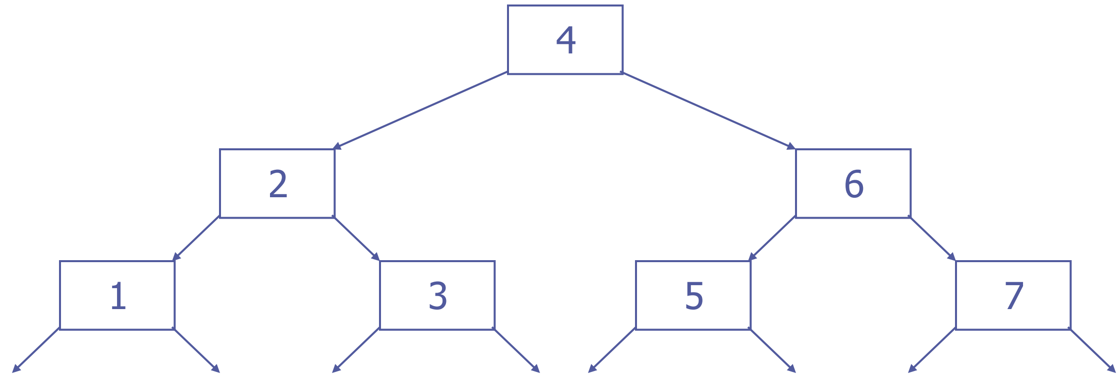

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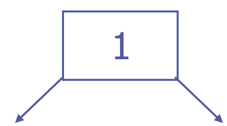

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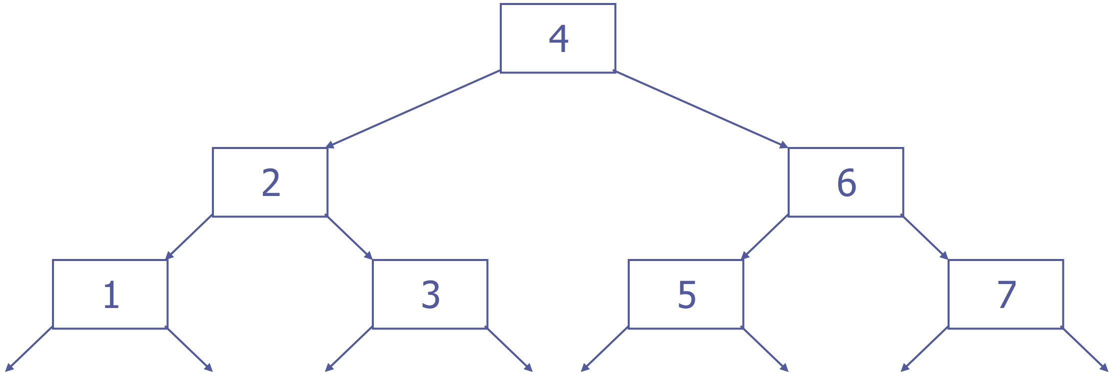

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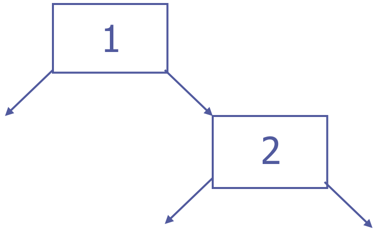

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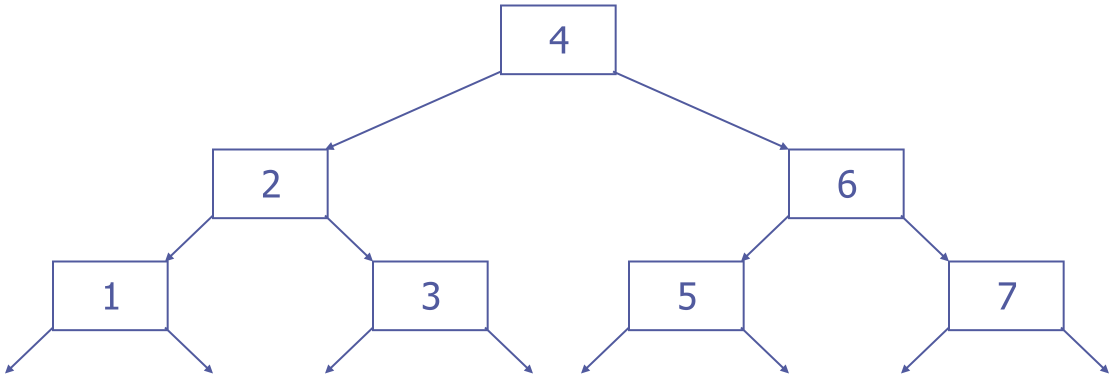

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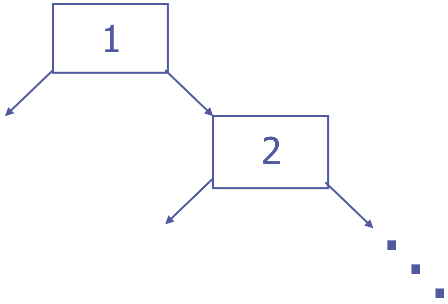

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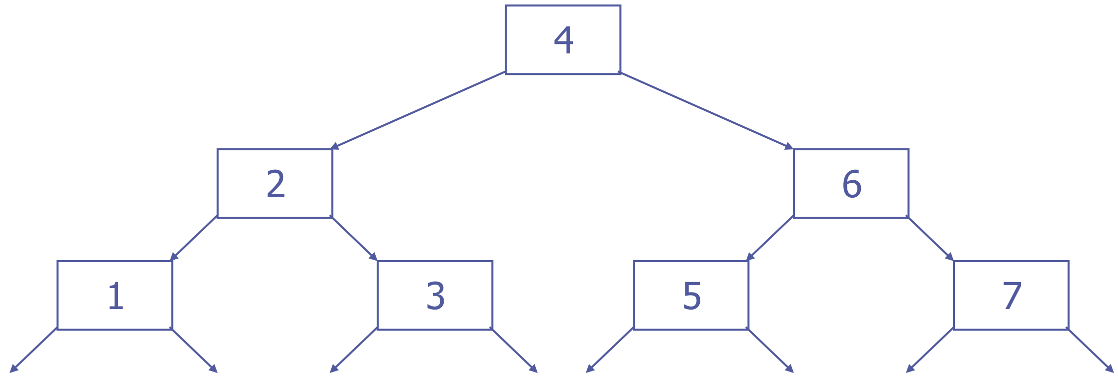

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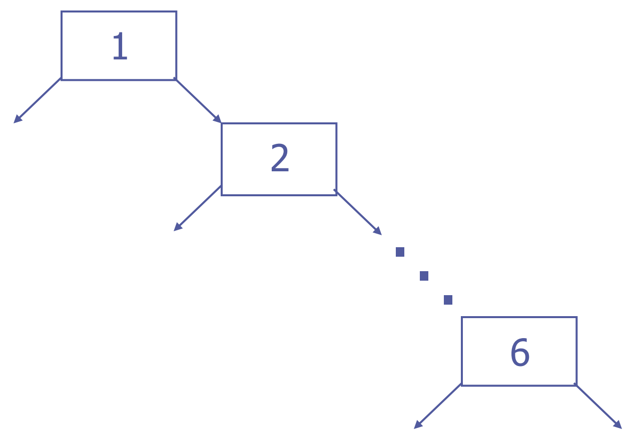

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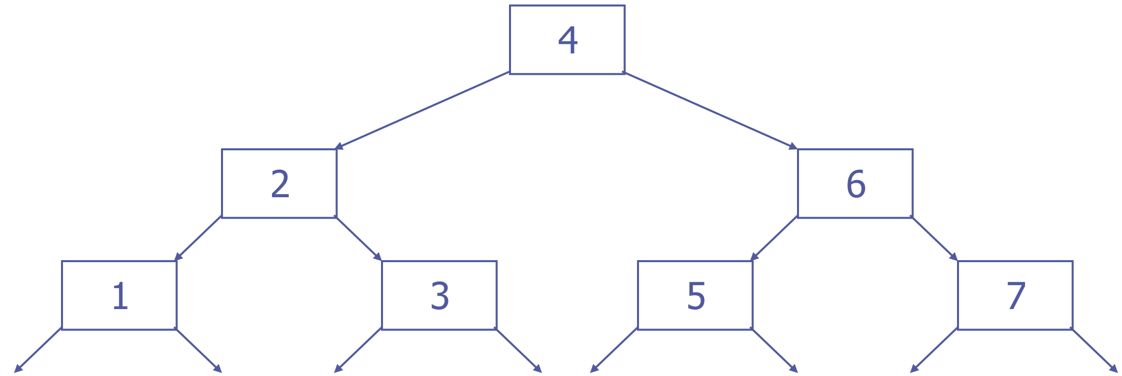

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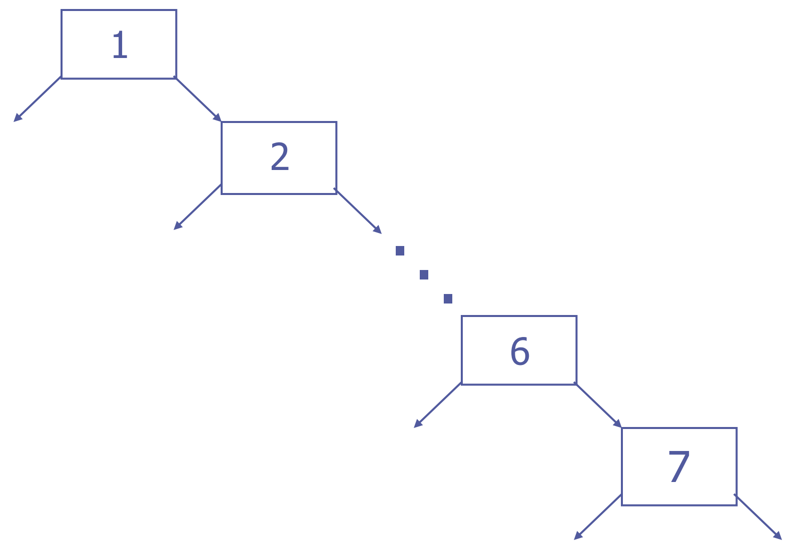

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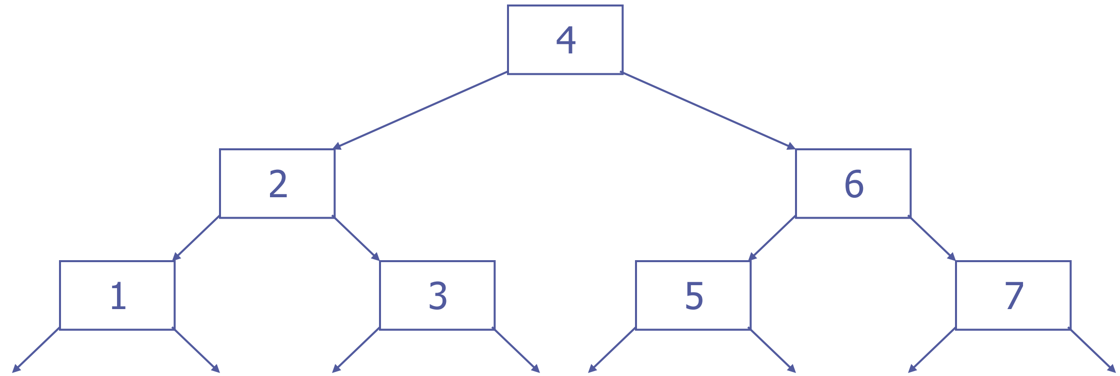

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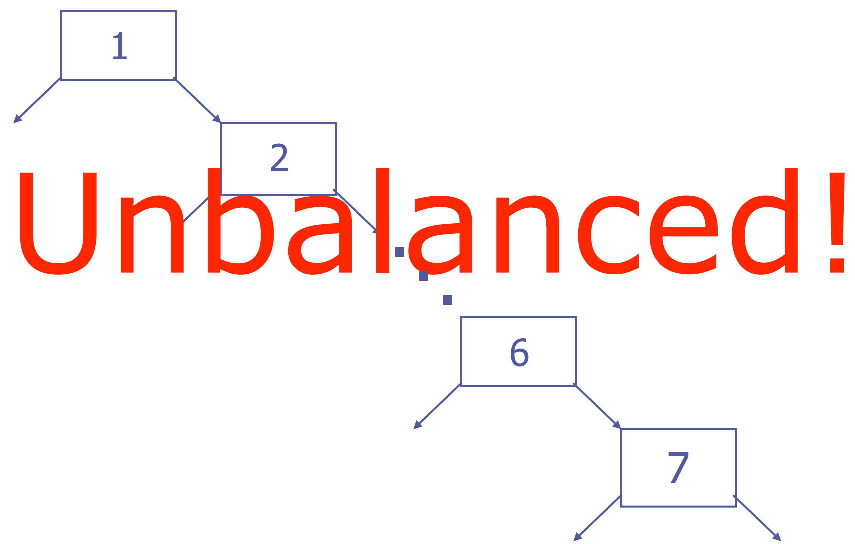

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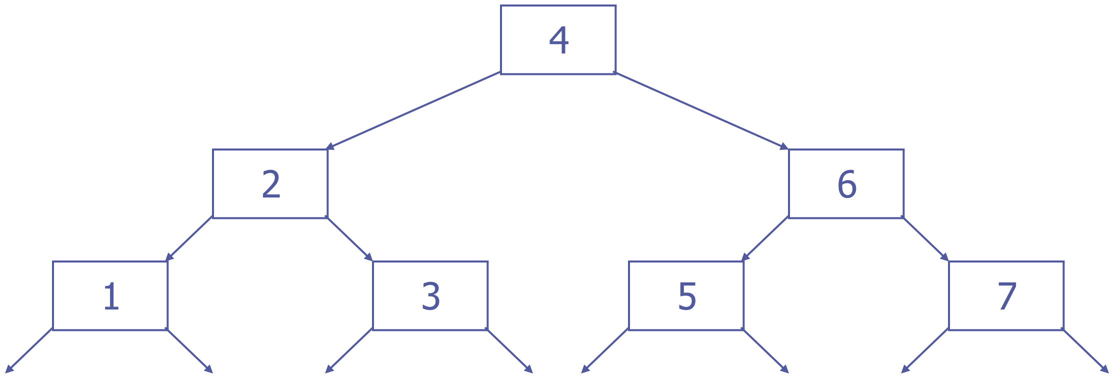

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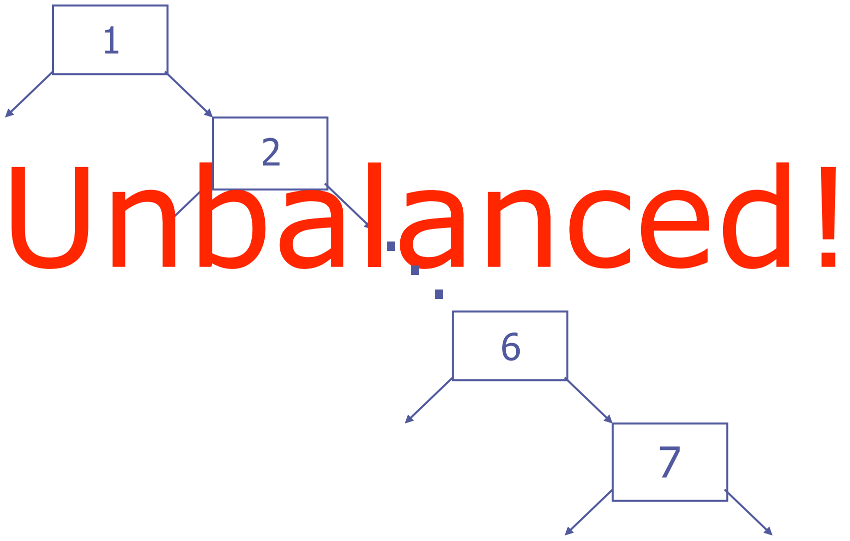
- Could have used lists!



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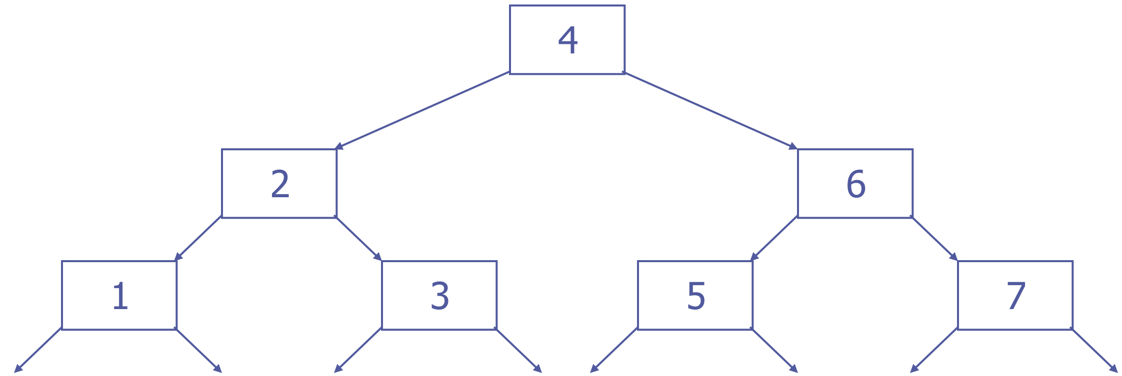

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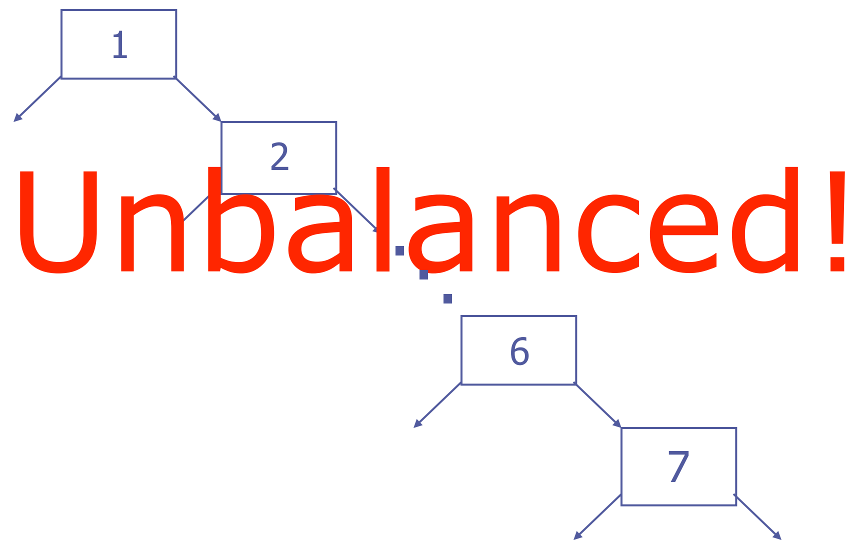

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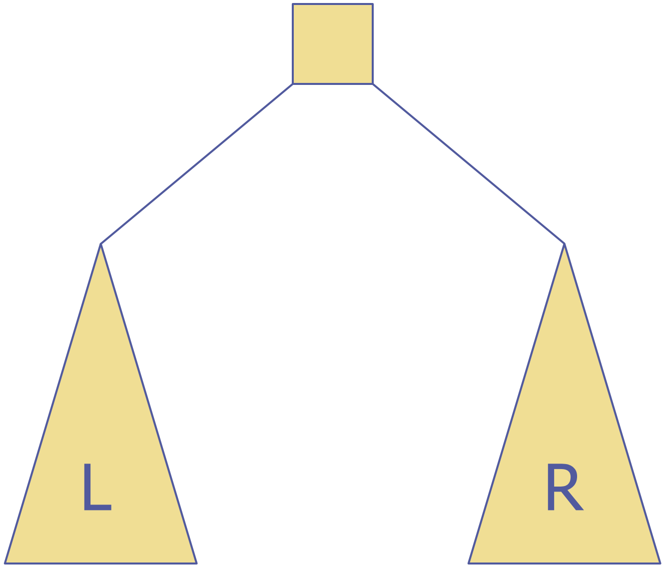
What does “balanced” mean?

Perhaps:



size L = size R

?

Too constraining!

- A balanced binary tree of height h has exactly n_h elements, where:

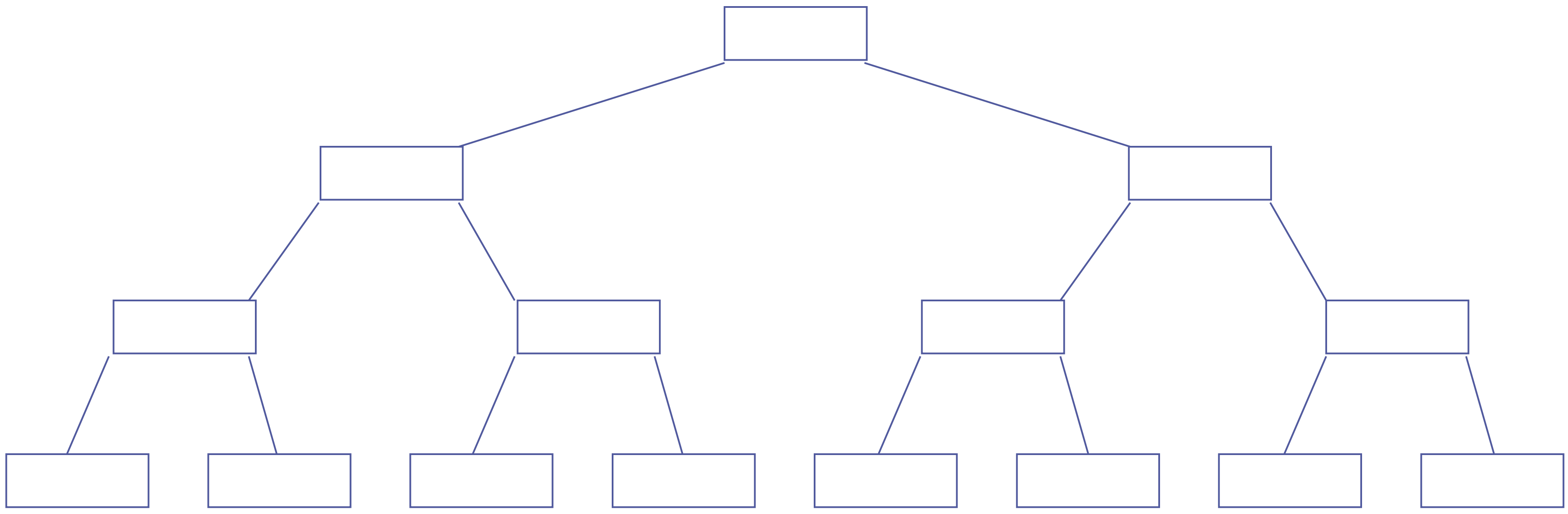
$$n_{-1} = 0 \text{ and } n_{(h+1)} = 1 + 2\, n_h;$$

- So if T is perfectly balanced, then:

$$\text{size } T \in \{0, 1, 3, 7, 15, 31, 63, \ldots, 2^h\text{-}1, \ldots\};$$

- There is no perfectly balanced tree with any other number of elements.

A perfectly balanced tree:



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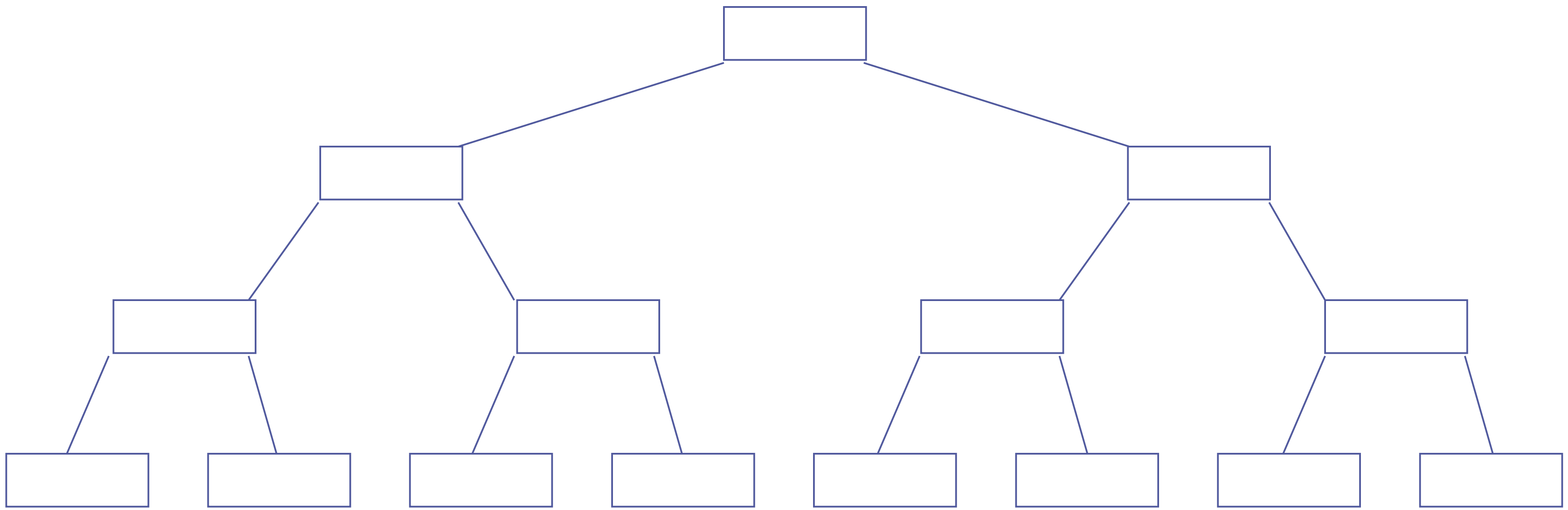

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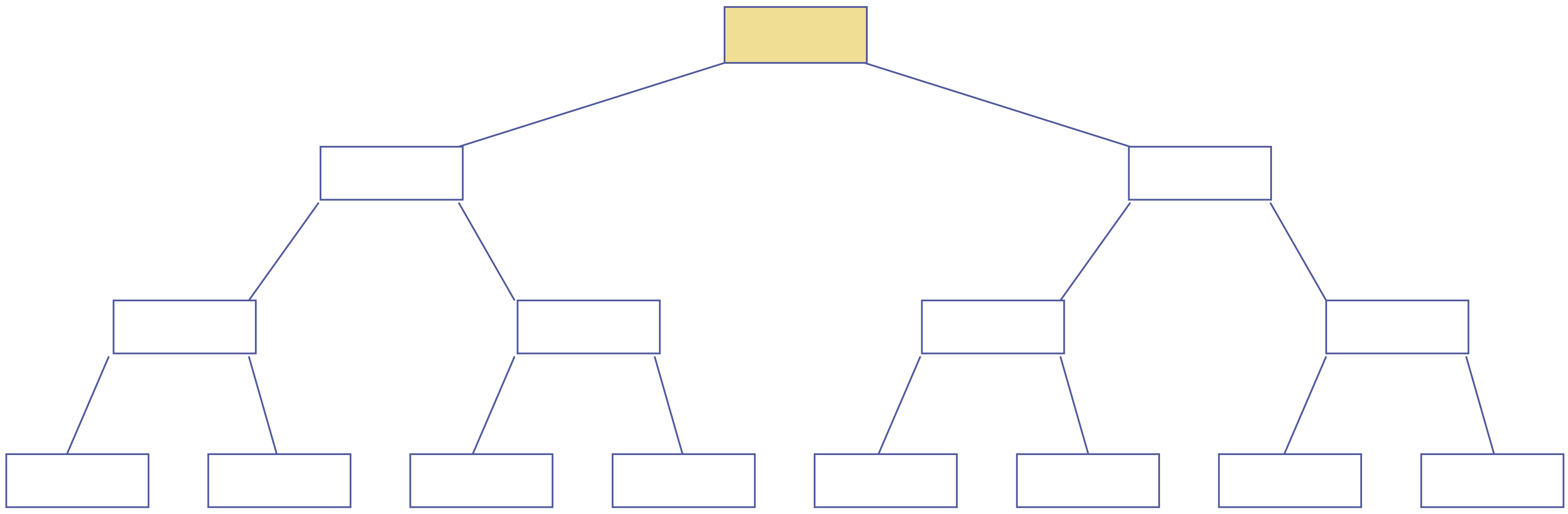

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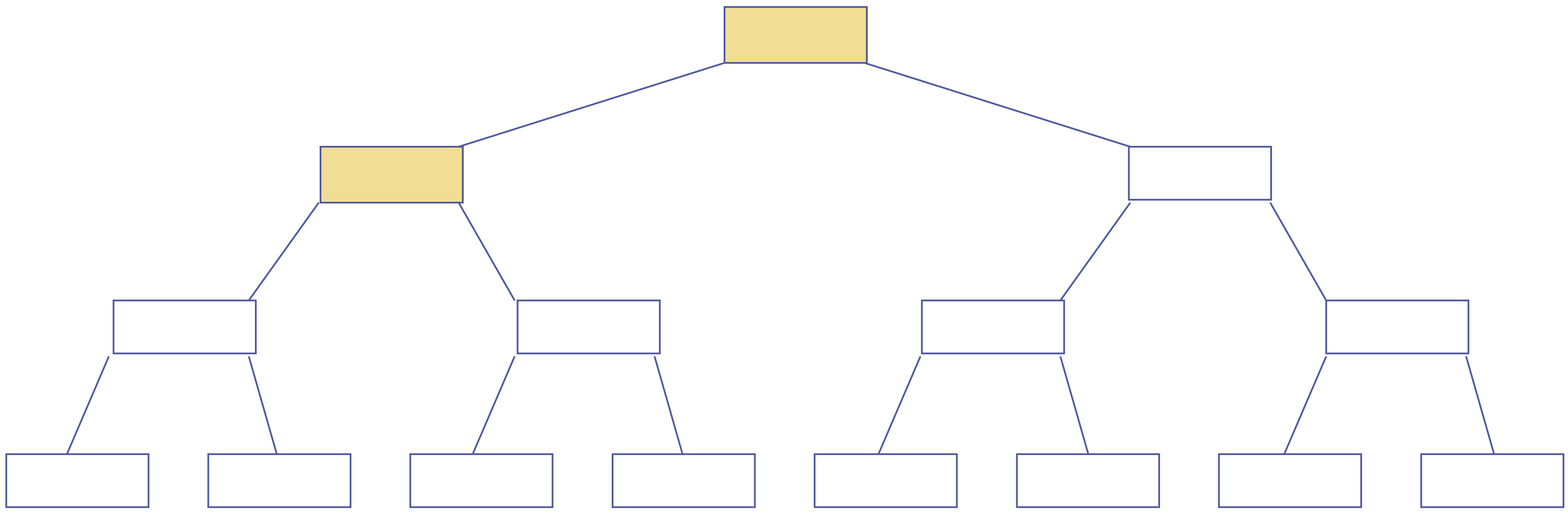

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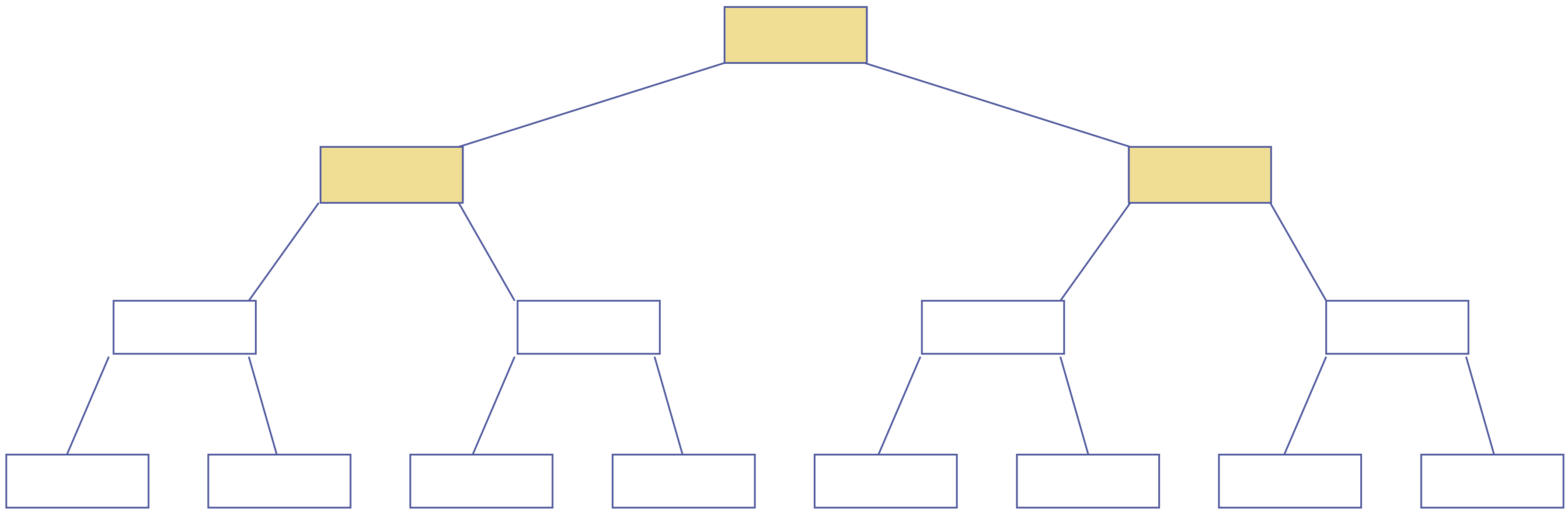

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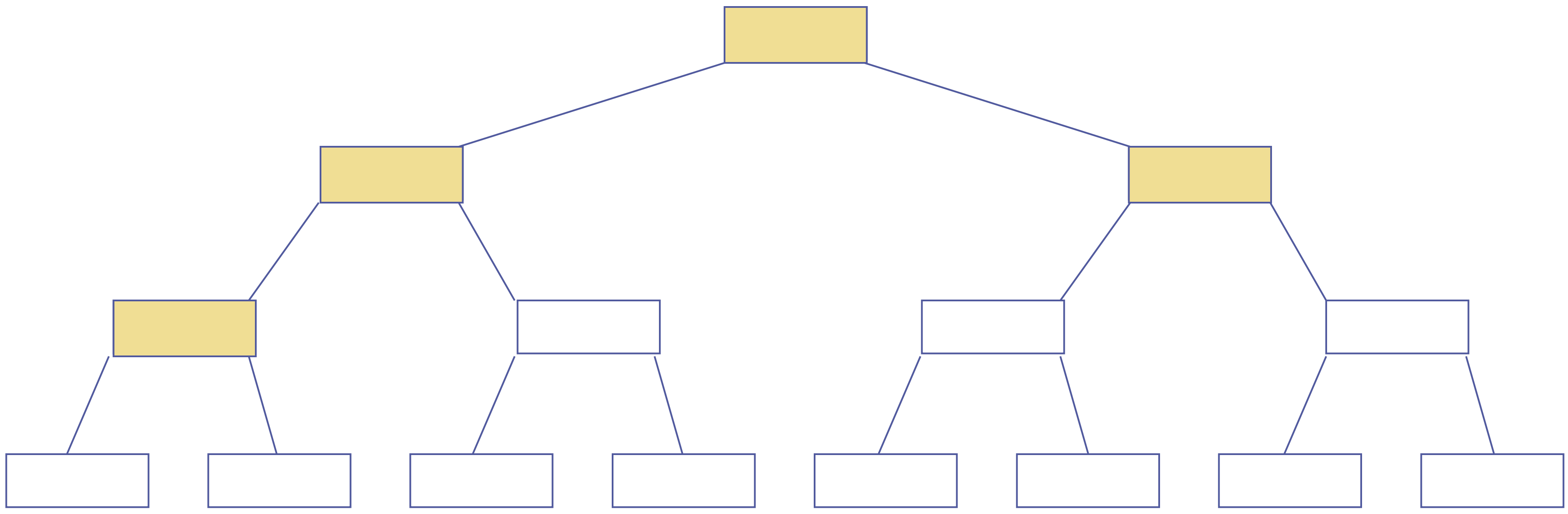

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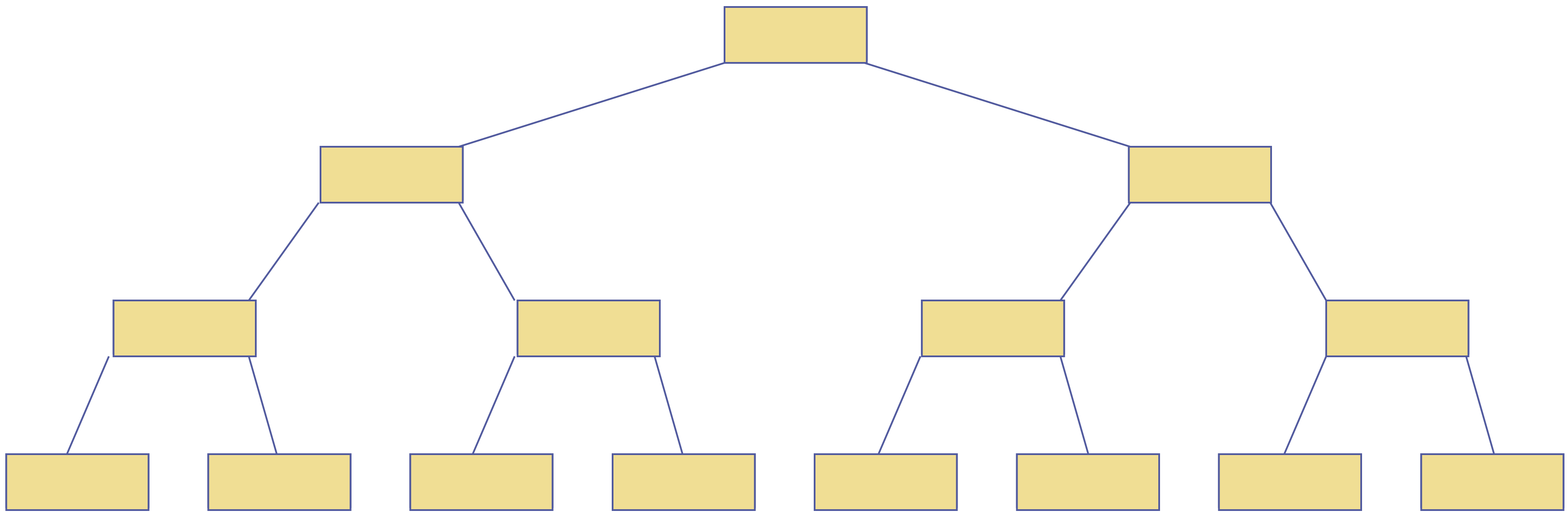

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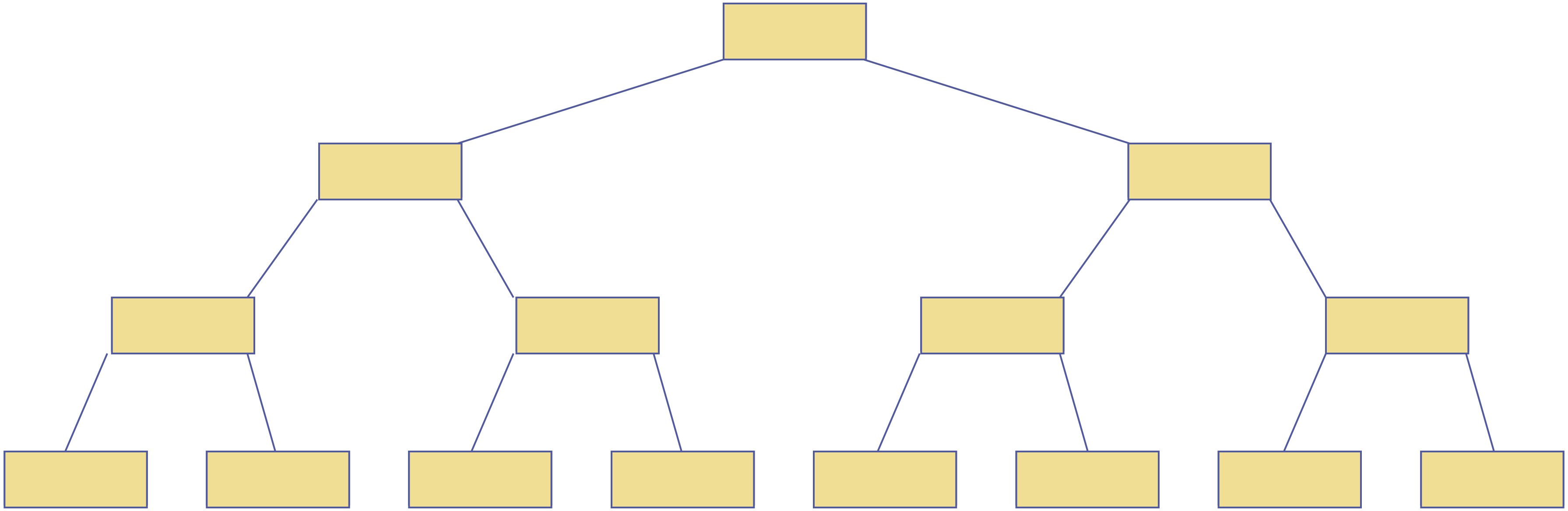
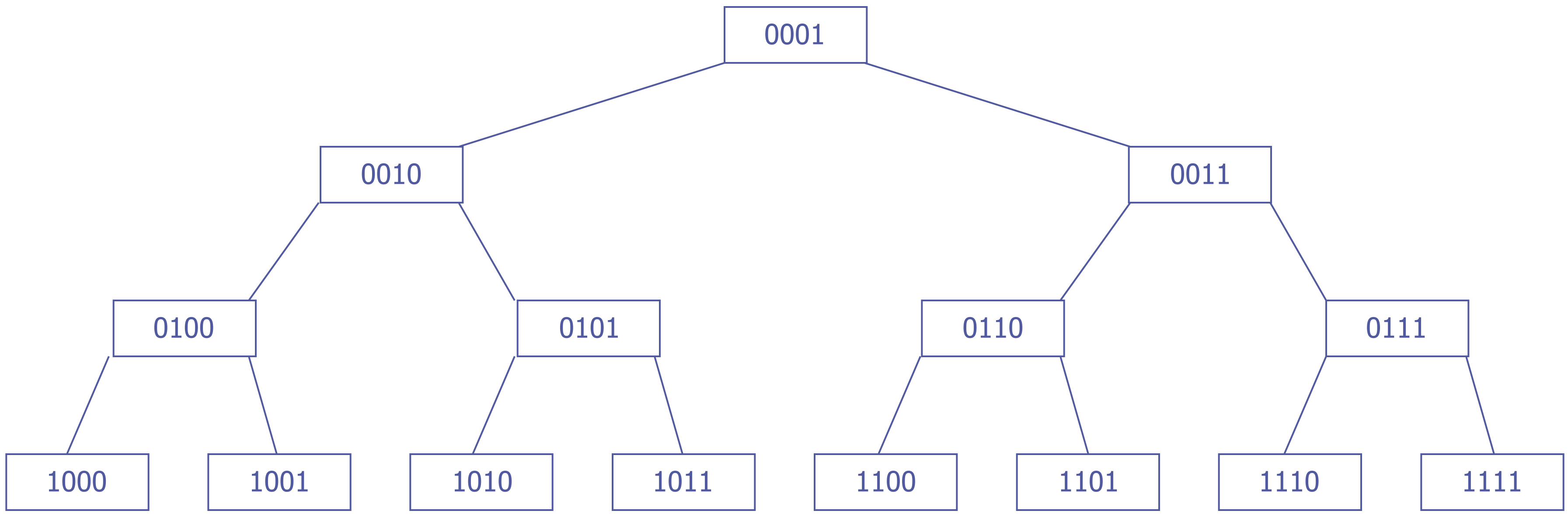


Think of this as an empty frame that we can fill with elements ...

... filling the rows up one at a time makes the tree as balanced as possible!

Number the nodes — in binary!



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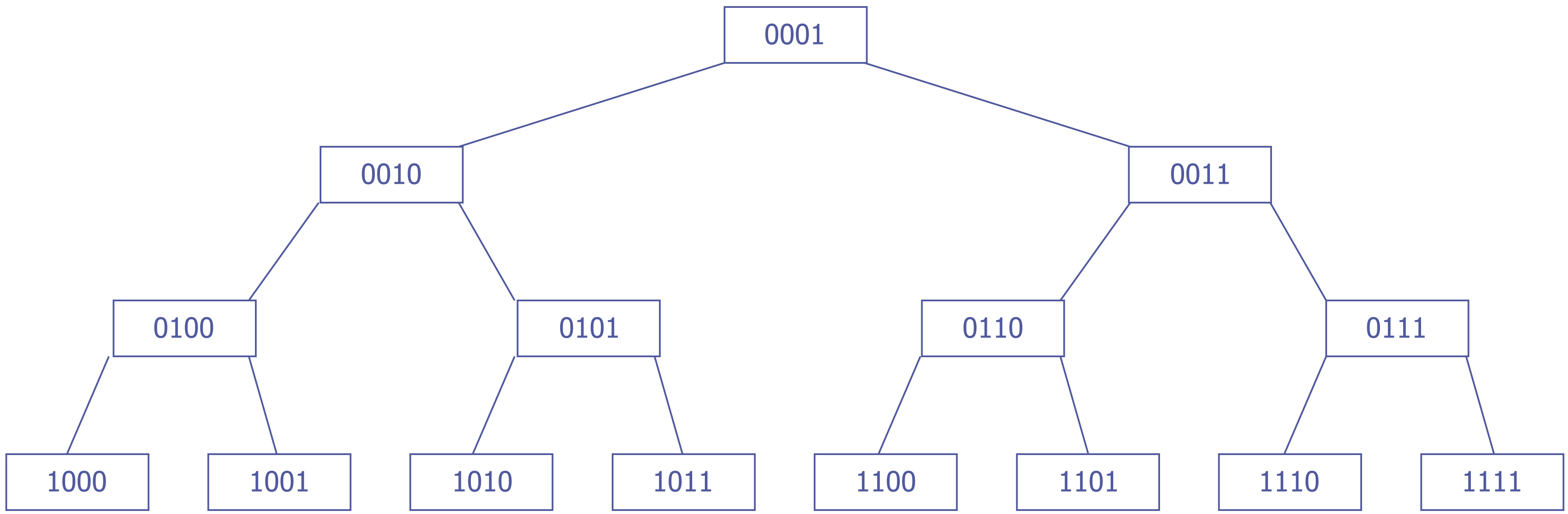

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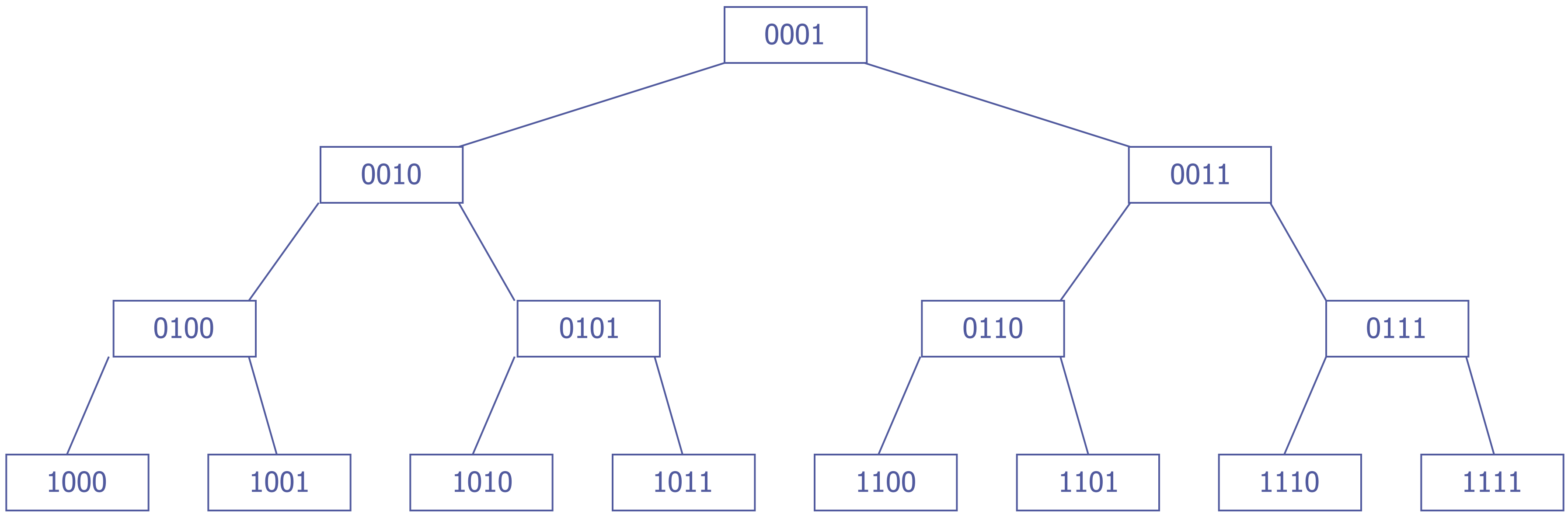

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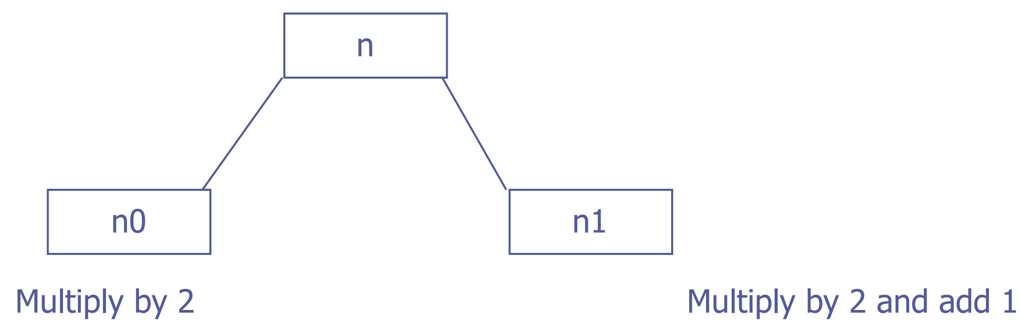


Embed a tree in an array

- A tree with $t < 2^n$ elements can be implemented using an array a and variable t:
  - elements a[1..t], (a[t +1 .. 2^n–1] are empty)
  - the root is held in position a[1]
  - left child of node a[i] is a[$2i$]
  - right child of node a[i] is a[$2i$+1]
  - parent of node a[i] is a[$\lfloor i/2 \rfloor$]

- True or False: all elements of the array with index $\geq 2^{n-1}$ represent leaf nodes

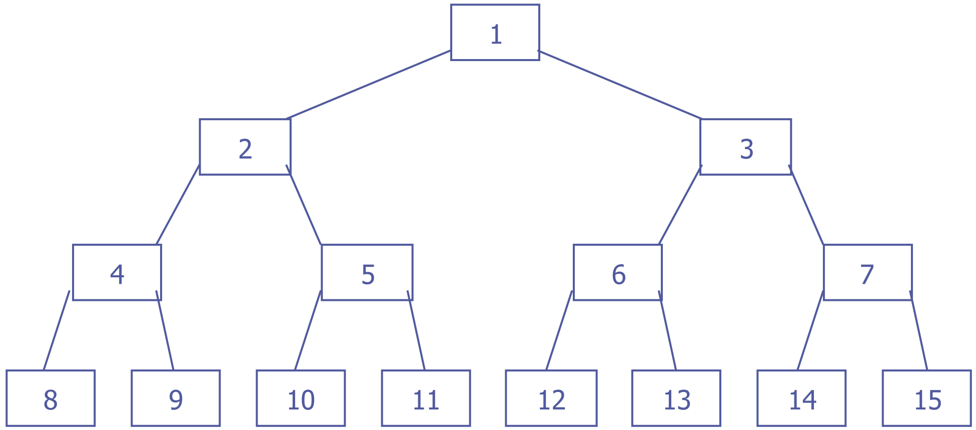
Too good to be true?

- So now we can build (almost) perfectly balanced binary trees with:
  - the smallest possible height for any number of elements stored;
  - O(1) complexity for addition.

- Where’s the flaw?

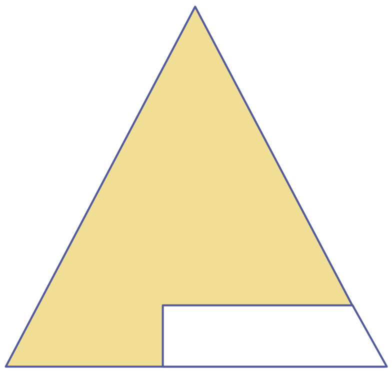
Out of order!

- Building a tree in this way does not give binary <u>search</u> trees:



- We <u>cannot</u> preserve the binary search tree invariant <u>and</u> retain O(1) time for insertion.

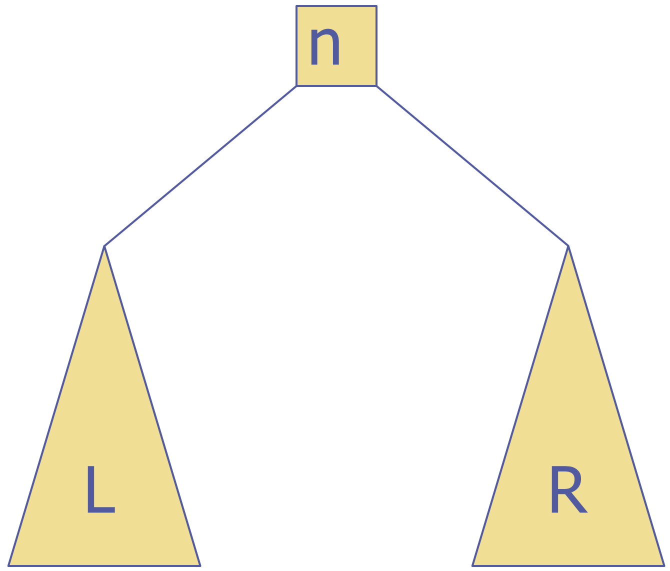
Properties of a Heap:



1. Shape Property:

   The binary tree is **essentially complete**, that is, all levels are filled except some of the rightmost leaves may be missing in the last level

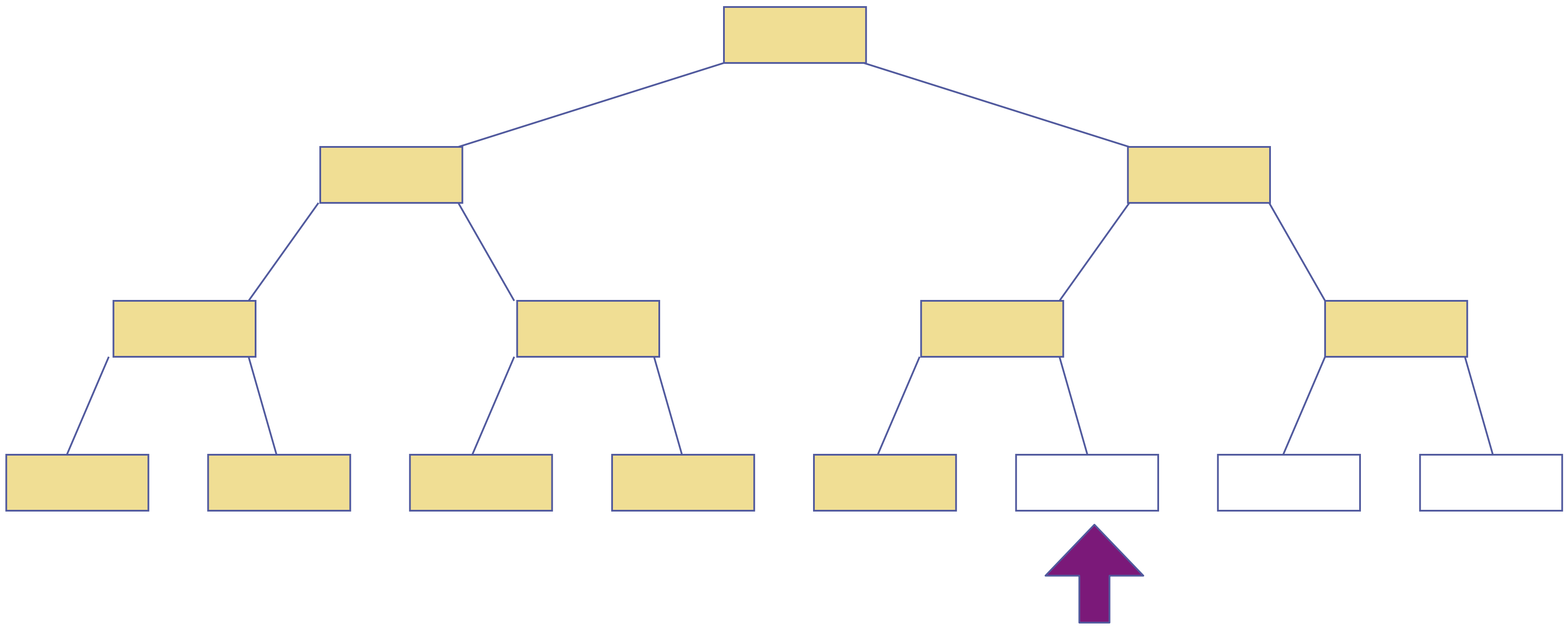
Properties of a Heap:



2. Parental dominance Property:

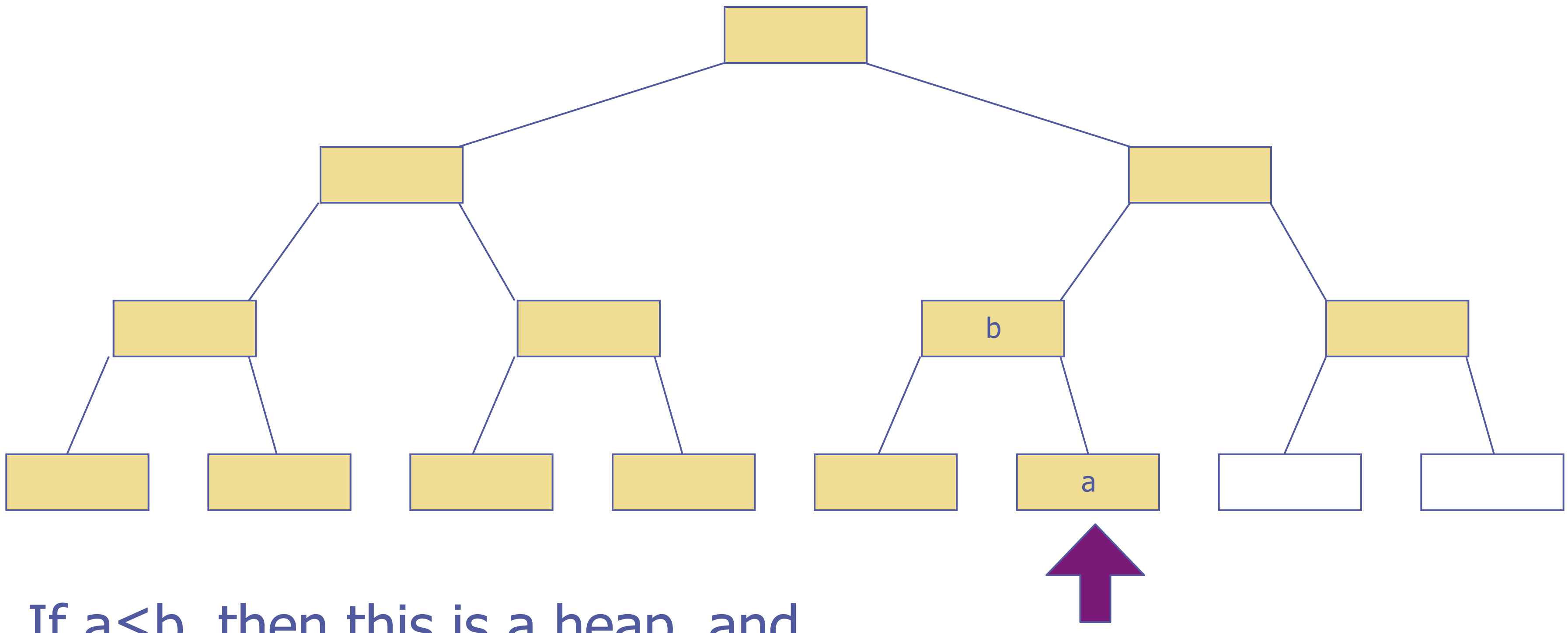
   The key in each node is greater than or equal to the keys of its children. So, all values in L are ≤n, and all values in R are <u>also</u> ≤n

Inserting an element:



The new element should be added here (takes O(1) time)

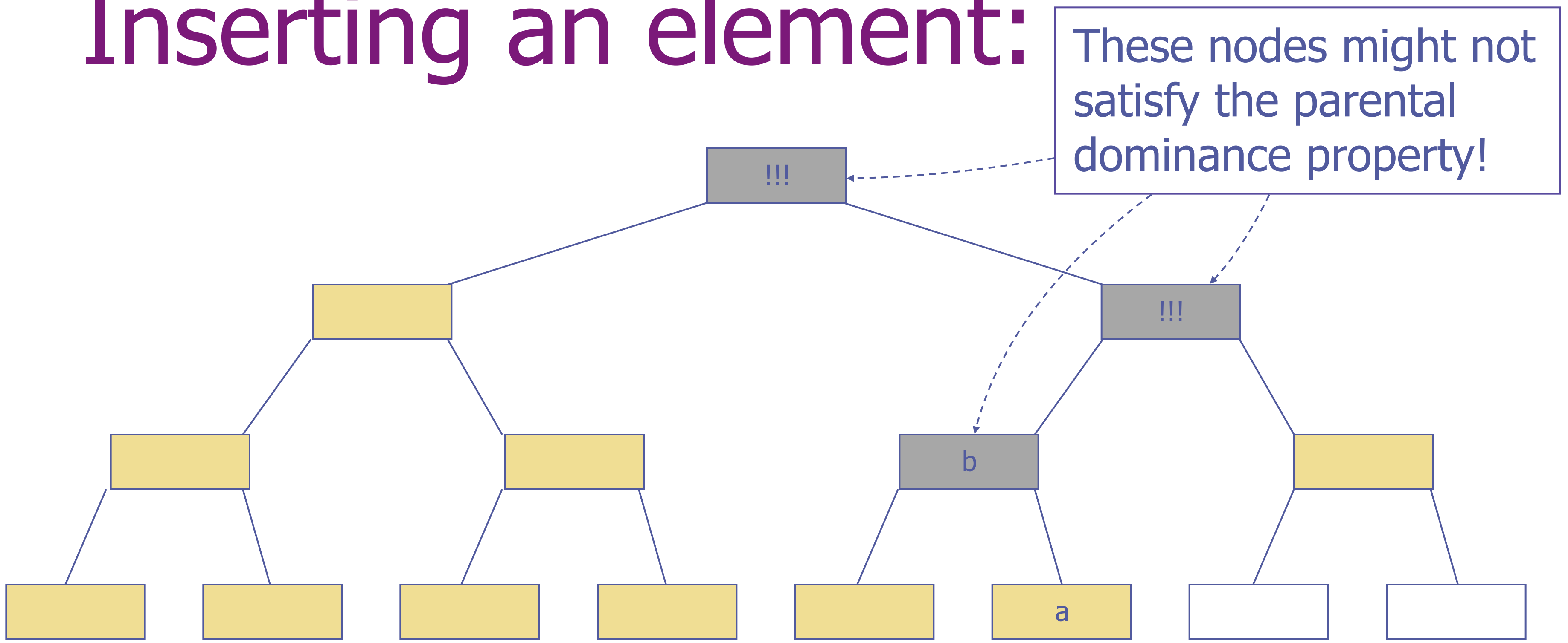
Inserting an element:



If a≤b, then this is a heap, and we are done!

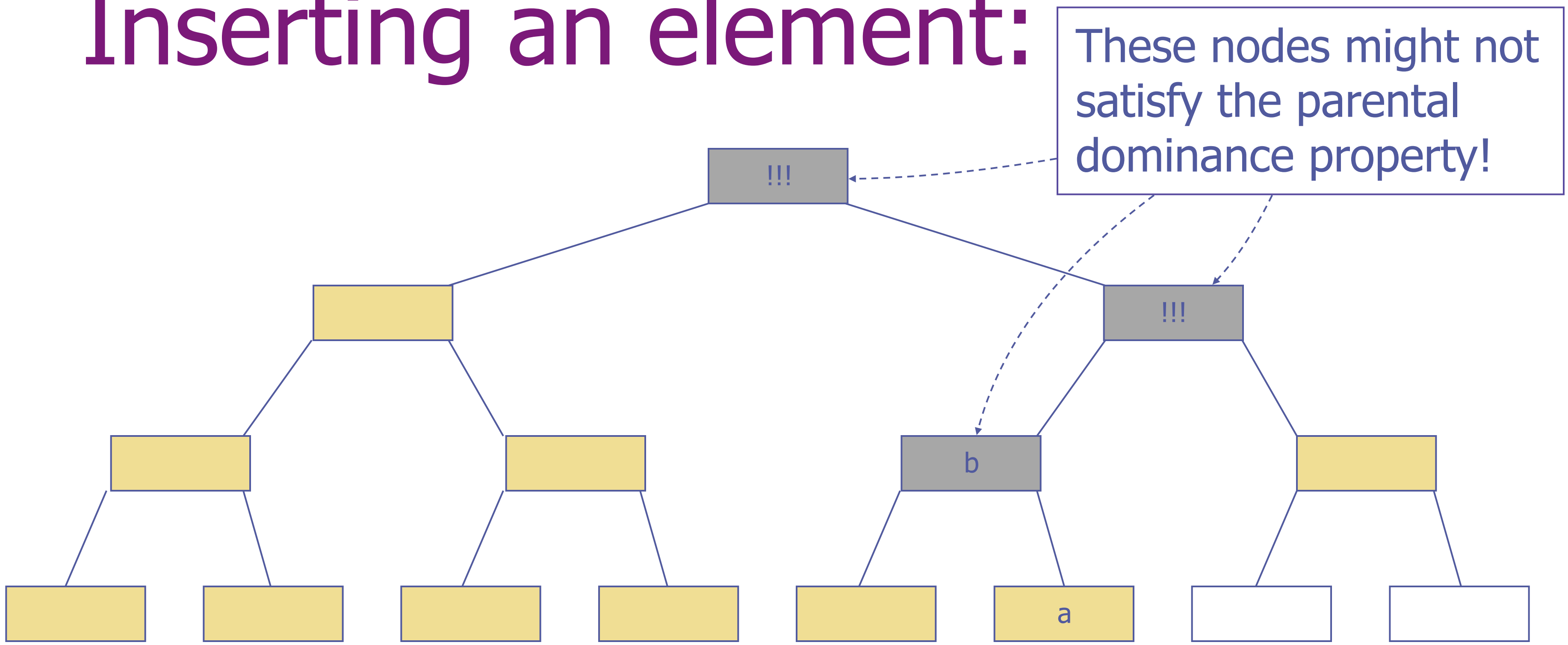
New value, a

Inserting an element:



But if a>b, then we need to do some work to restore the heap property.

Inserting an element:

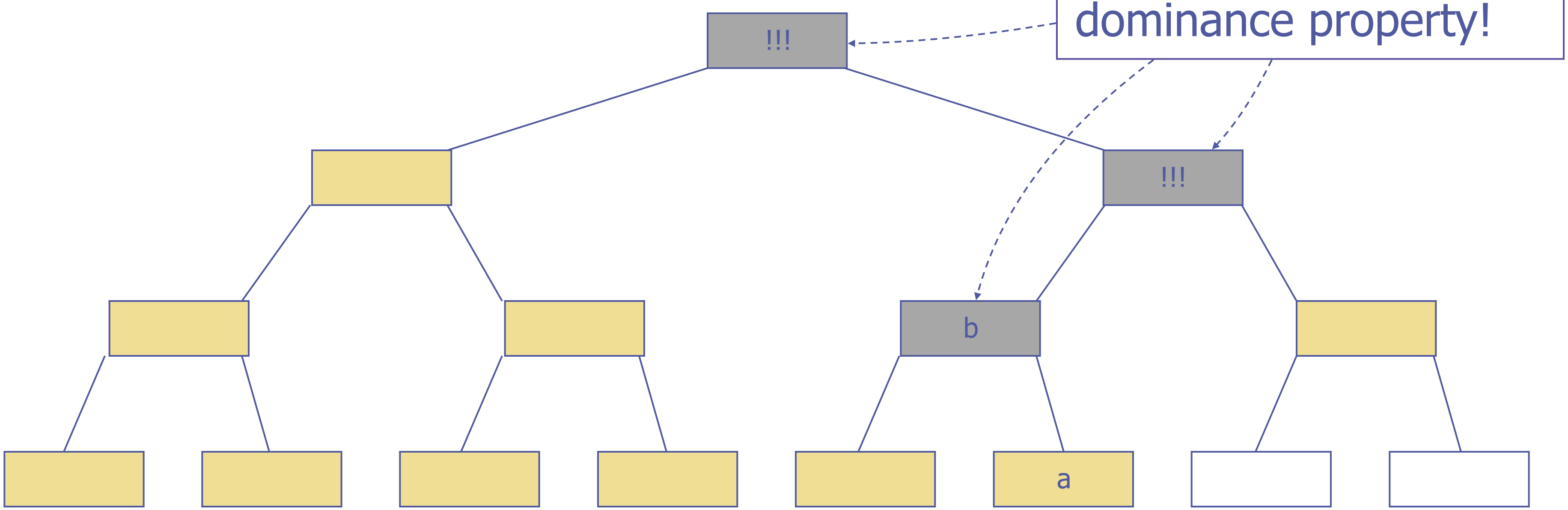


But if a>b, then we need to do some work to restore the heap property.

Start by swapping a and b ...

Inserting an element:

These nodes might not satisfy the parental dominance property!



Repeat until we're done.

Takes O(log n) time: we have to worry about the nodes on only one path in the tree.

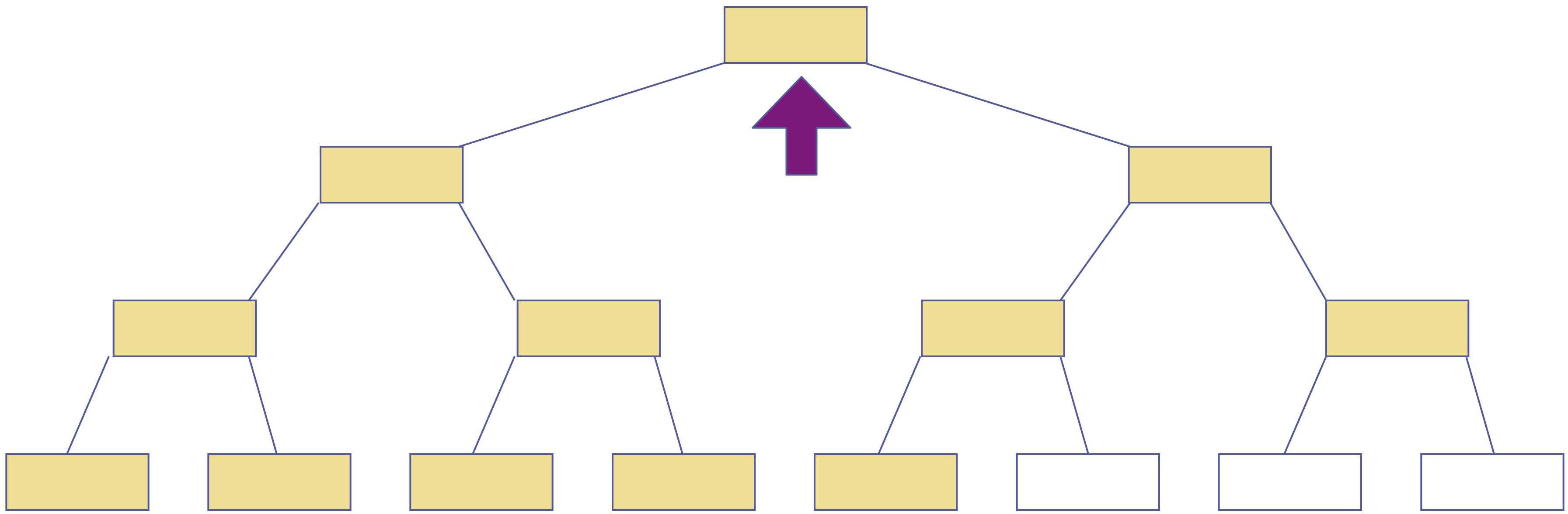
Implementation:

```
heapInsert(value) {
    size ← size + 1
    int i ← size;
    while (i>1 ∧ h[parent(i)]<value) do {
        h[i] ← h[parent(i)]
        i    ← parent(i)
    }
    h[i] ← value;
}
```

`h[]` is an array containing the heap elements;

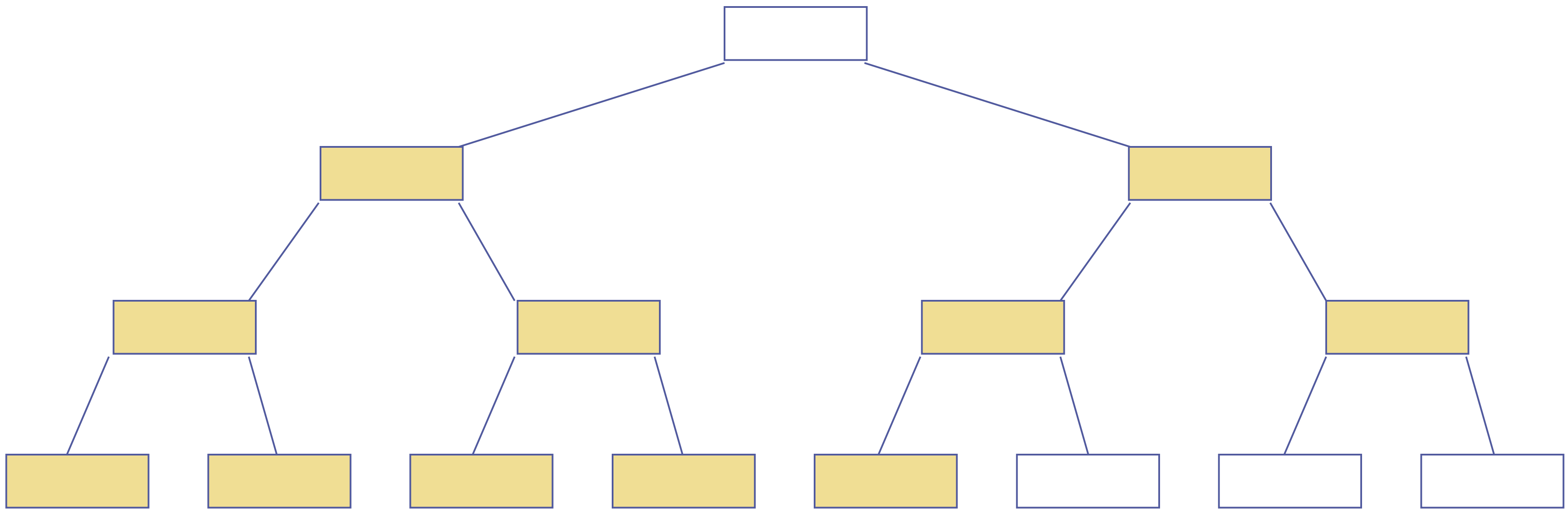
`size` is the number of entries in the heap that have been used.

Removing maximal element:



Finding the maximum element is easy! (takes O(1) time)

Removing maximal element:



We can fill the gap with the last value in the array (takes O(1) time)

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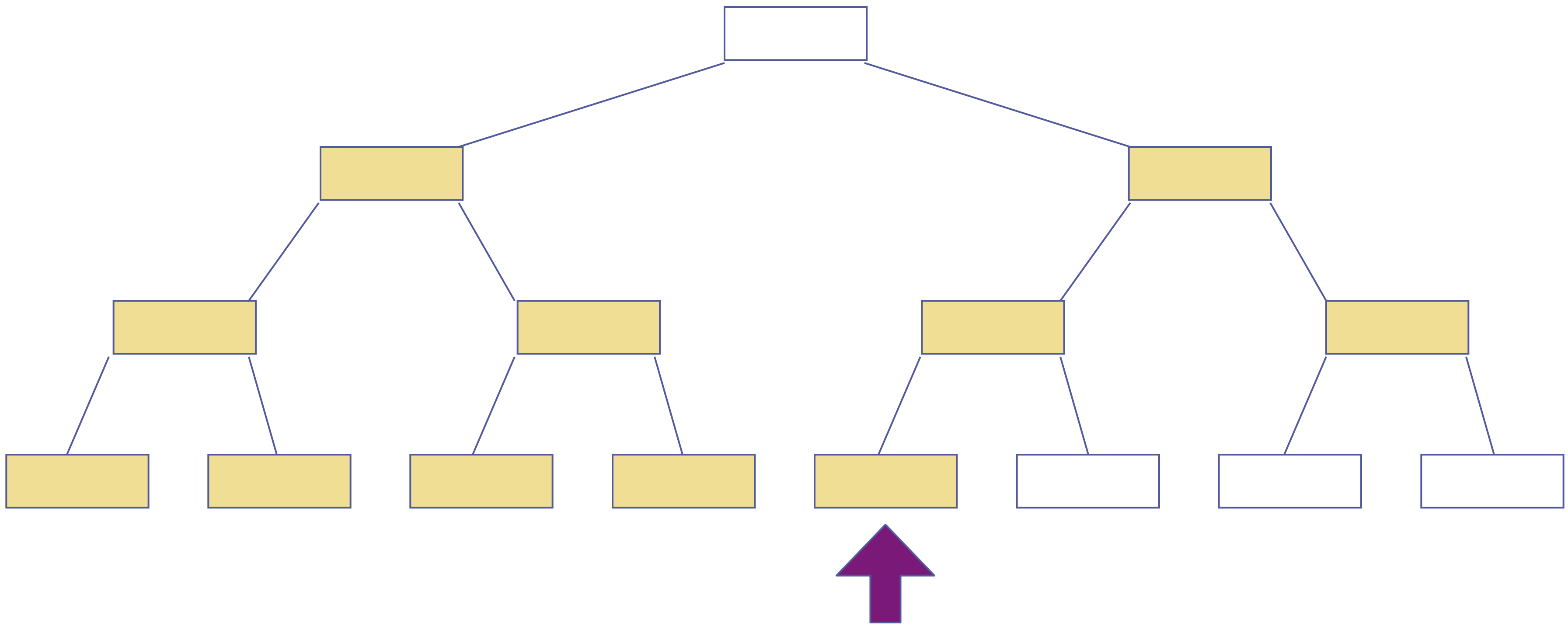

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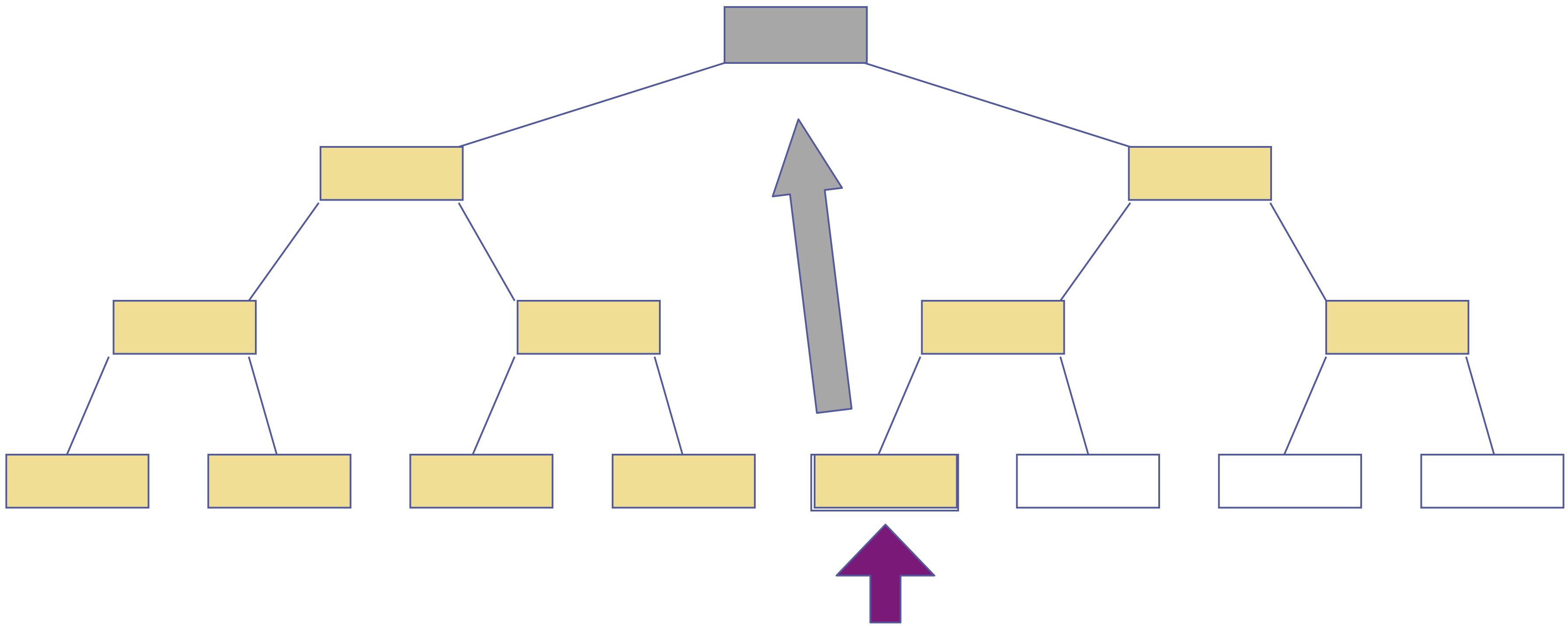

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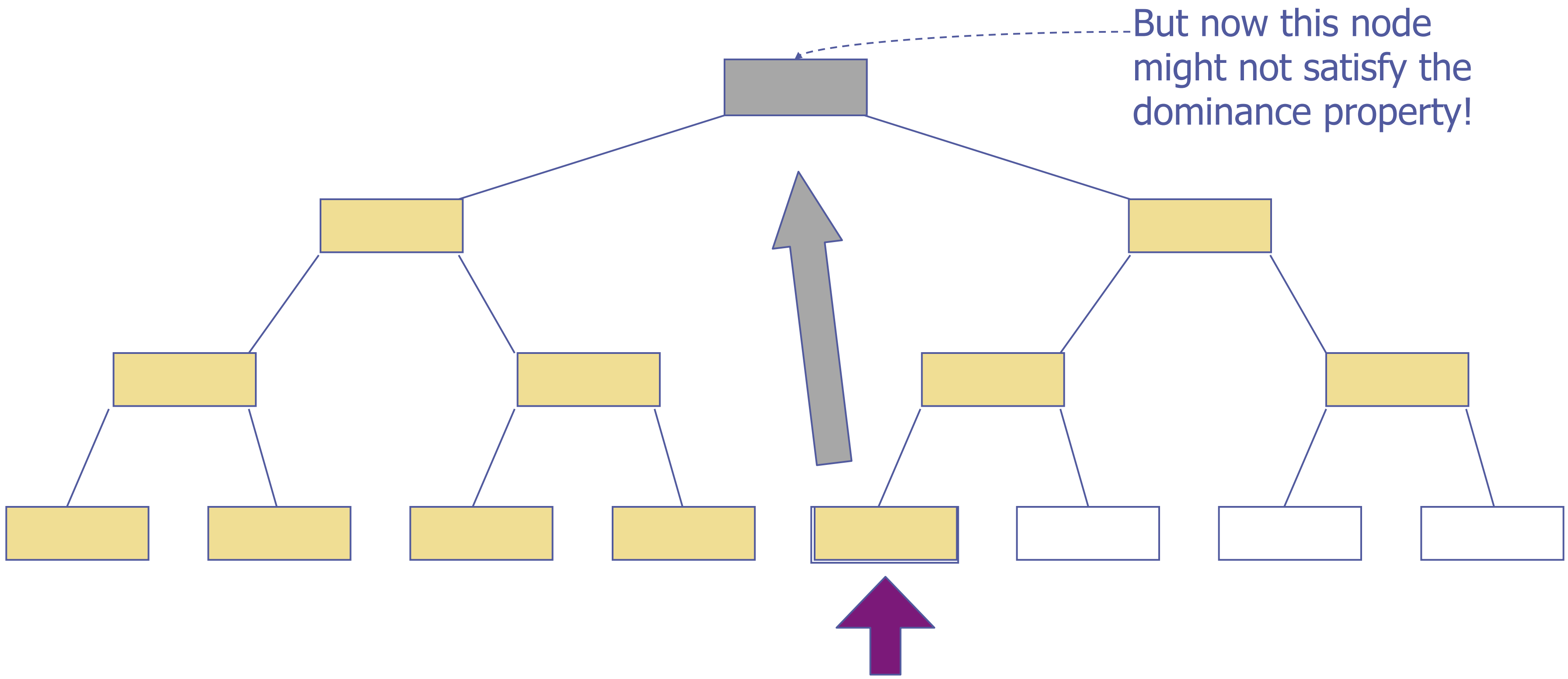

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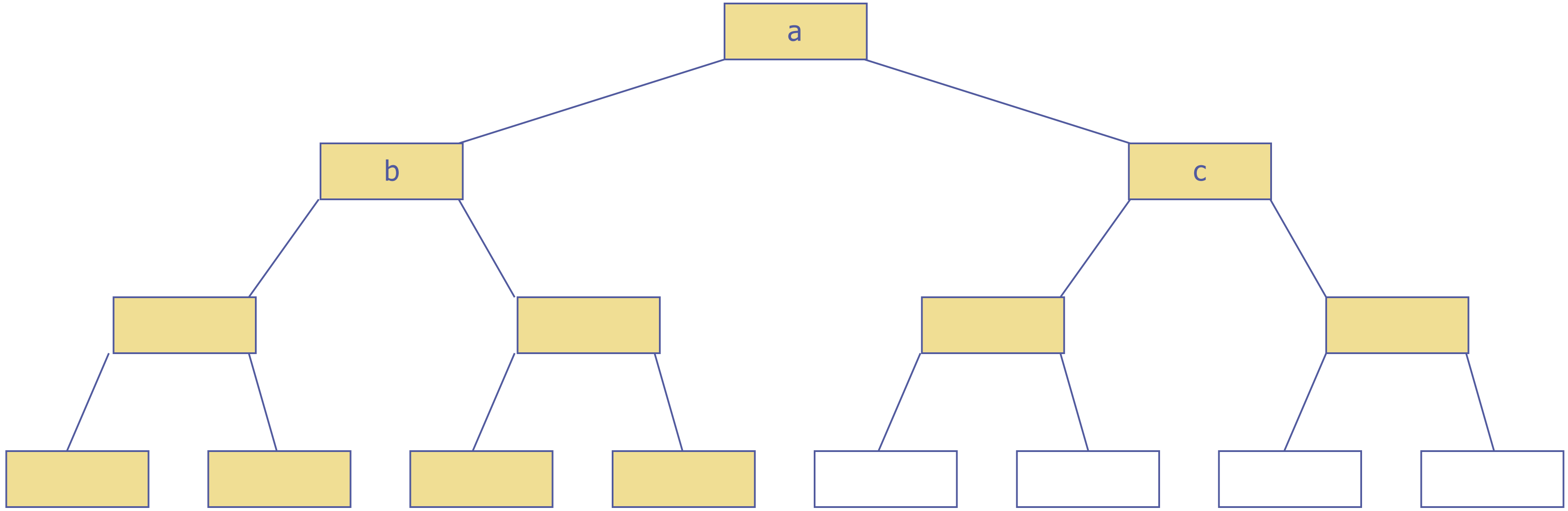

If a>b and a>c, then this is a heap, and we are done!

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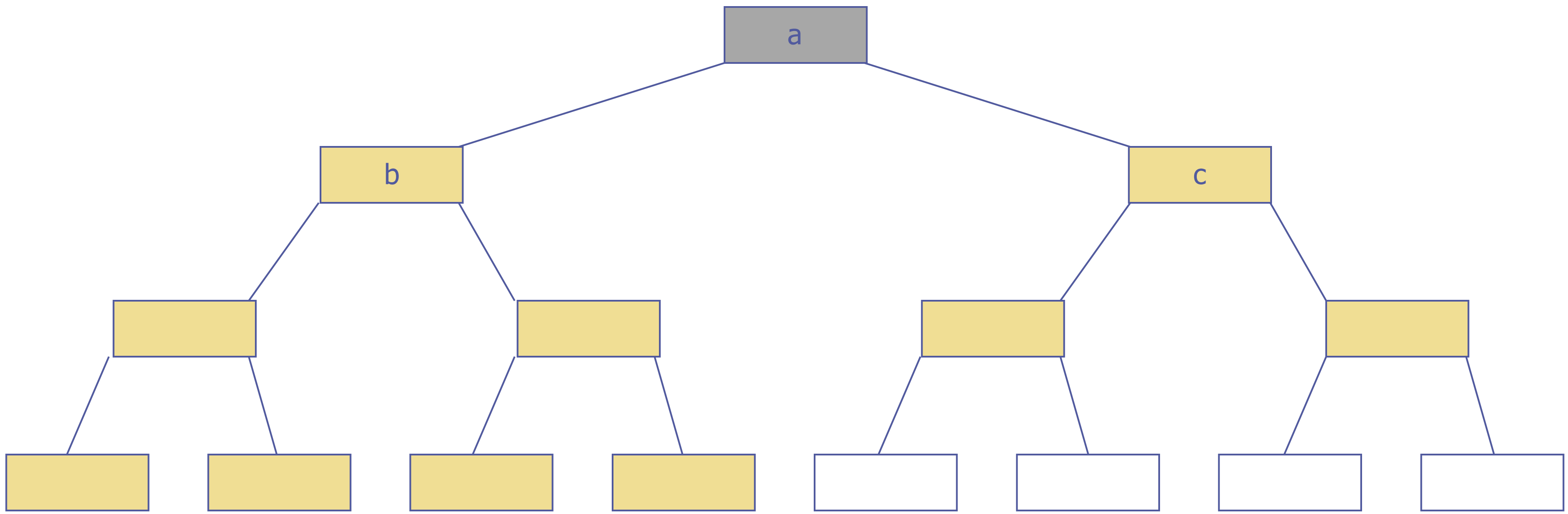
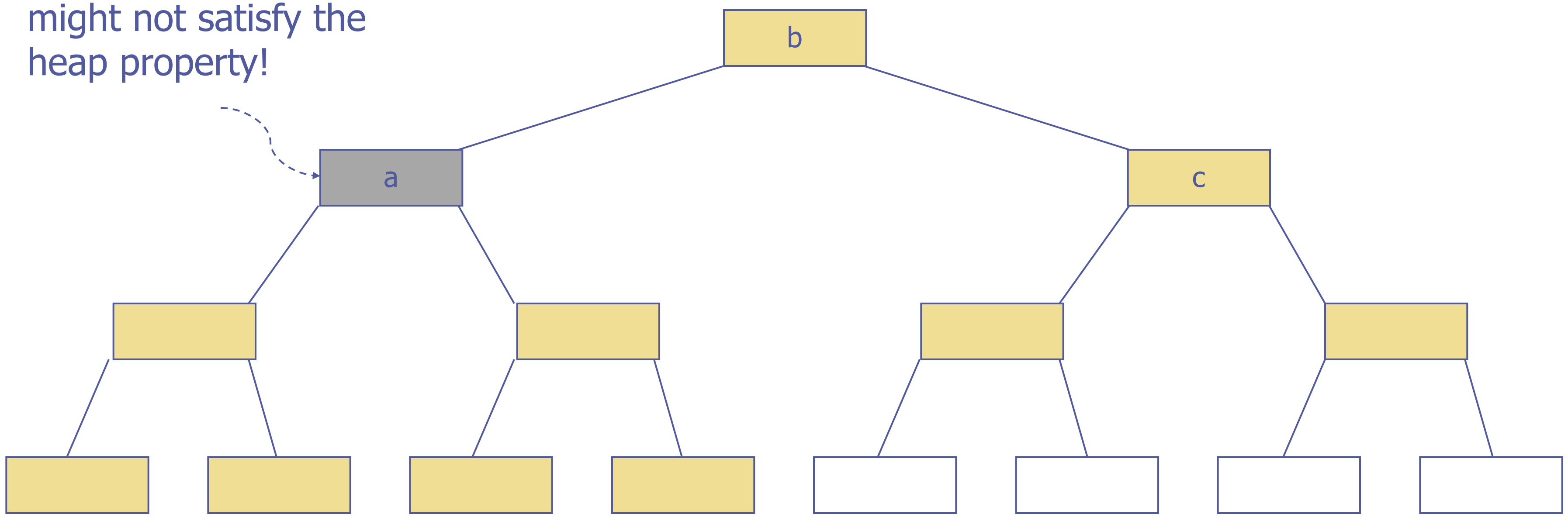


Otherwise, suppose b>a and b>c.

Then we can swap a with b …

Removing maximal element:

But now this node might not satisfy the heap property!



Repeat until we're done.

Takes $O(\log n)$ time: we have to worry about the nodes on only one path in the tree.

Implementation:

```
heapExtractMax() {
    size ← size - 1
    int max ← h[1];
    h[1]    ← h[size];
    heapify(1);
    return max;
}
```

Implementation:

```
heapify(i) {
   l ← left(i); r ← right(i);
   largest ← i;
   if (l≤size) {
      if (h[l]>h[i])
         largest ← l;
      if (r≤size ∧ h[r]>h[largest])
         largest ← r;
   }
   if (largest≠i) {
      h.swap(i, largest);
      heapify(largest);
   }
}
```

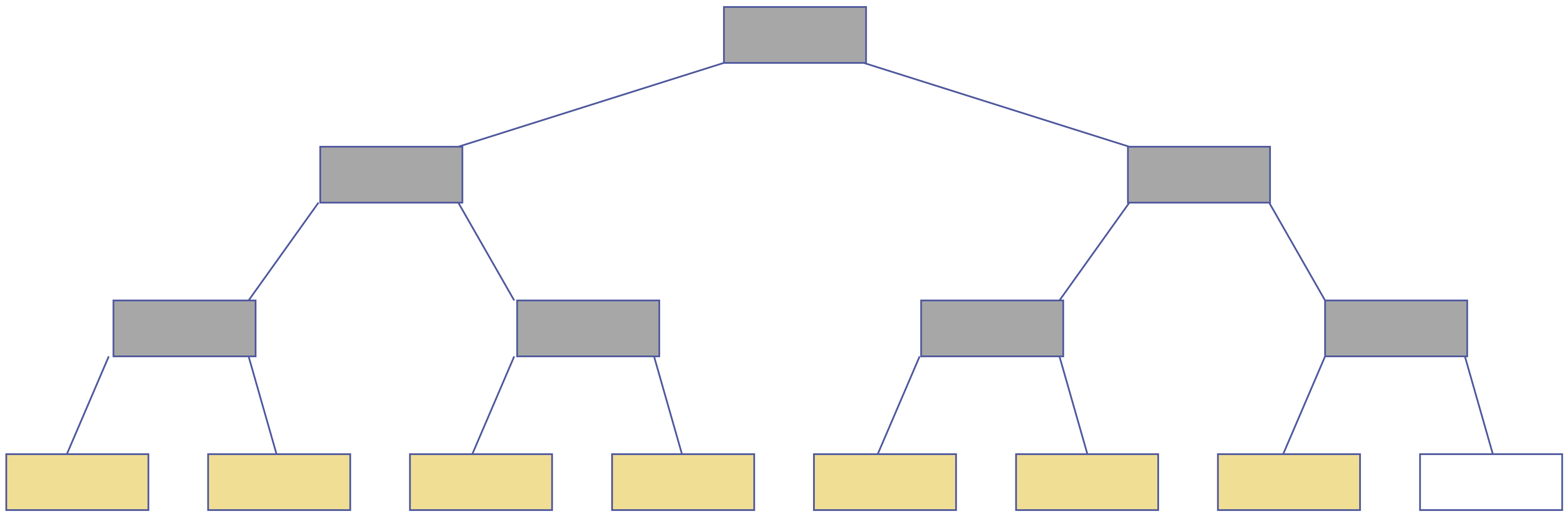
Priority queues:

- A priority queue is a variation on the queue data structure with a "highest-priority first out" policy.

- More concretely, a priority queue supports operations to:
  - Add an element, and
  - Remove highest priority element.

- Heaps can be used as an implementation of priority queues—one of the most common uses of heaps in practice.

Building a heap:



Suppose we start with an arbitrary array of values.

Run `heapify` on each of the interior nodes, starting at the bottom, and working back to the root. Now we have a heap!

Implementation:

```
buildHeap() {
    size ← h.length;
    for i from size/2 downto 1 do {
        heapify(i);
    }
}
```

Complexity:

- To a first approximation: there are $O(n)$ calls to `heapify`, and $O(\log n)$ steps for each such call, giving a total:

$$O(n \log n)$$

Complexity:

- To a first approximation: there are $\mathrm{O}(n)$ calls to `heapify`, and $\mathrm{O}(\log n)$ steps for each such call, giving a total:

$$\mathrm{O}(n \log n)$$

  - But we can do better than this!
  - Many of the calls to `heapify` involve trees with heights that are $< \log n$.

- The total cost of buildHeap is:

$$\sum_{h=0}^{\lceil \lg n \rceil} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h)$$

- Simplifying:

$$\sum_{h=0}^{\lceil \lg n \rceil} \frac{n}{2^{h+1}} O(h) = O\left(n \sum_{h=0}^{\lceil \lg n \rceil} \frac{h}{2^{h+1}} \right)$$

$$\leq O\left(n \sum_{h=0}^{\infty} \frac{h}{2^{h}} \right) = O(n)$$

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Spanning Trees

Spanning Trees

- If e is a minimum-weight edge in a connected graph, then e must be an edge in <u>at least one</u> minimum spanning tree

  - True or False?

Spanning Trees

- If e is a minimum-weight edge in a connected graph, then e must be an edge in all minimum spanning trees of the graph
  - True or False?

Spanning Trees

- If every edge in a connected graph G has a distinct weight, then G must have exactly one minimum spanning tree
  - True or False?

Kruskal’s Algorithm

Building bridges:

- Suppose that we want to link a group of n small islands together with bridges.
- There will be many possible ways to do this, each corresponding to a connected graph, with the islands as vertices and bridges as edges.
- What is the minimum number of bridges that we will need to build?

Spanning trees:

- A spanning tree T of a connected graph G = (V,E) is a subgraph of G that is:
  - connected;
  - acyclic;
  - includes all of V as vertices.

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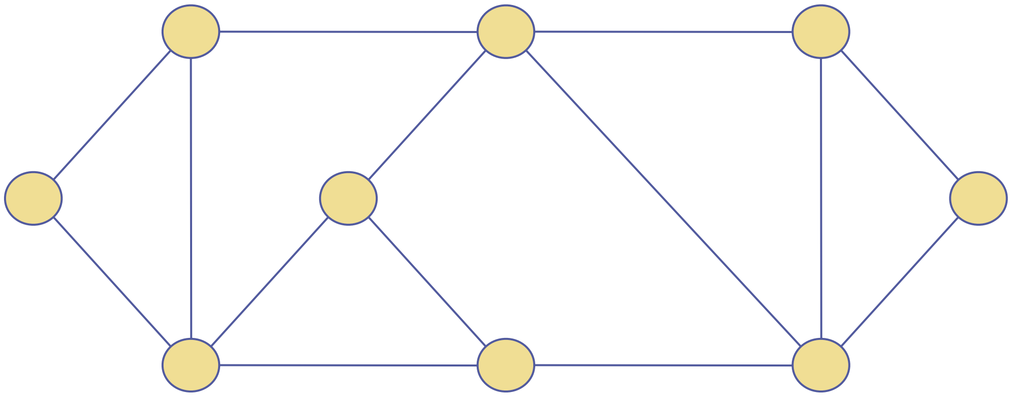

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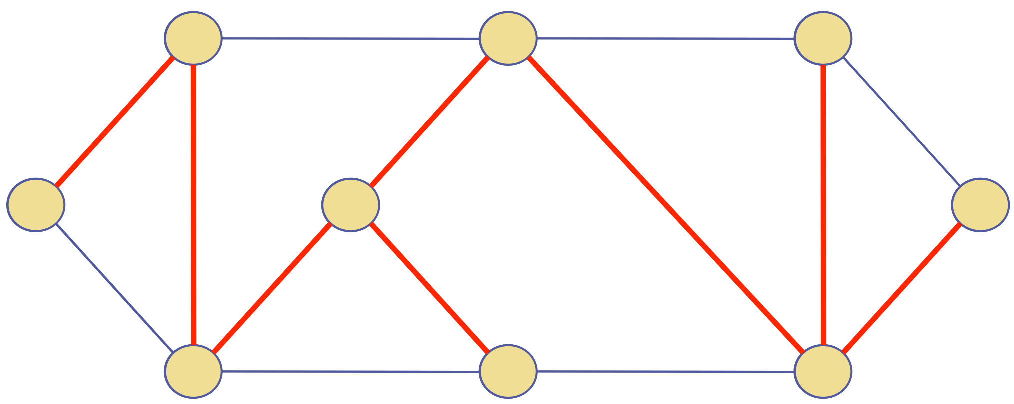

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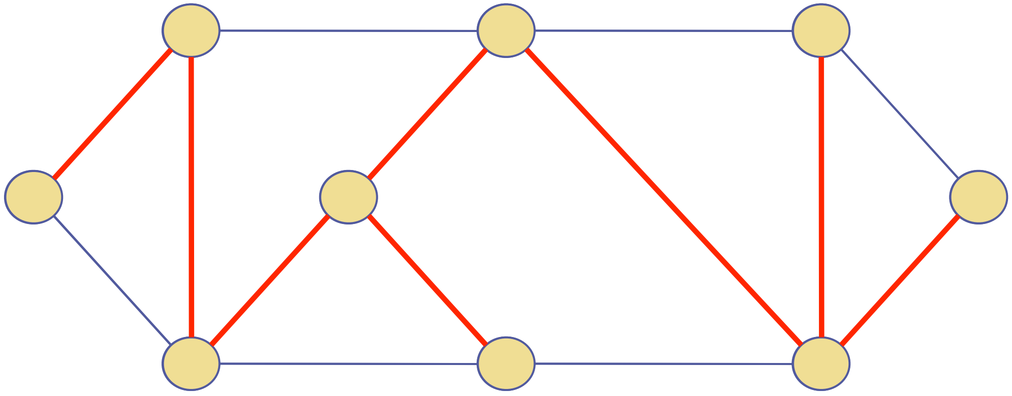


- Any spanning tree has |V|−1 edges.

Growing a forest:

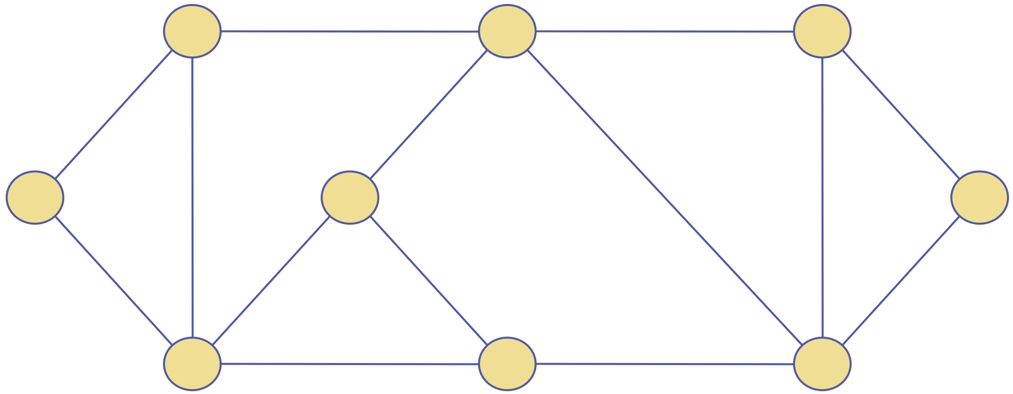
- Find a spanning tree for connected graph G=(V,E):

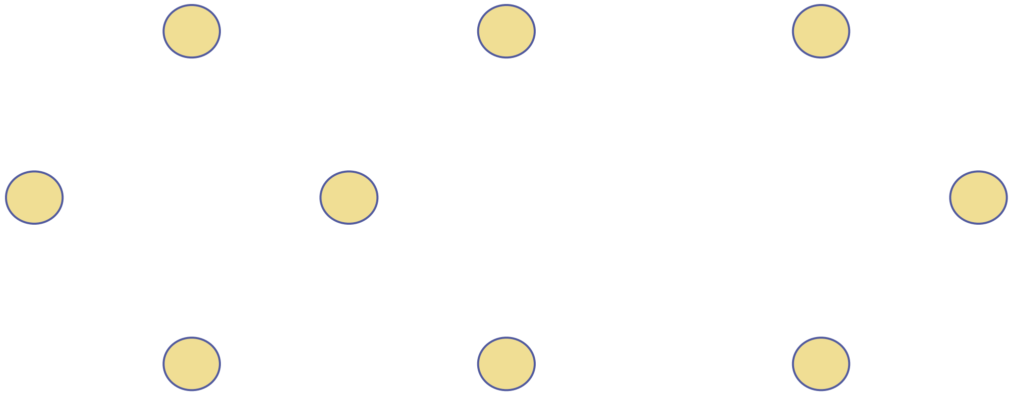
```
partition V into |V| singleton sets of the form {v}.
let E_T be an empty set of edges.
for each edge (u,v) in E:
   let S_u be the set containing u
   let S_v be the set containing v
   if S_u ≠ S_v, then
      replace S_u and S_v with S_u ∪ S_v
      add (u,v) to E_T
return (V, E_T) as the spanning tree
```

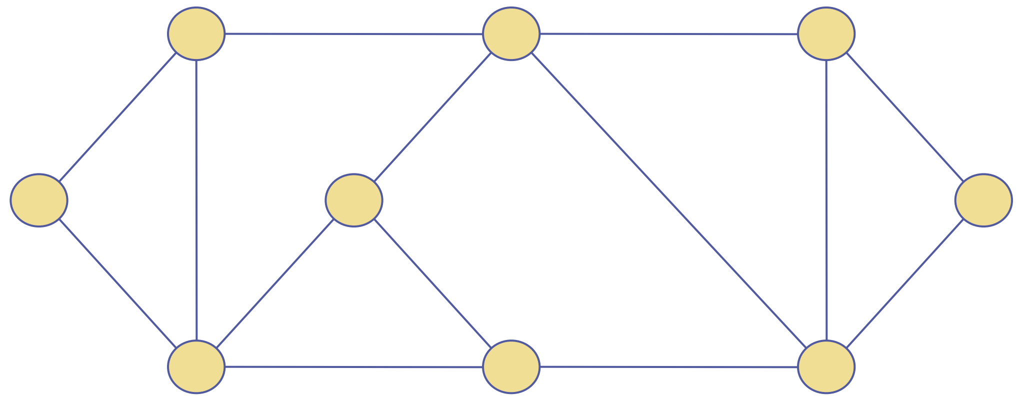
- We start with |V| sets …

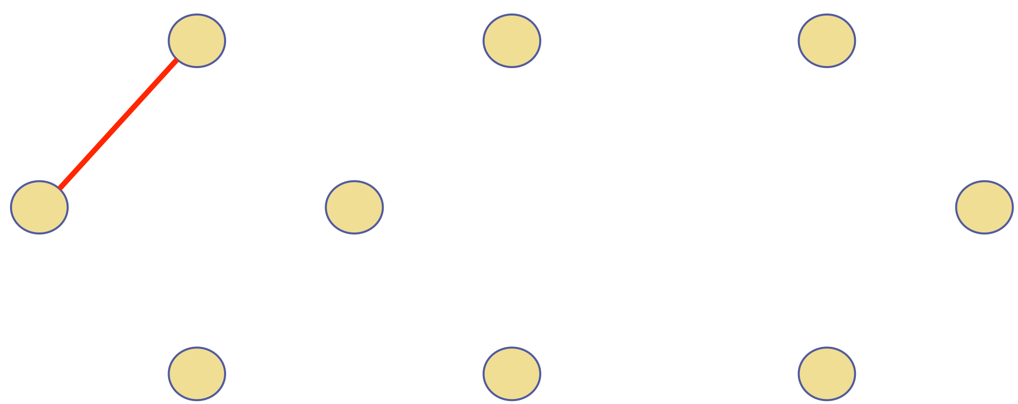
  … we end up with just 1 set.

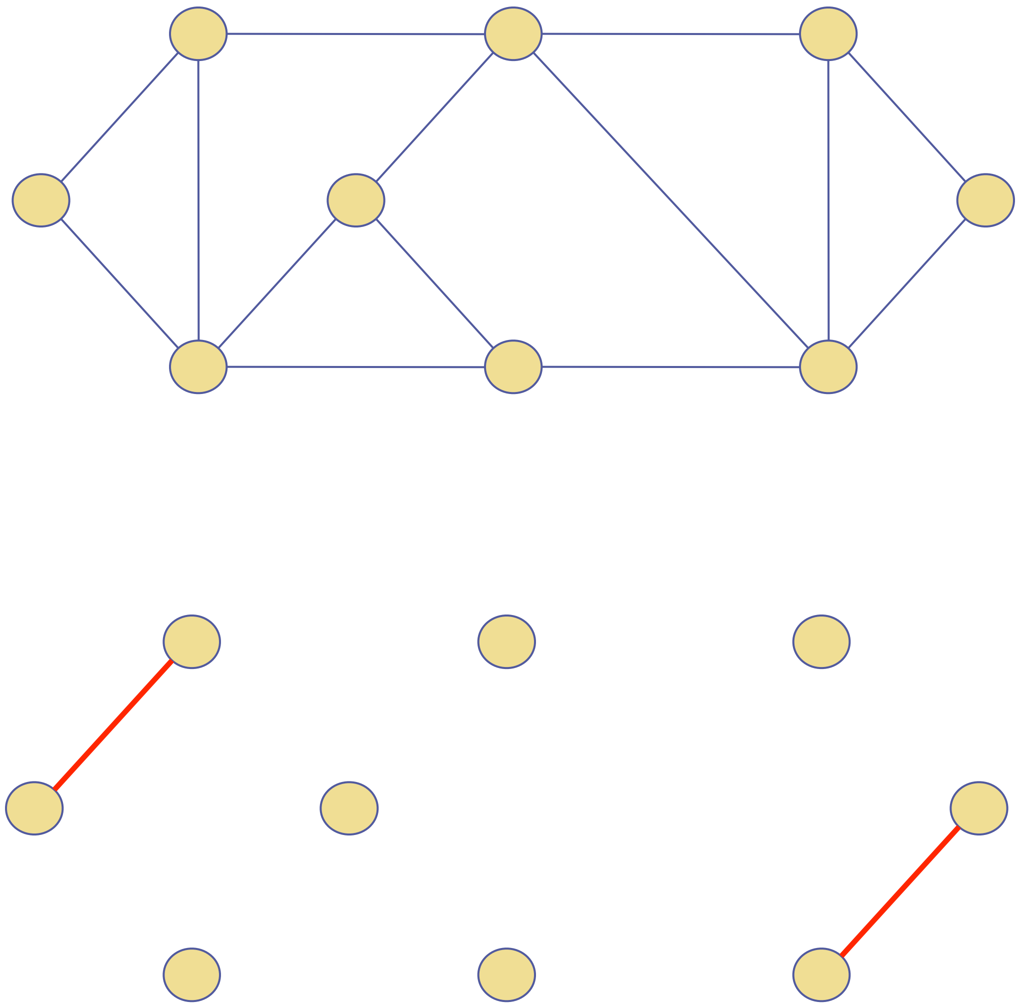
- Hence: |V|−1 unions, |V|−1 edges added to E_T.

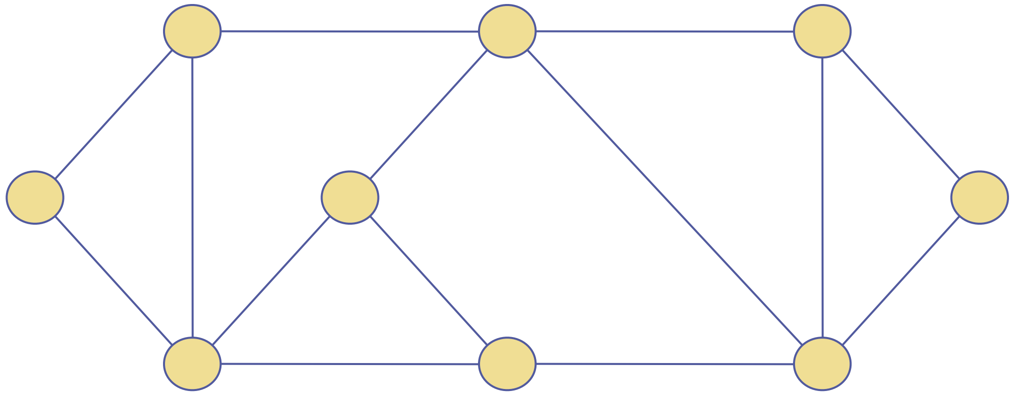


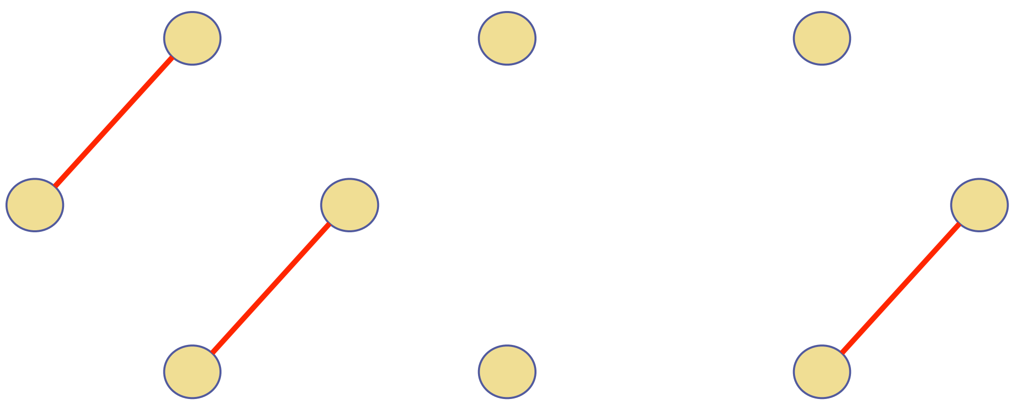


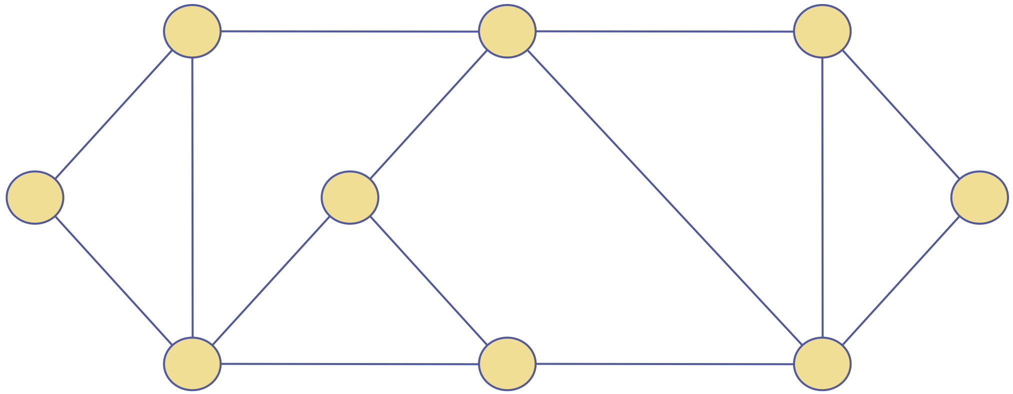


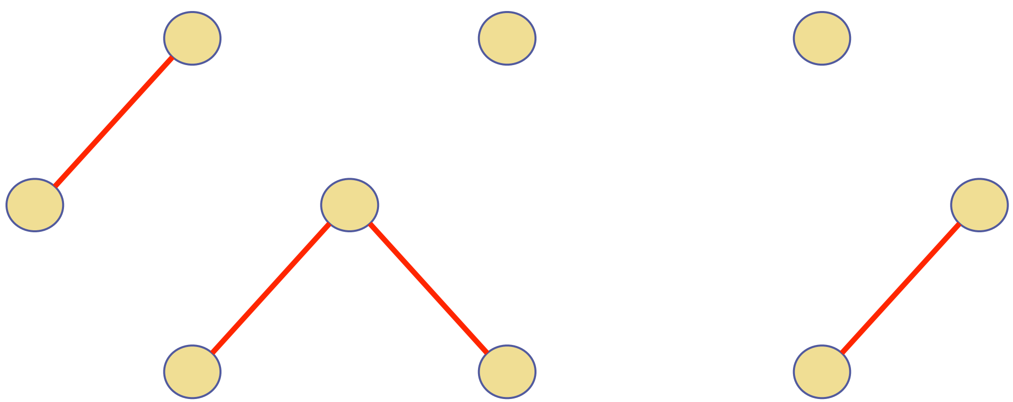


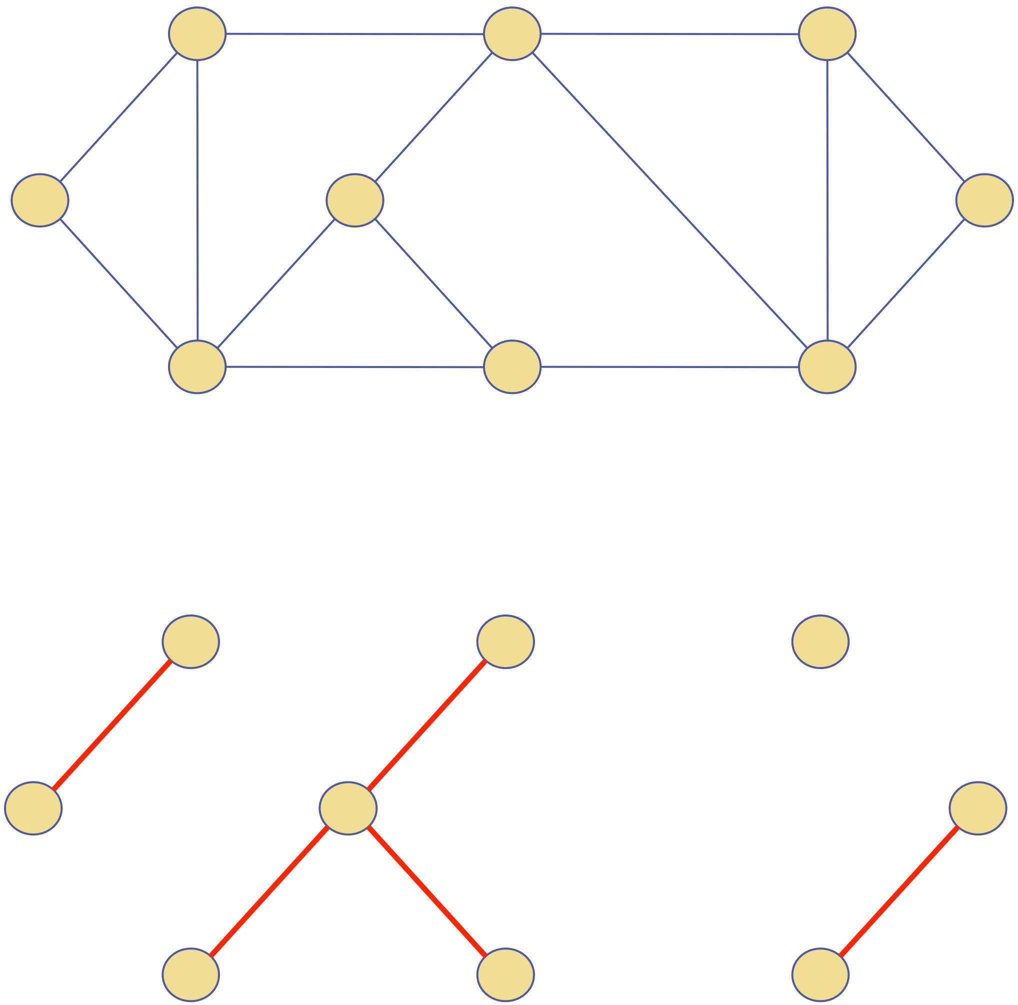


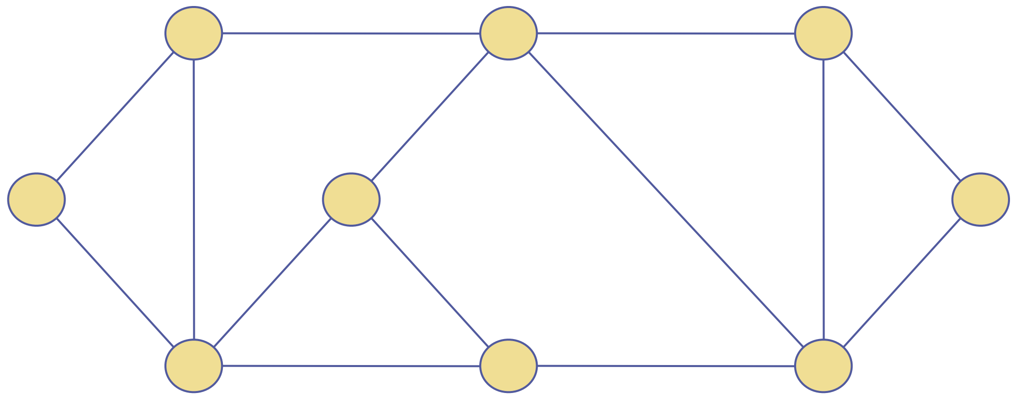


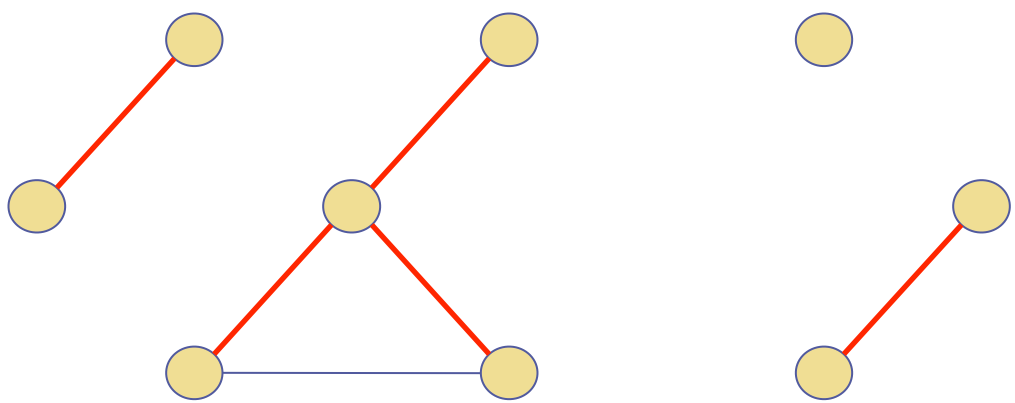


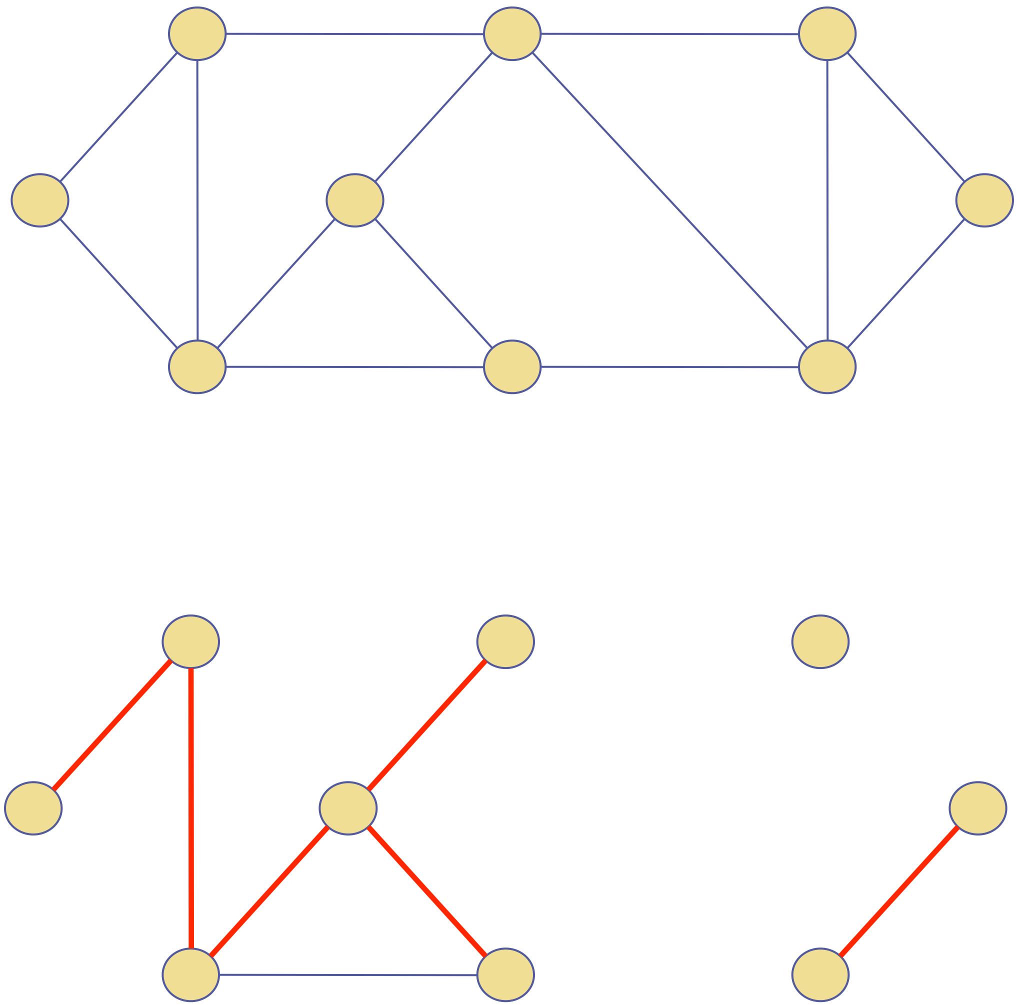


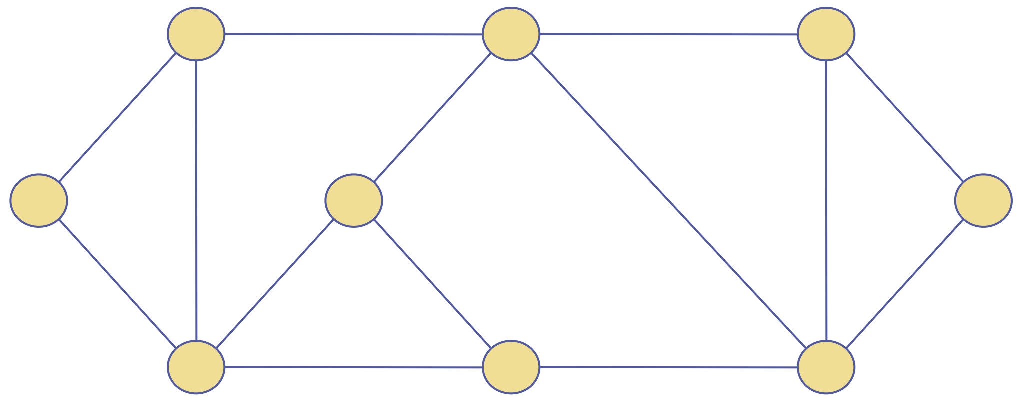


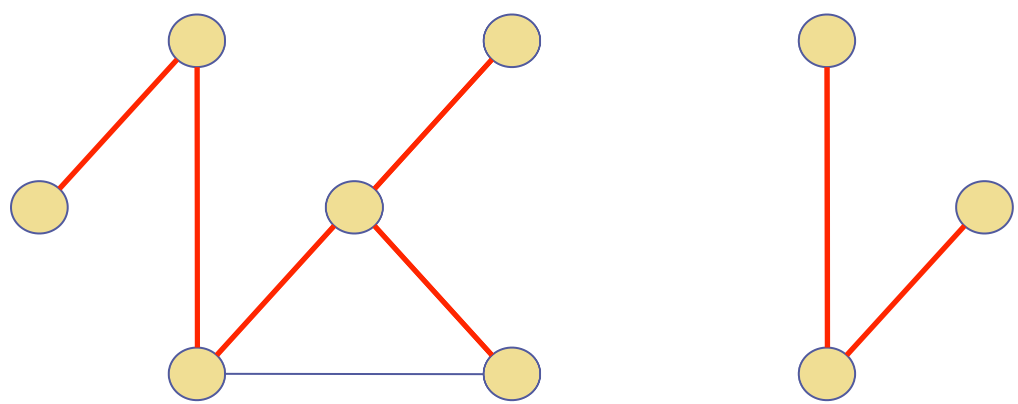


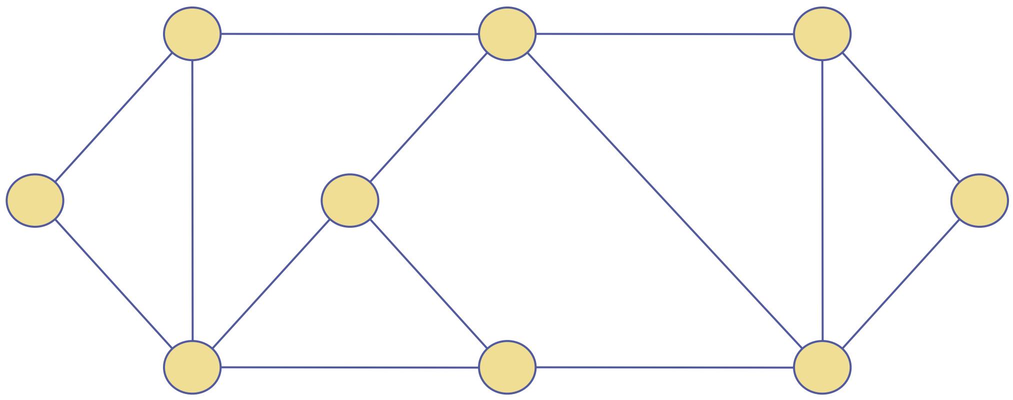


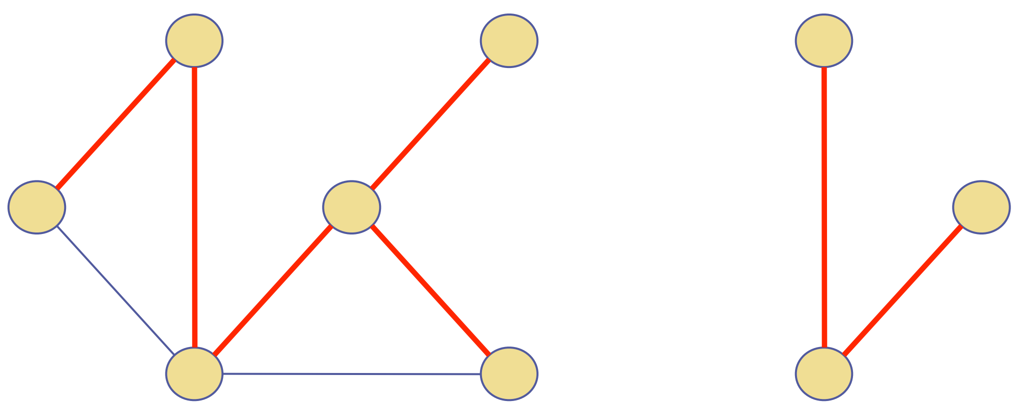


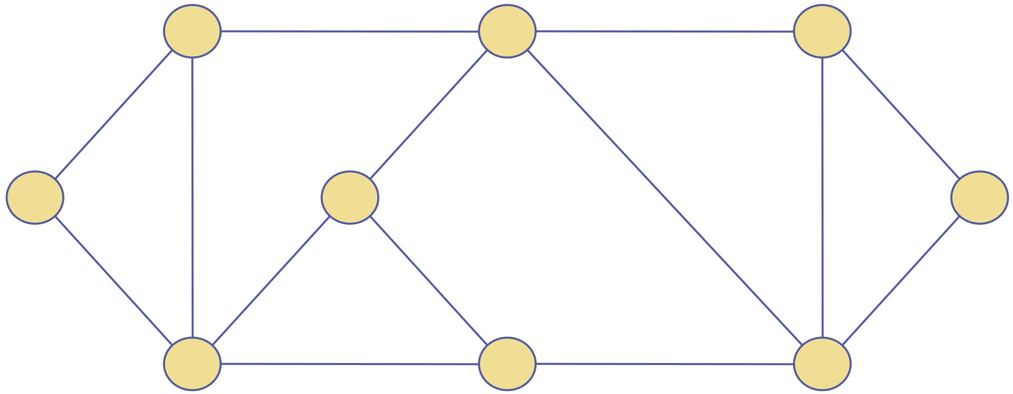


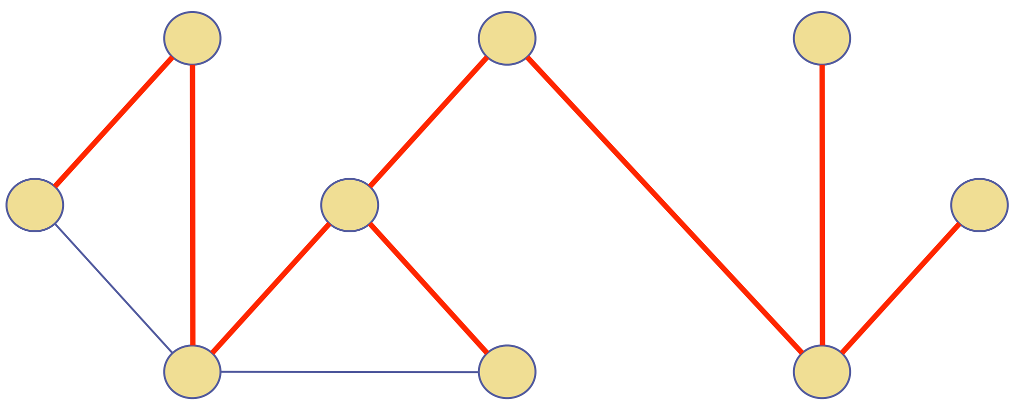


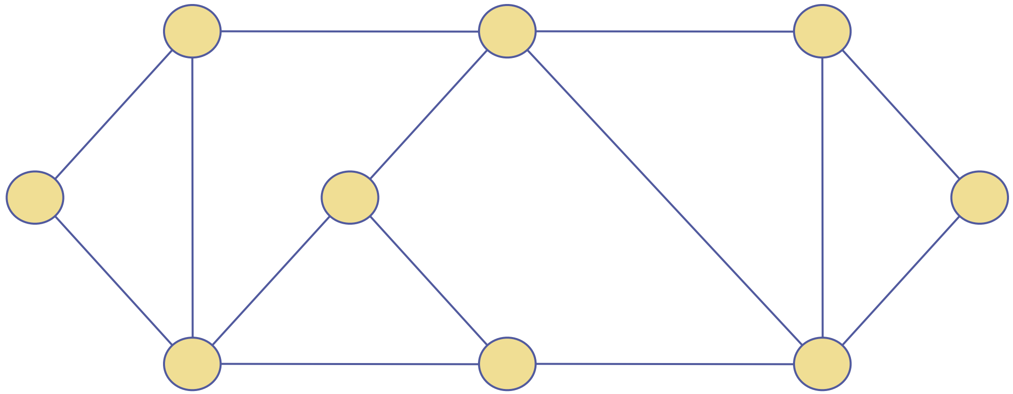


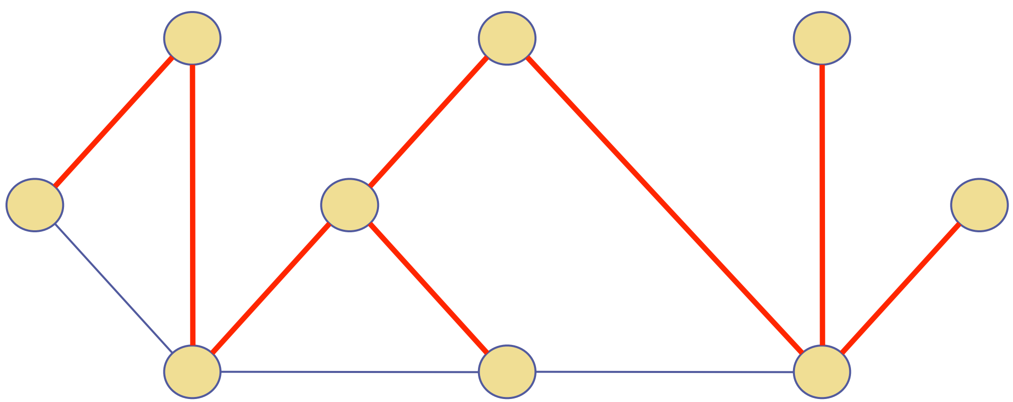


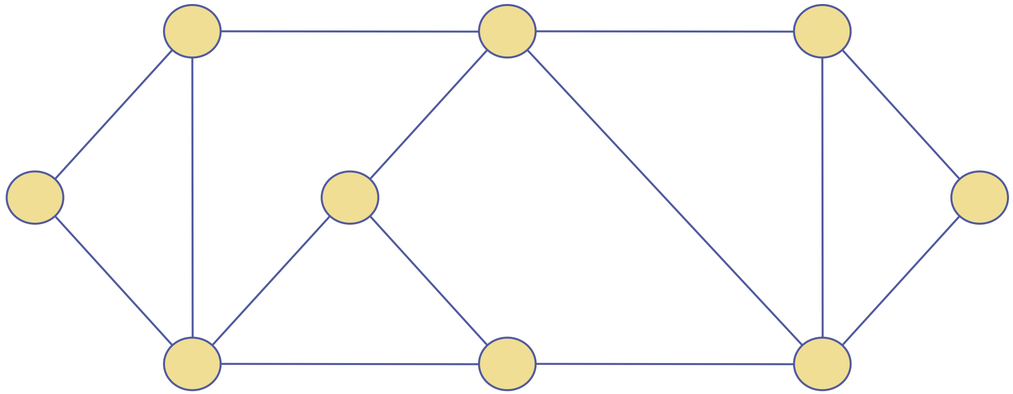


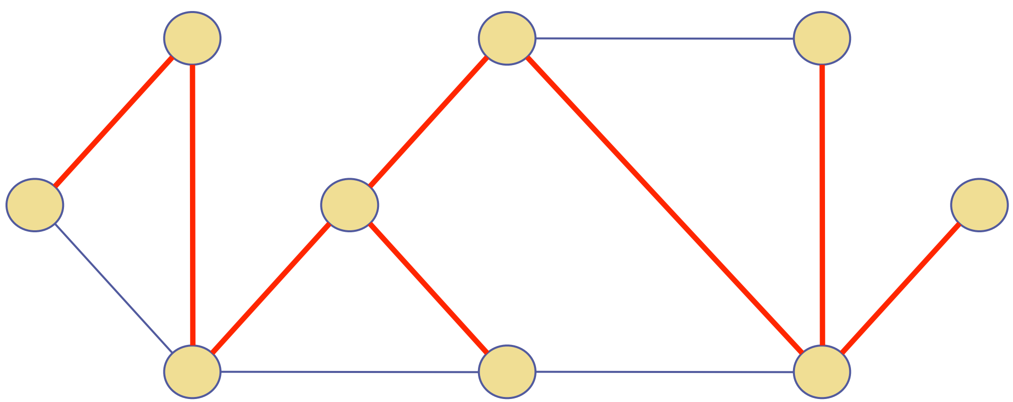


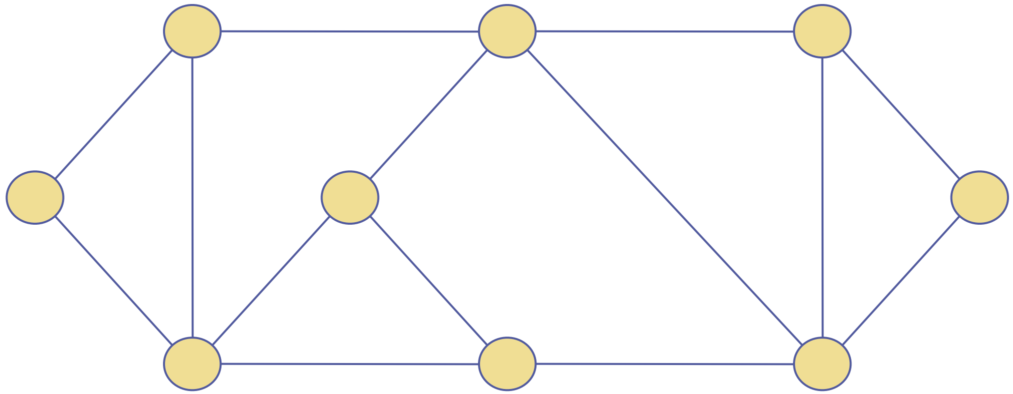


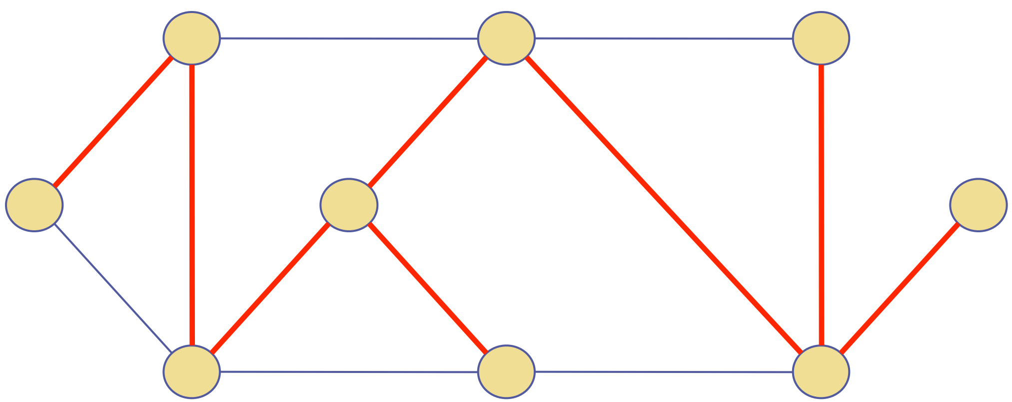


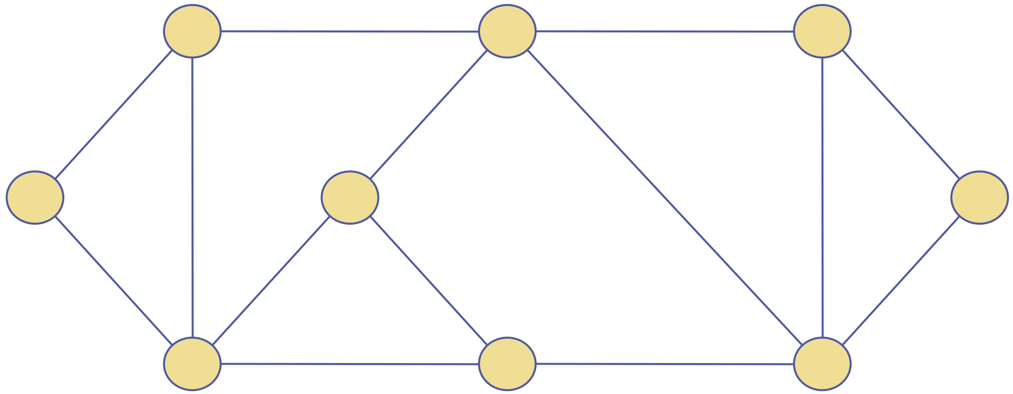


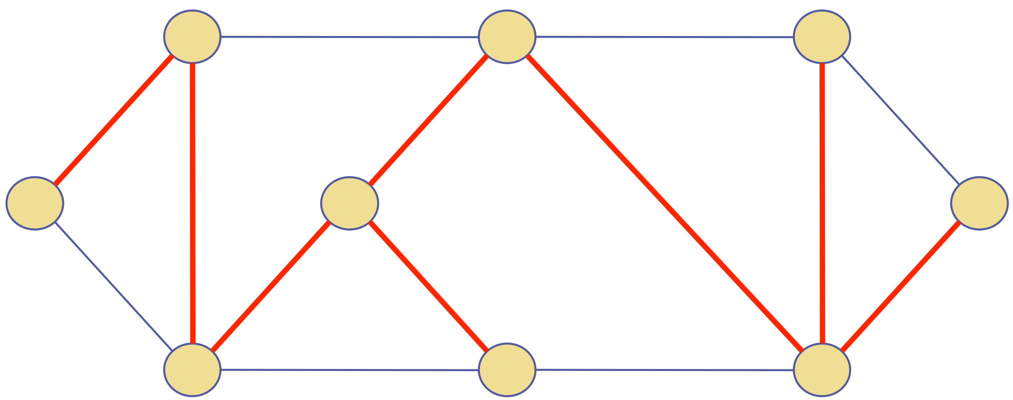












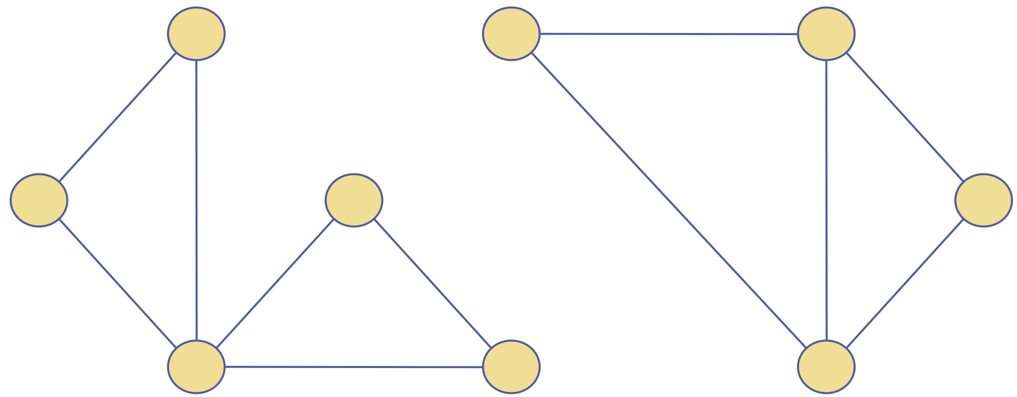
Calculating connected components:

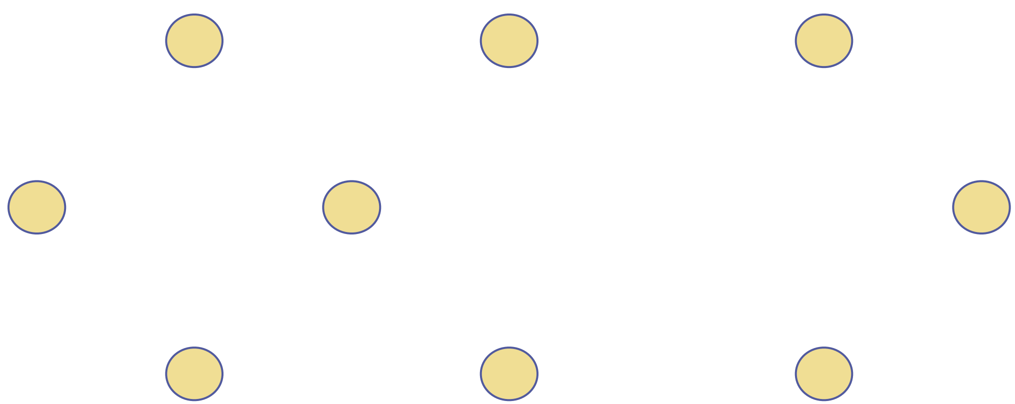
- What if G=(V,E) is not connected?

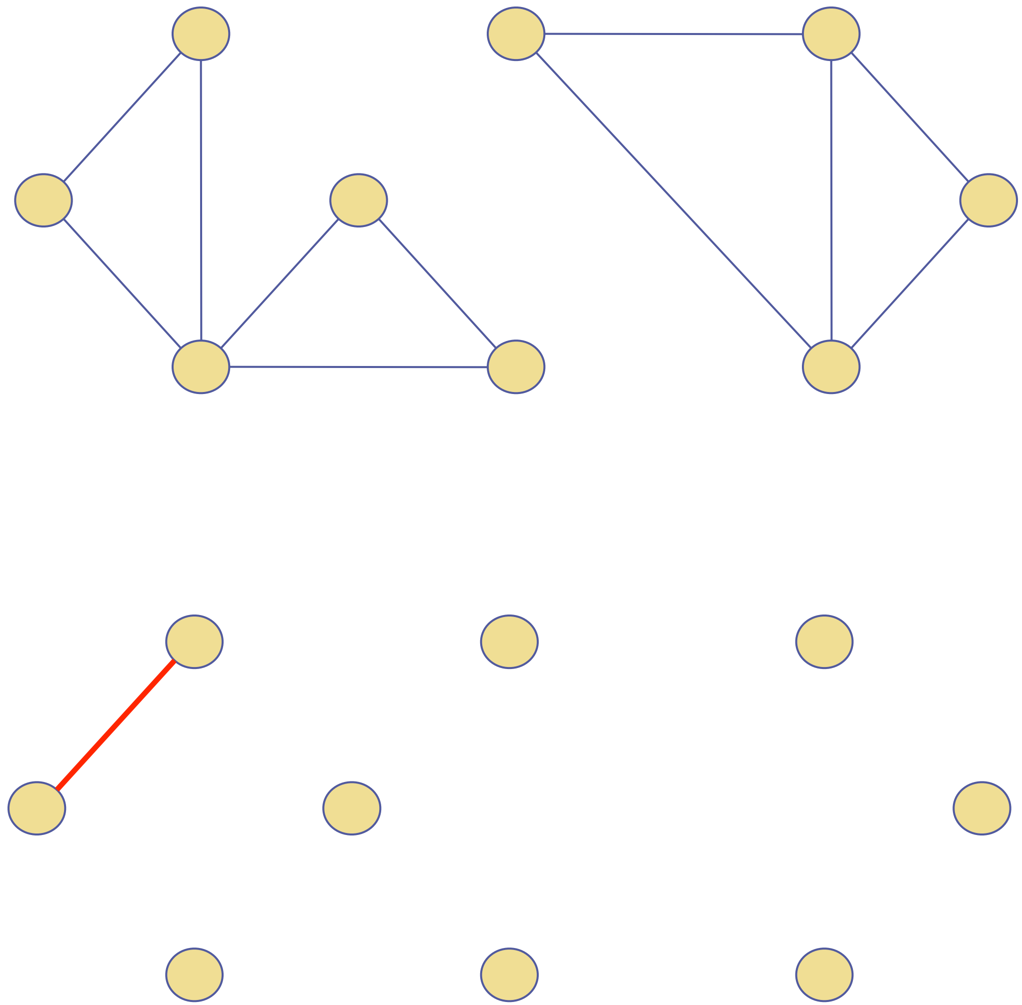
```
partition V into |V| singleton sets of the form {v}.
let E_T be an empty set of edges.
for each edge (u,v) in E:
   let S_u be the set containing u
   let S_v be the set containing v
   if S_u ≠ S_v, then
      replace S_u and S_v with S_u ∪ S_v
      add (u,v) to E_T
```

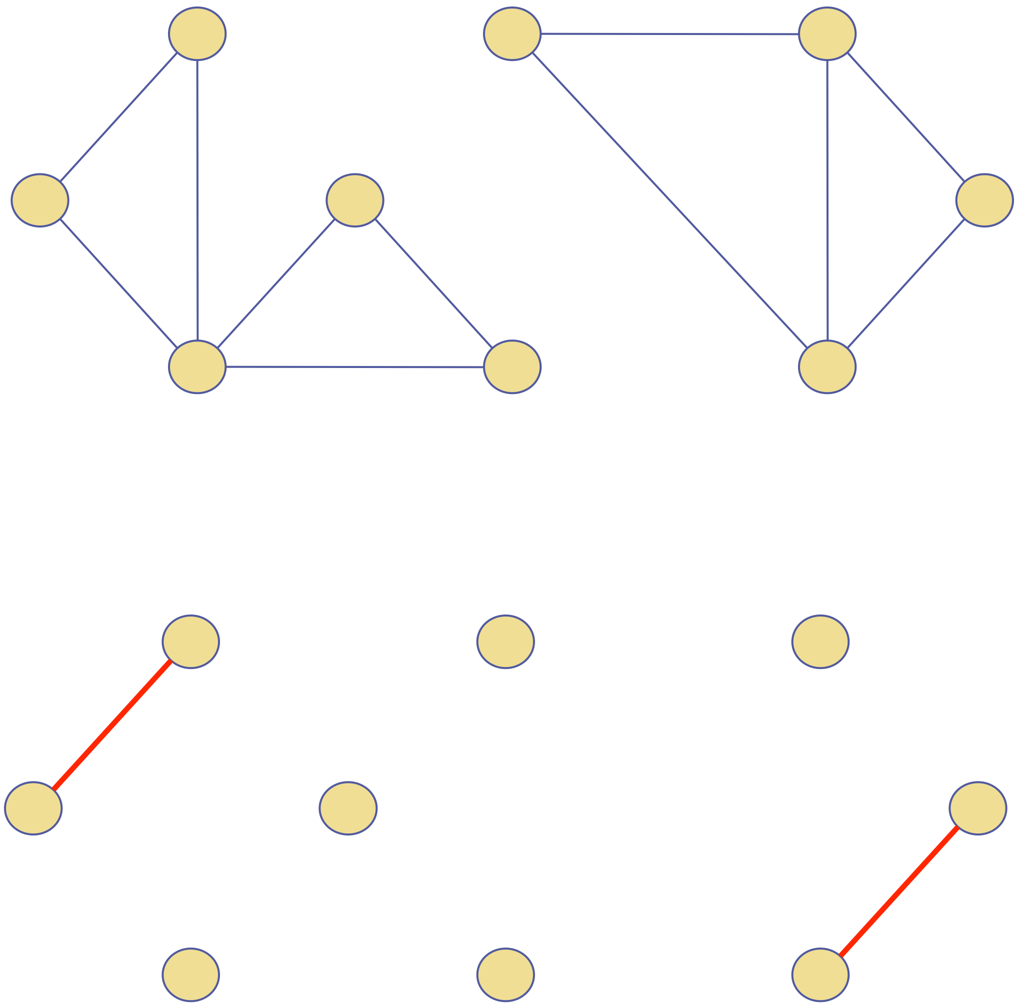
NB: exactly the same algorithm as before, but repeated for convenience!

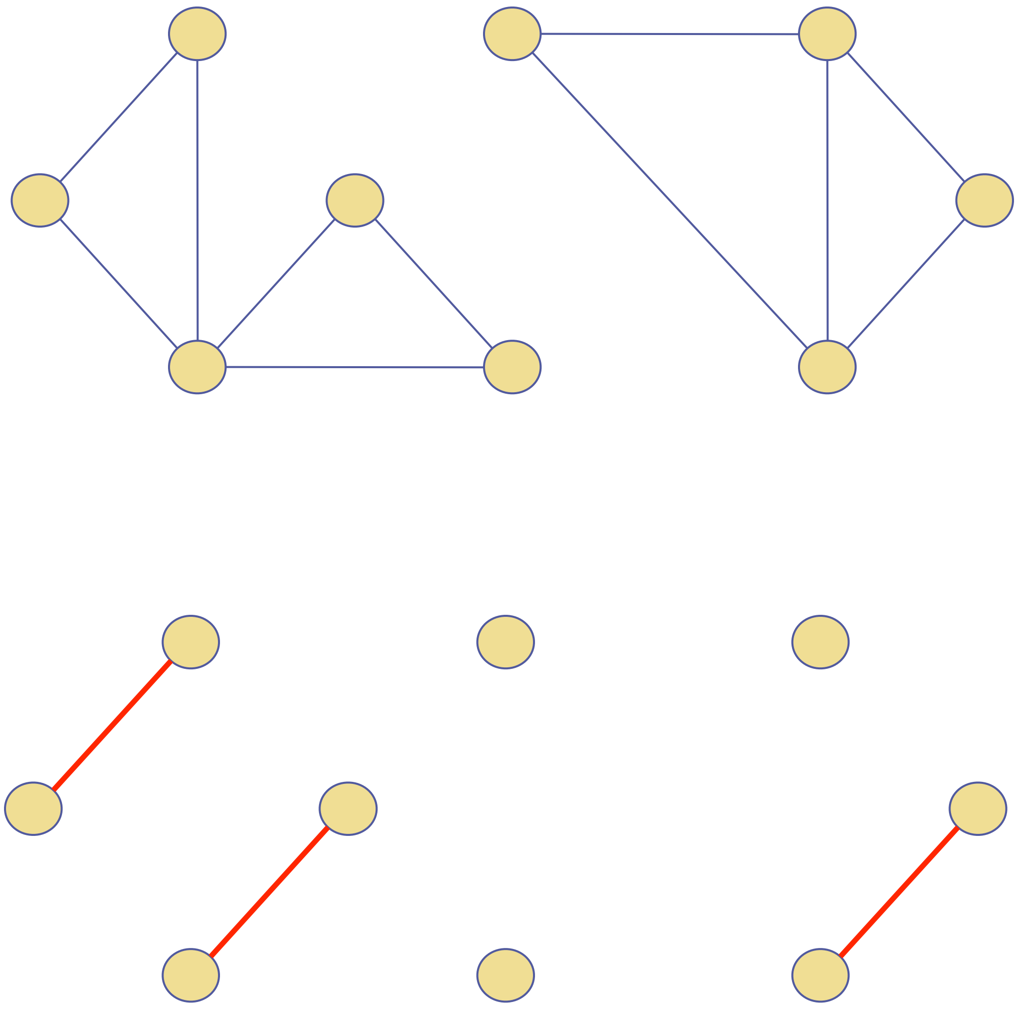
- We end up with c distinct sets S_u, where c is the number of connected components of G;
- E_T is a spanning forest for G, with $|V|-c$ edges.

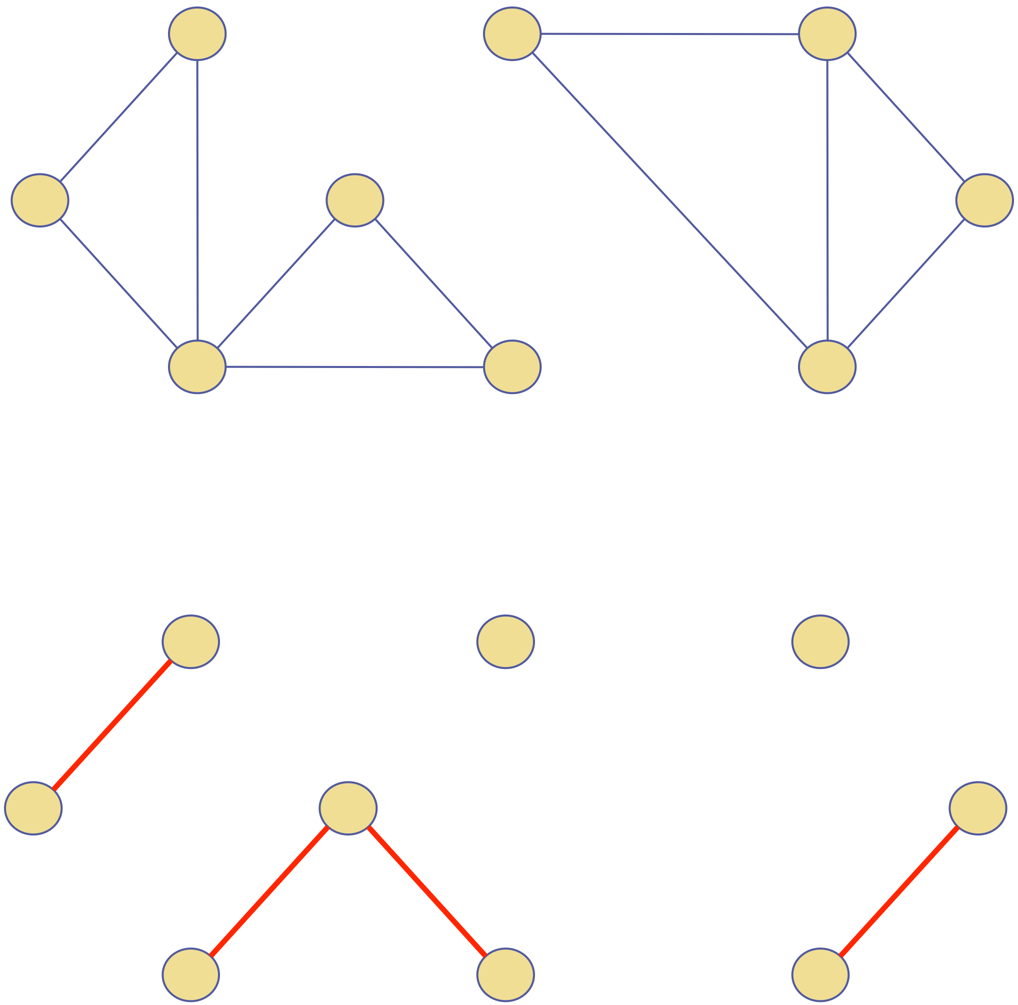


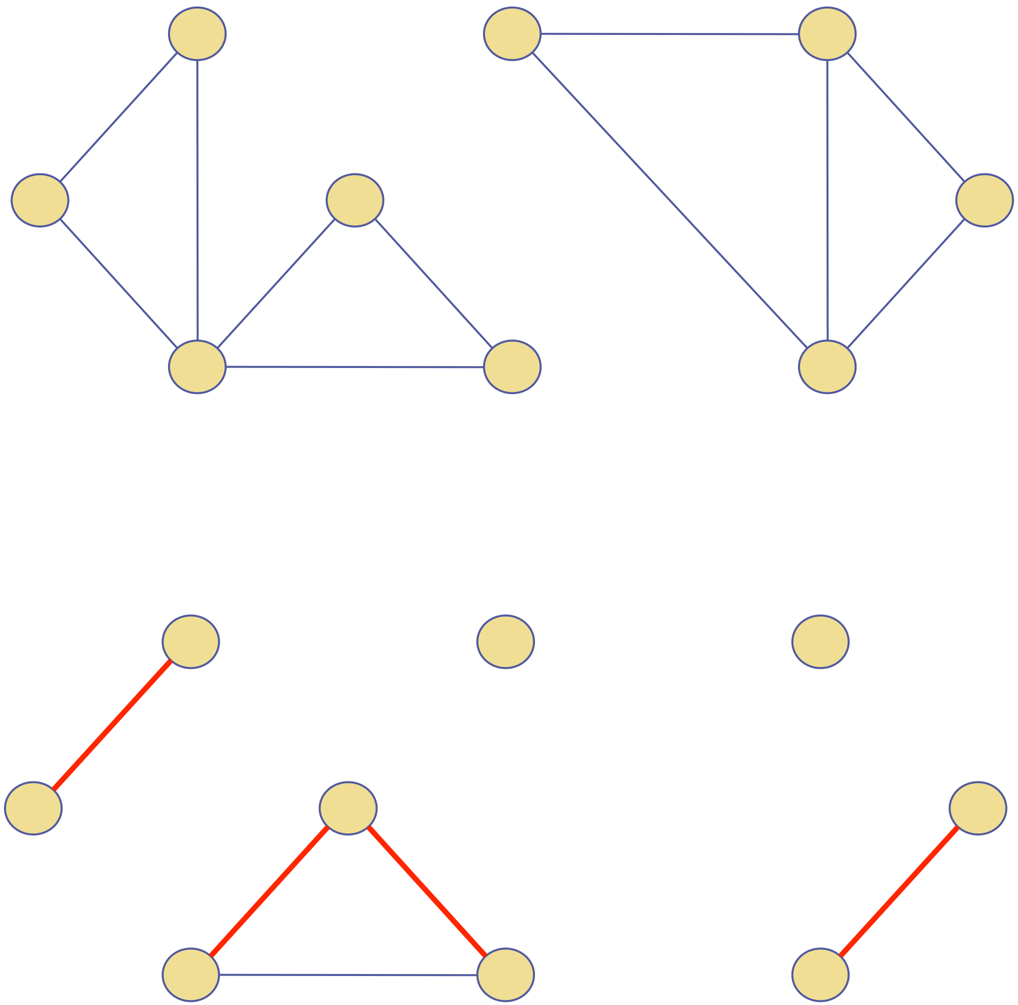


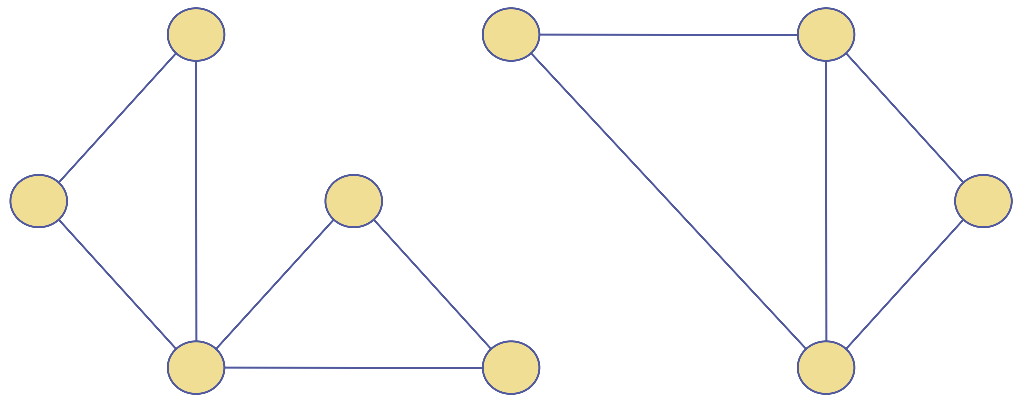


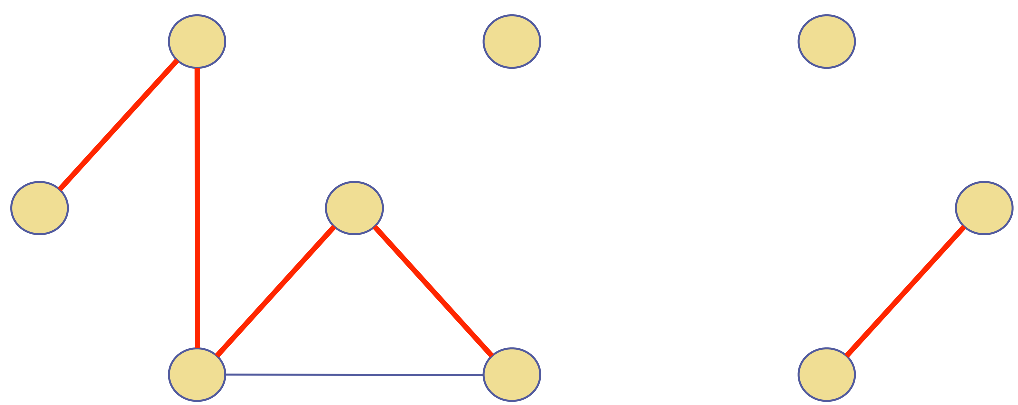


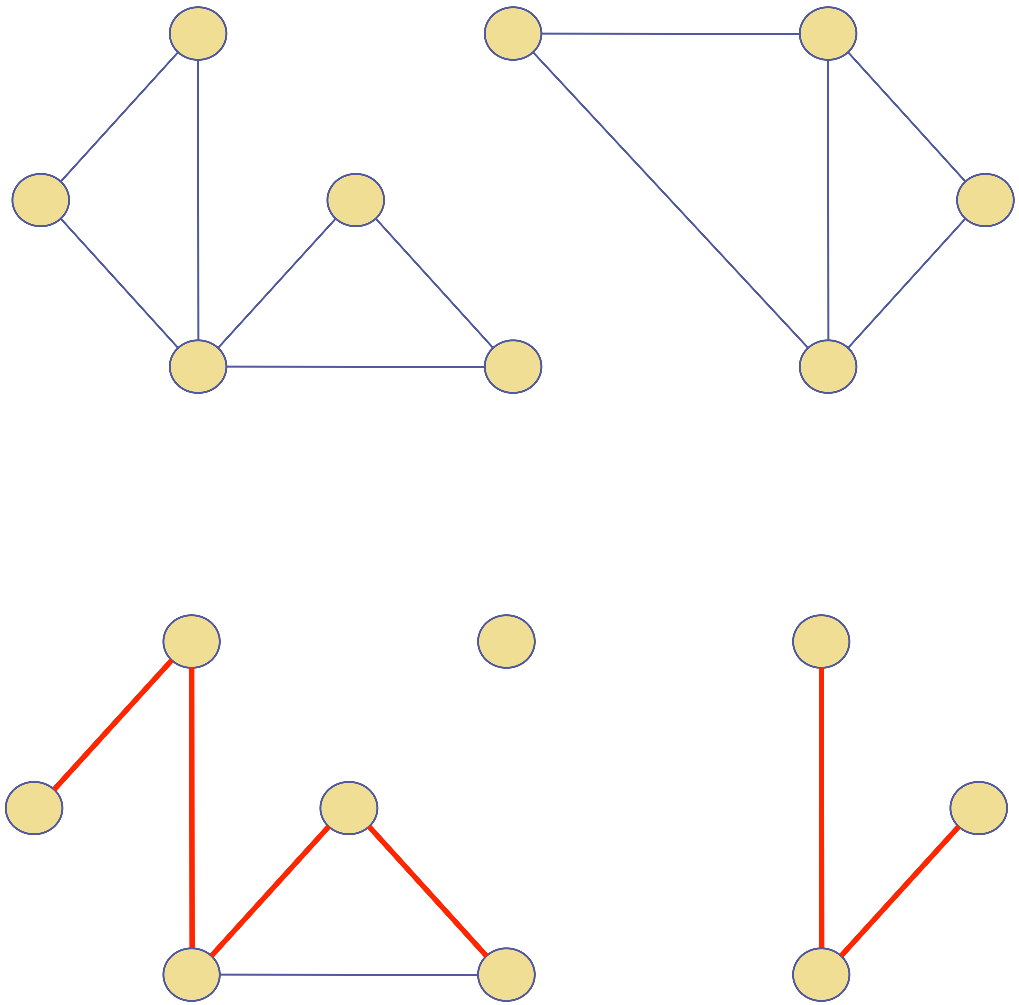


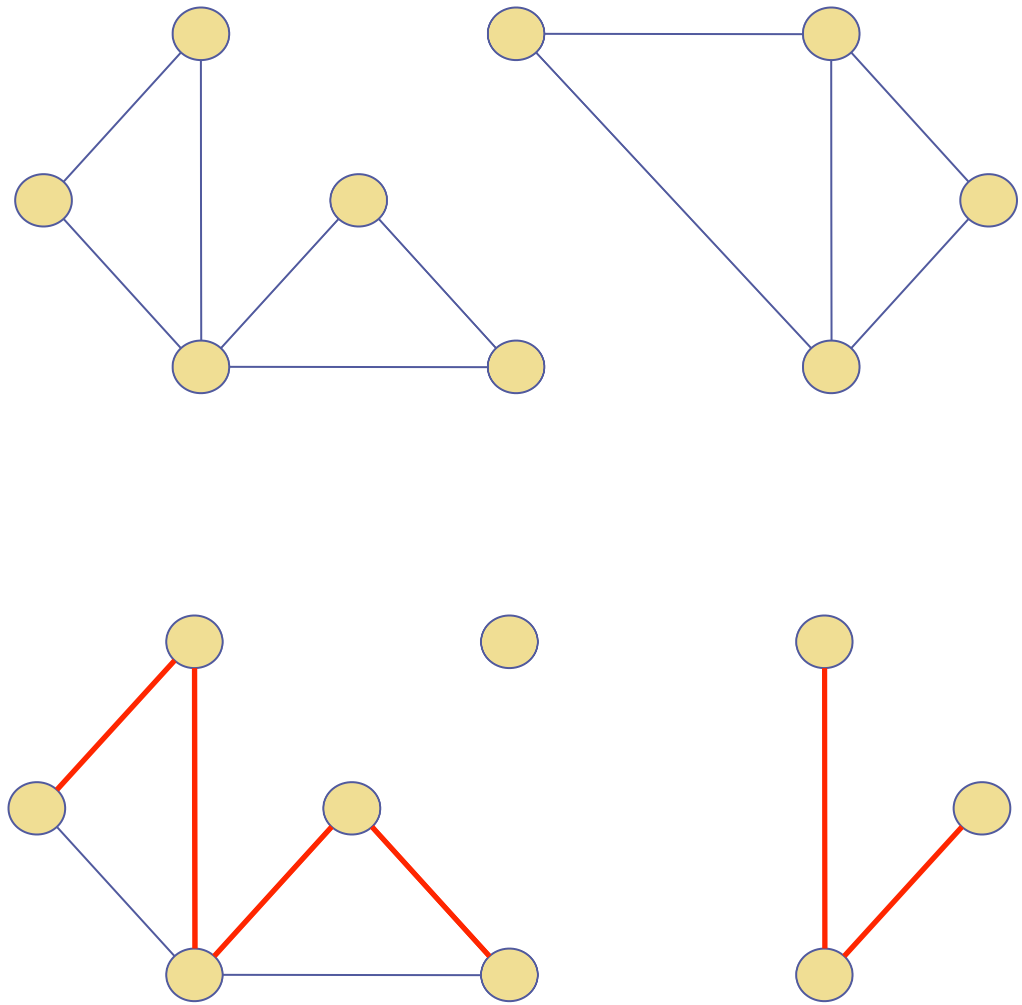


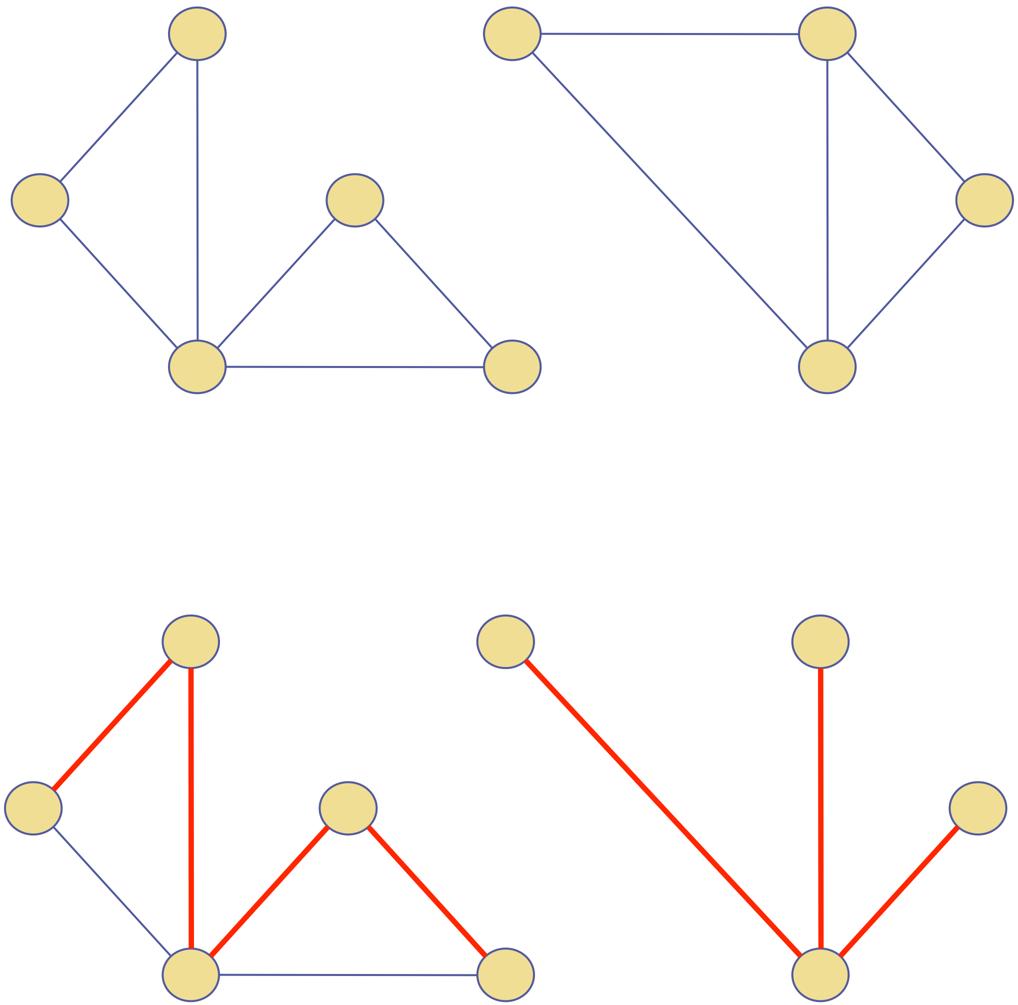


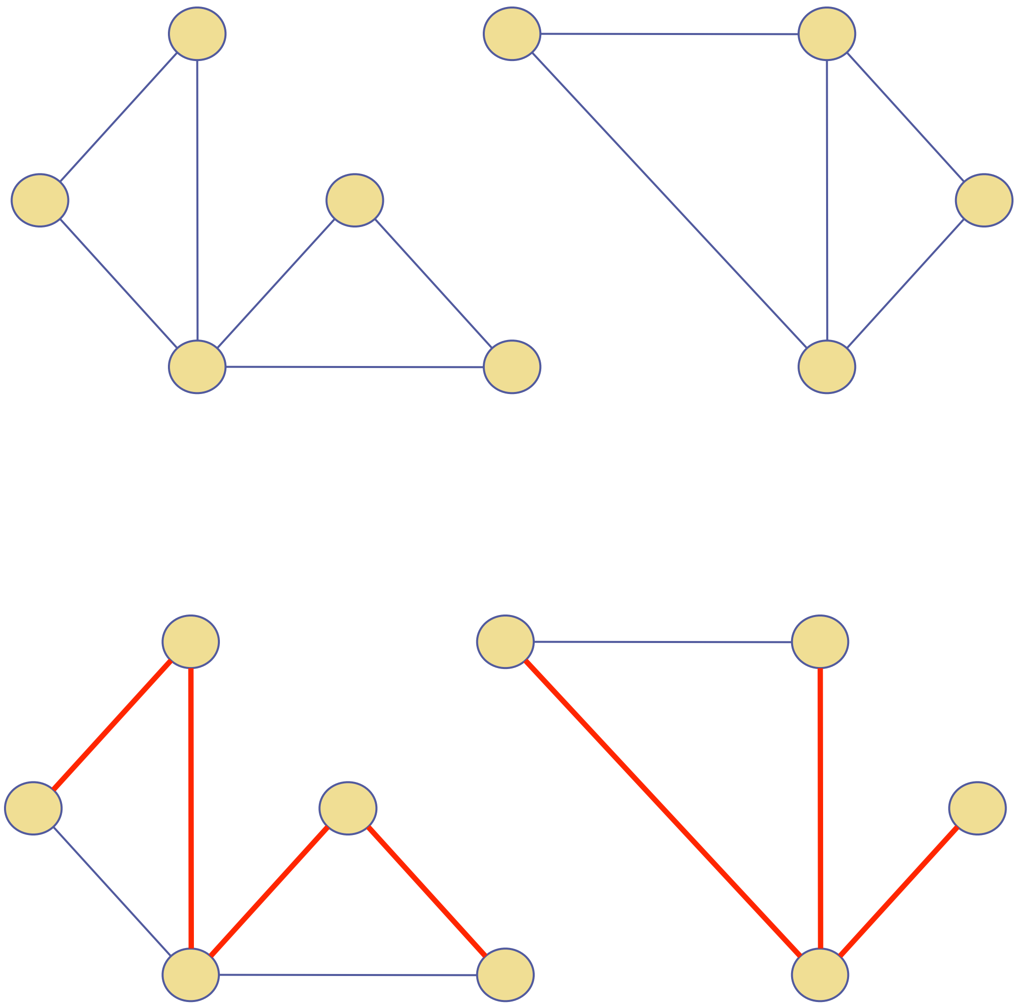


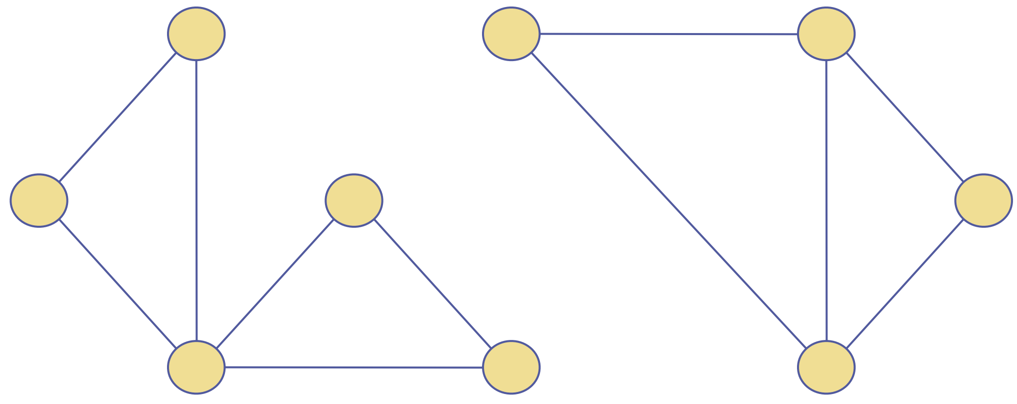


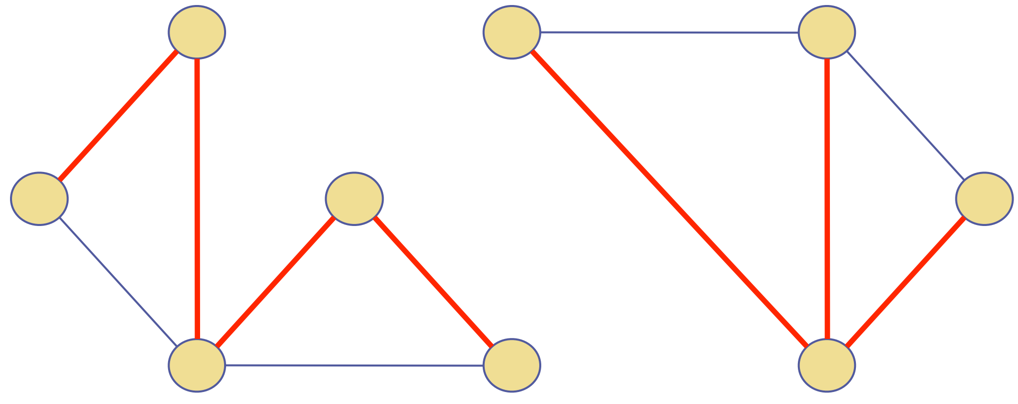






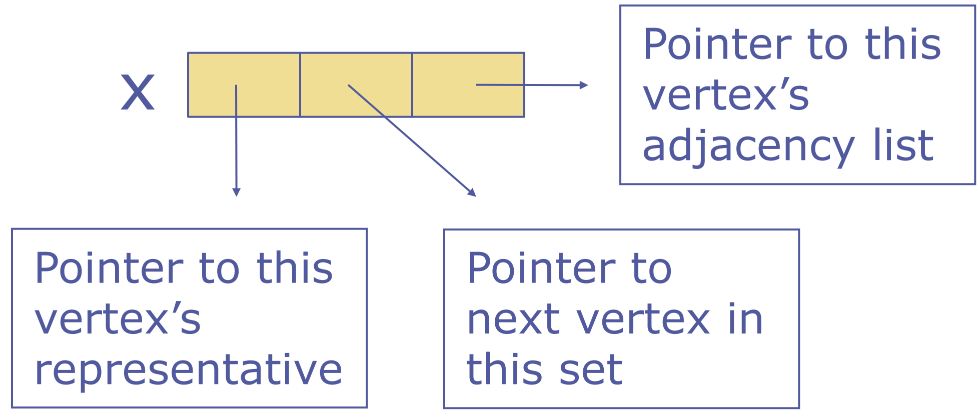




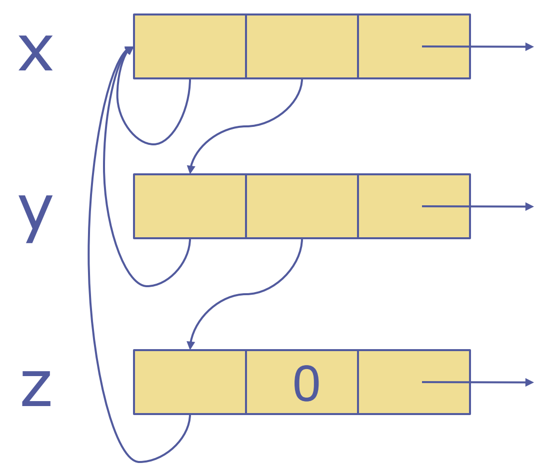


Union-find:

- The operations we need are:
  - Make a singleton set;
  - Test if two sets are equal;
  - Union two sets together.

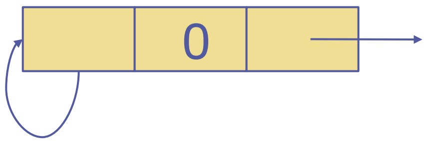


- There is a simple data structure that we can use to implement these operations.

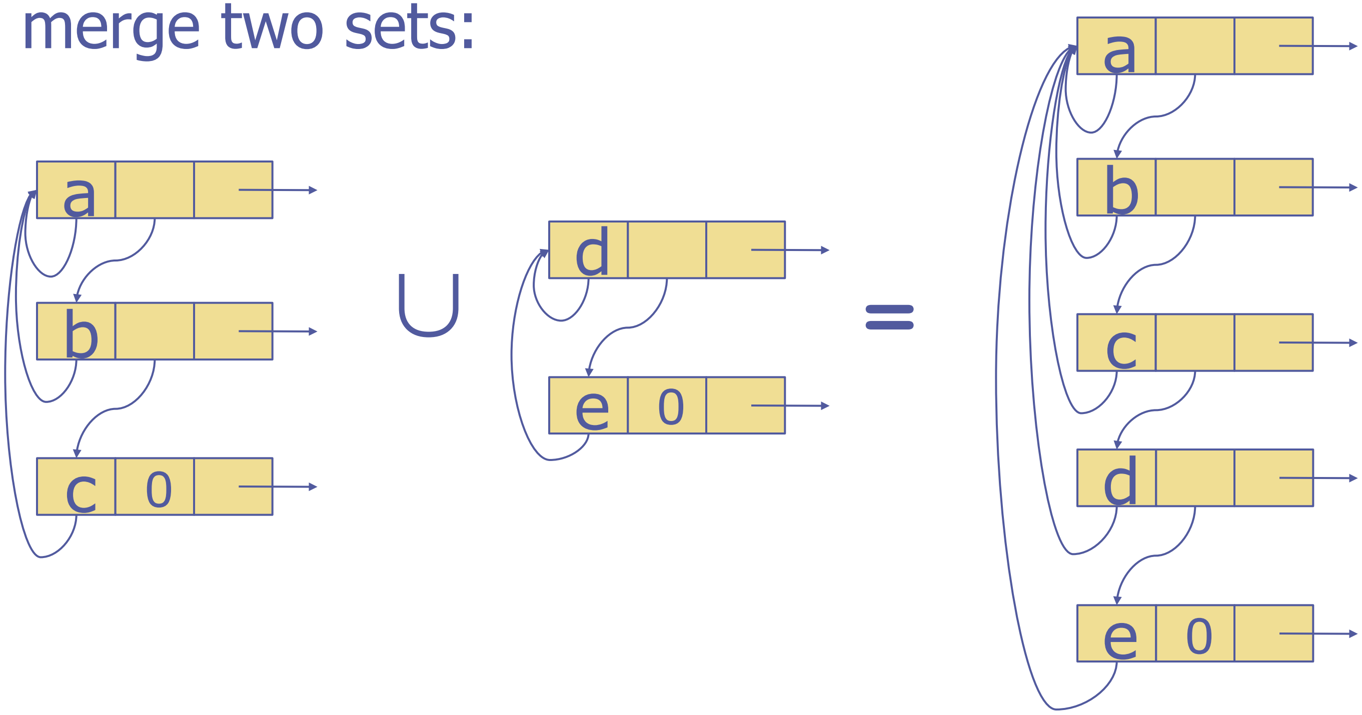


Implementation:

- To make a singleton set:

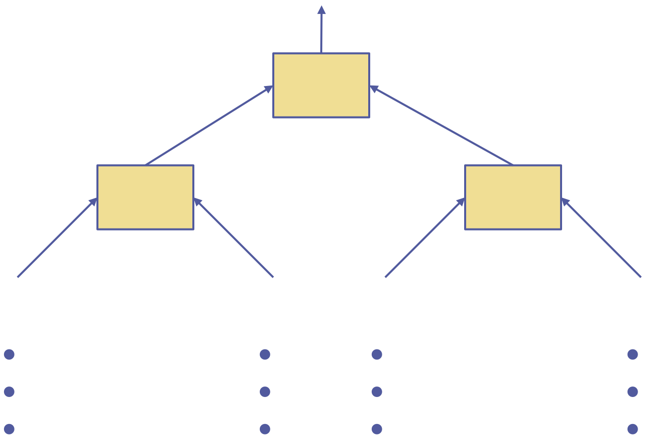


- To test if two sets are the same:
  - Test if the representatives are the same.
- To merge two sets:



Complexity:

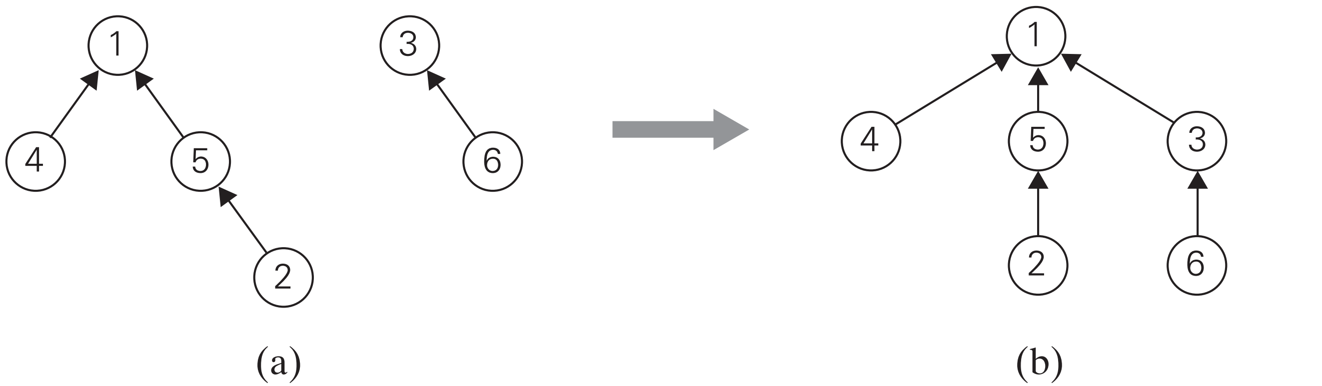
- A sequence of m operations can take $\Theta(m^2)$ time (amortized time per operation is $\Theta(m)$)
- More sophisticated variations are possible, with better complexity bounds.
- A tree based approach
  - Optimization heuristics:
    - Union by rank
    - Path compression



- See Levitin §9.2 or CLRS Chapter 21 for more details.

Quick Union:

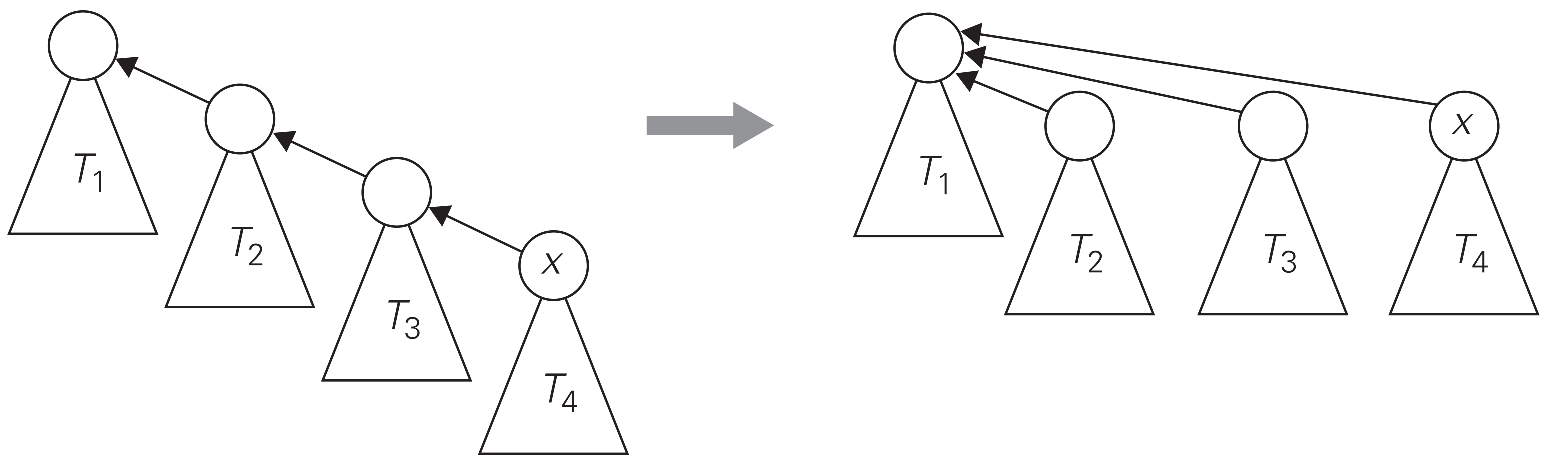
- Uses Tree-based representation of sets
  - root of tree used as representative of set



Tree representing {1, 4, 5, 2} and {3, 6}

After union(5,6)

Path Compression



- Amortized cost can be reduced by updating pointers to point directly to the root when they are queried.
- See Levitin §9.2 or CLRS Chapter 21 for more details.

Back to ...

Kruskal’s Algorithm

Growing a tree:

- Suppose that we have a connected graph G=(V, E) and pick an arbitrary vertex r ∈ V:

```
let W ← {r}, E_T ← empty set;

while (W≠V) do {
    find an edge (u,v) with u∈W and v∉W;

    W ← W ∪ {v};
    E_T ← E_T ∪ {(u,v)};
}
```

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```

How many times will this loop execute?

Growing a tree:

- Suppose that we have a connected graph G=(V, E) and pick an arbitrary vertex $r \in V$:

Invariant: (W, E_T) is a connected, acyclic subgraph of G

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There must always be such an edge, otherwise G would not be connected.

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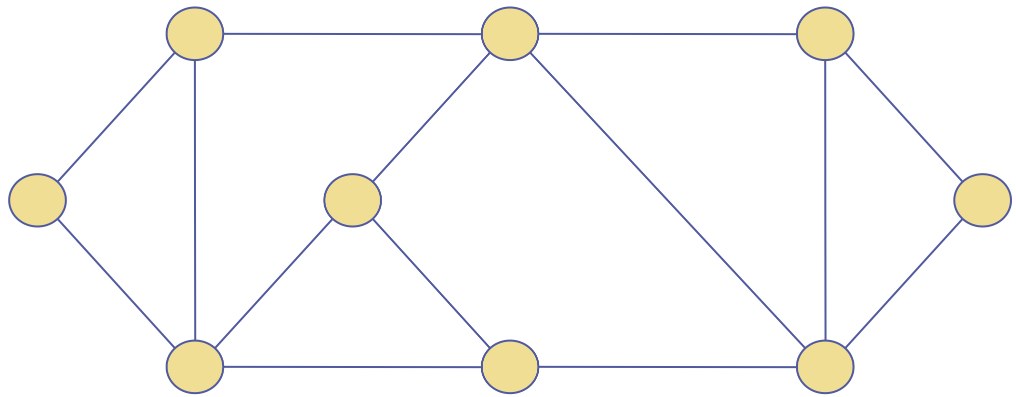
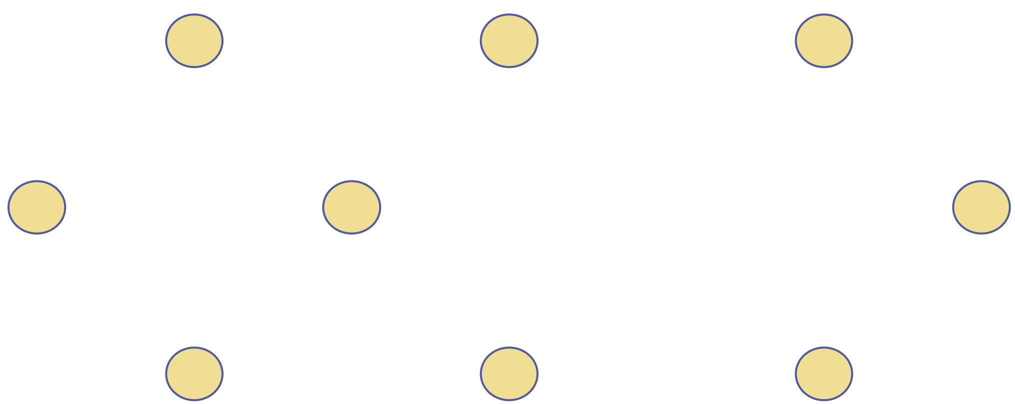
Invariant: (W,E_T) is a connected, acyclic subgraph of G

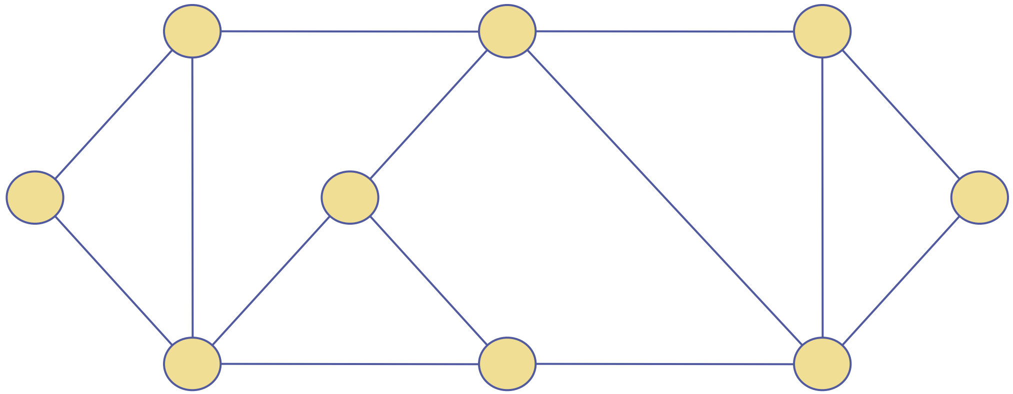
There must always be such an edge, otherwise G would not be connected.

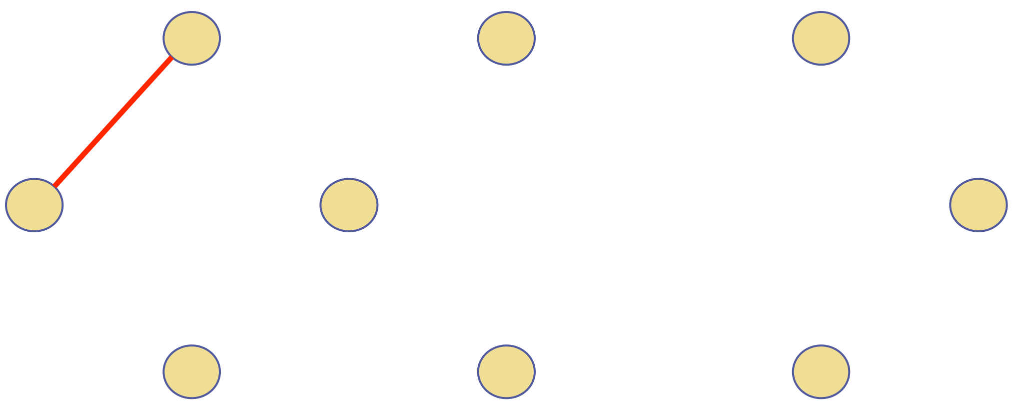
How many times will this loop execute?

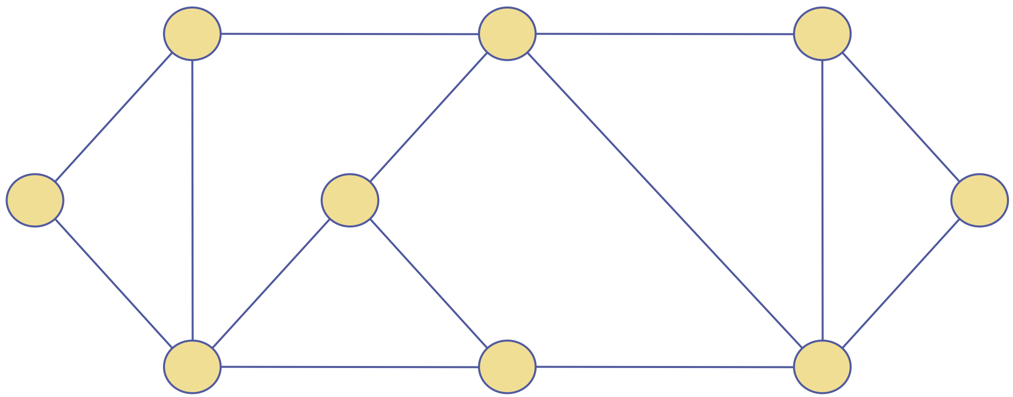
We add a total of |V|-1 edges to E_T

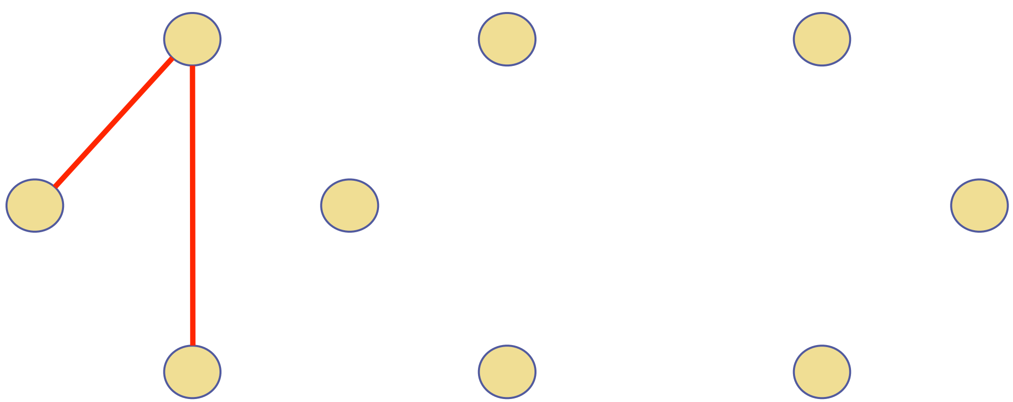


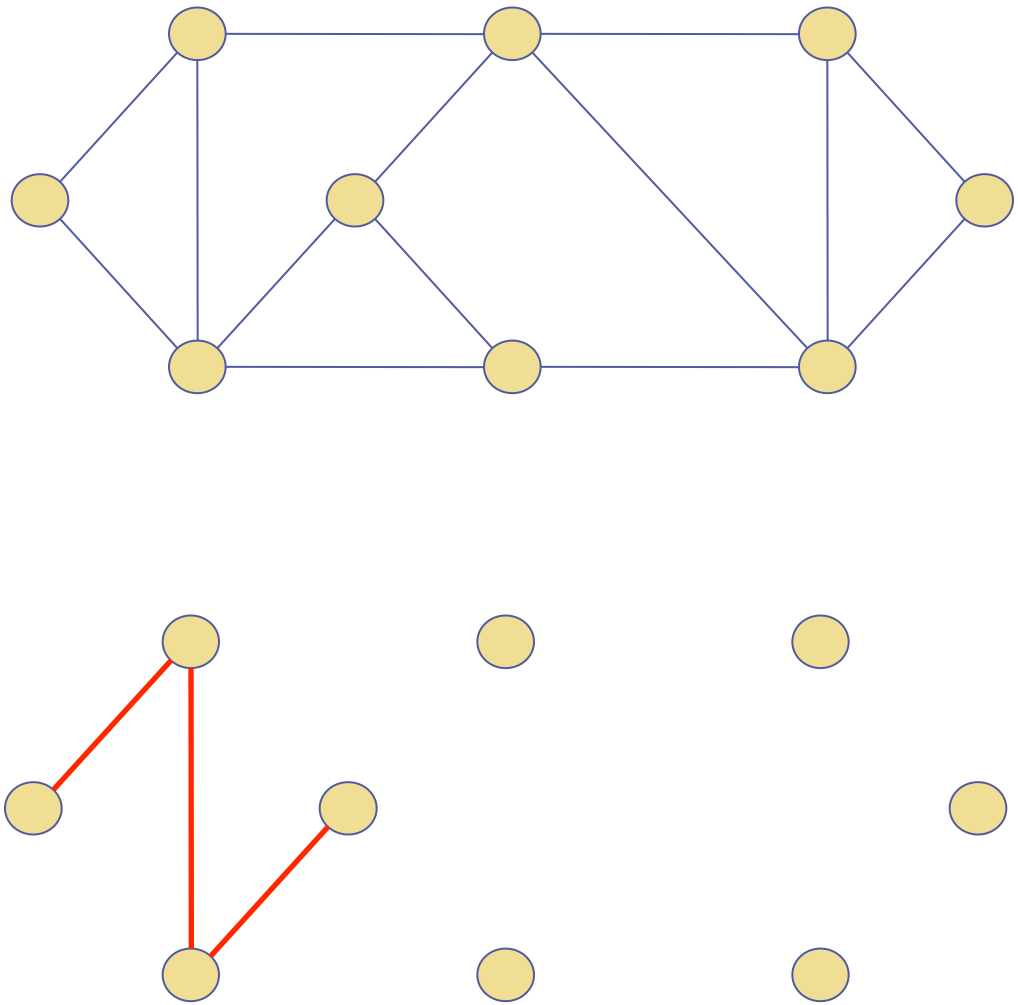


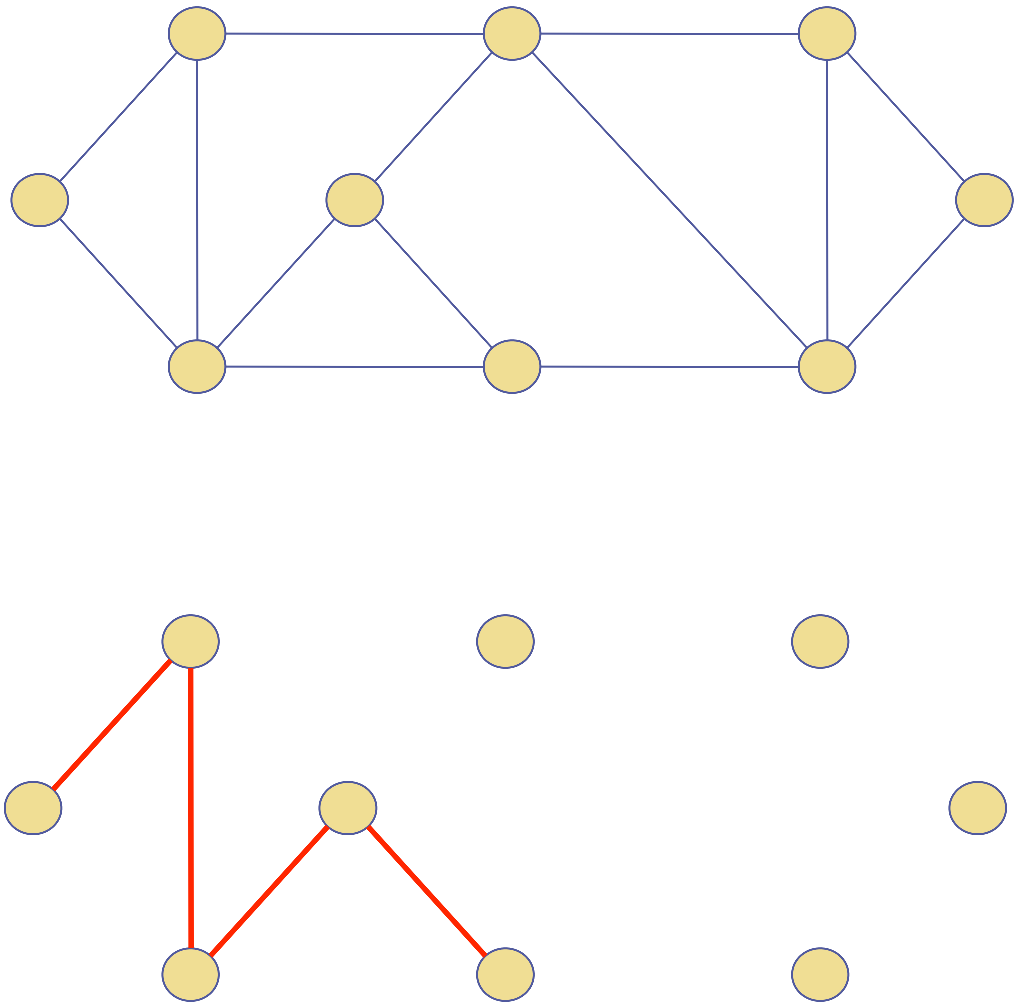


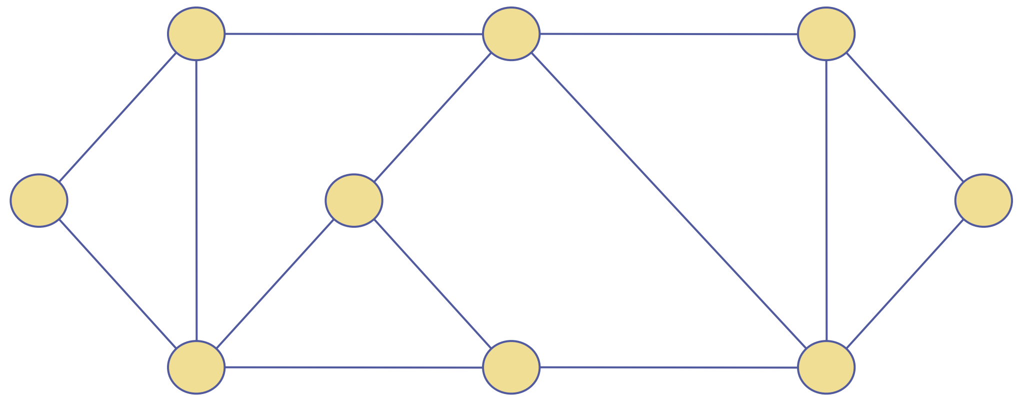


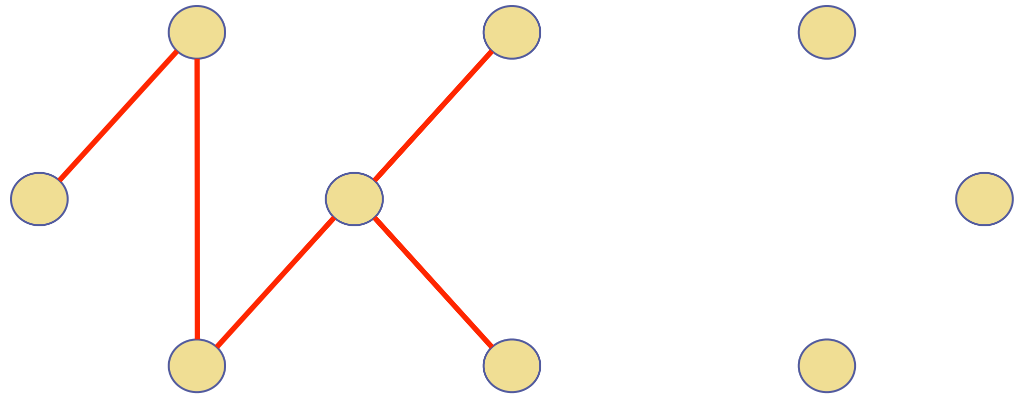


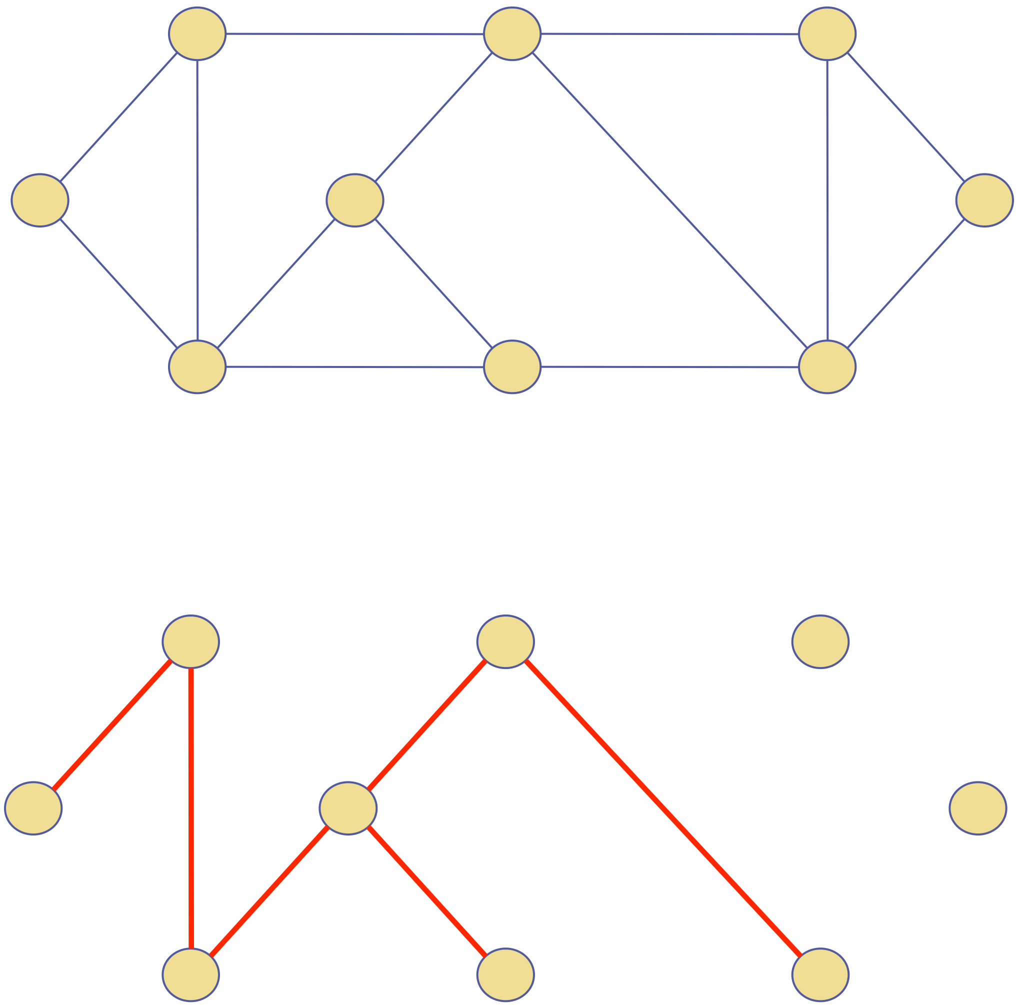


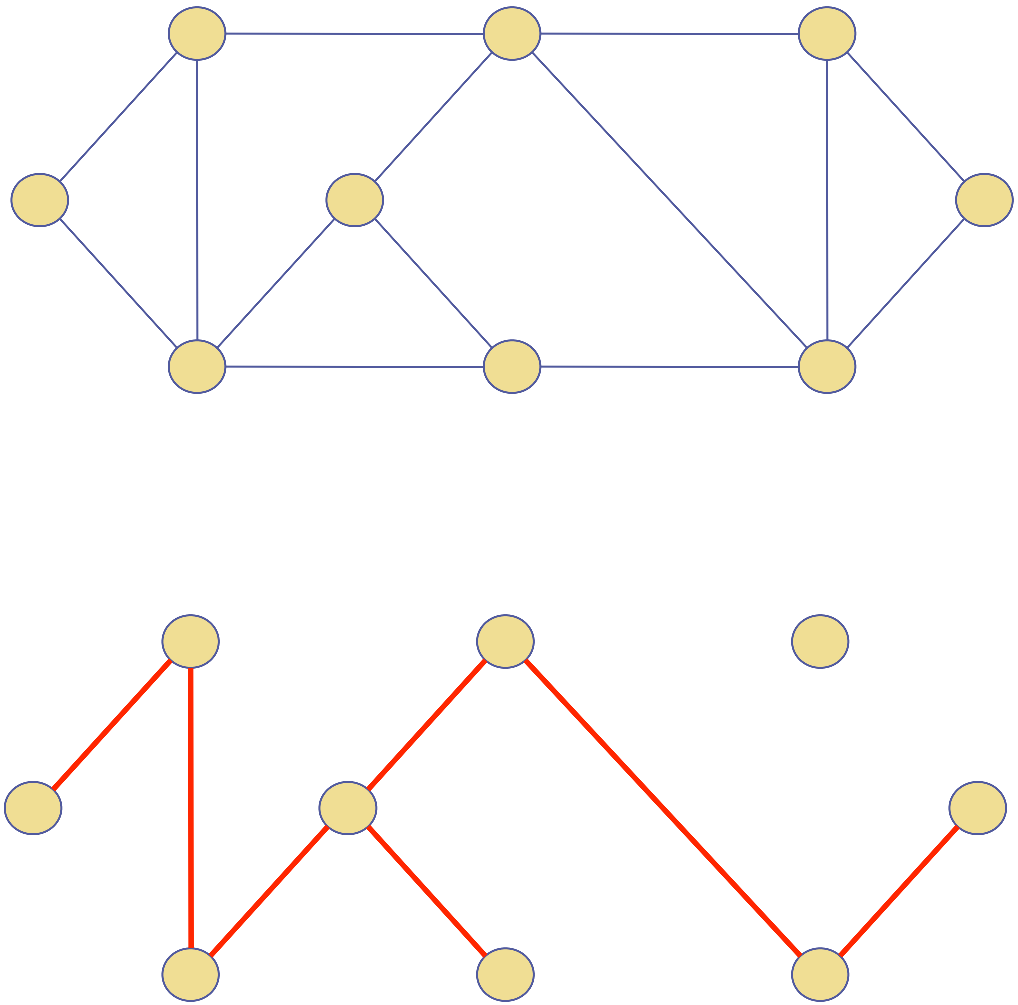


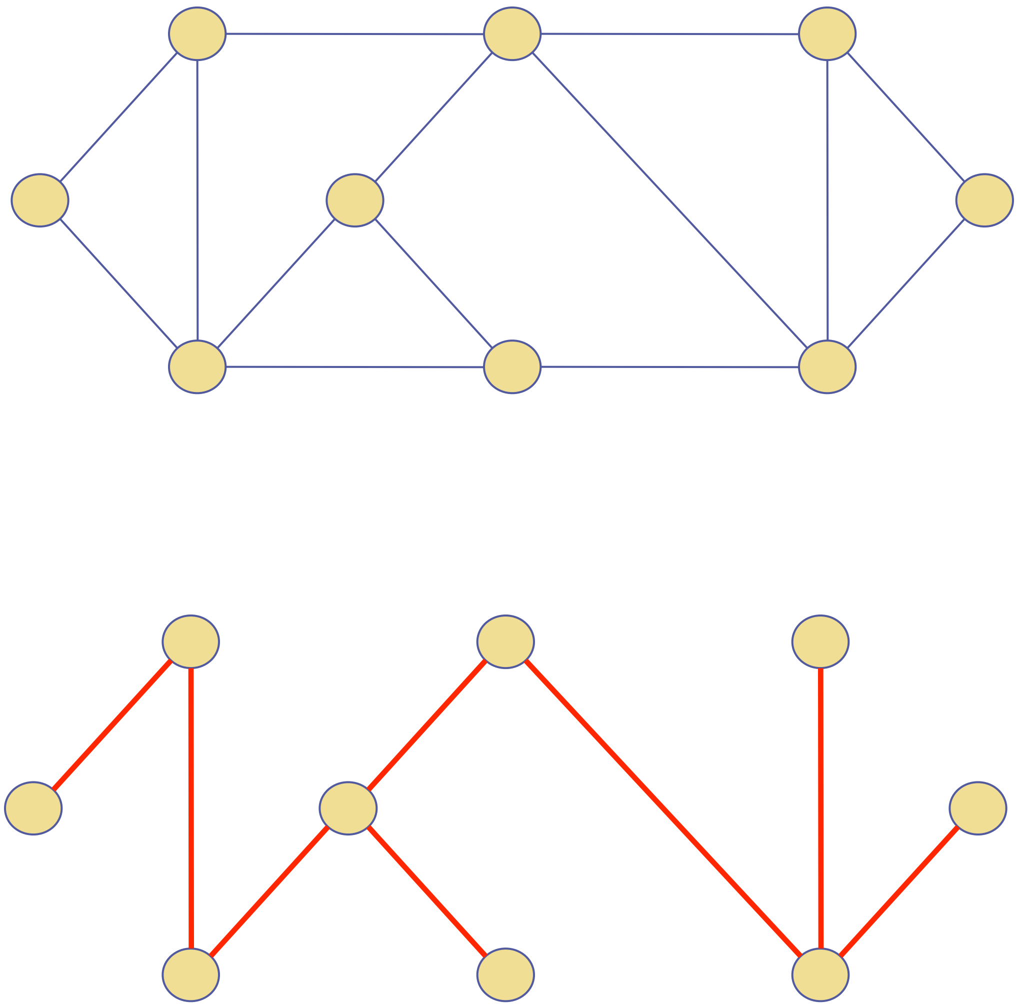












Minimum Spanning Trees

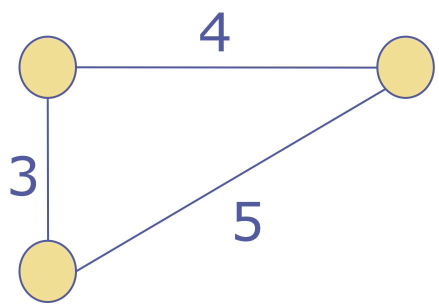
Back to bridge building …

- To link a group of n small islands together with bridges, we will need to build at least (n-1) bridges; any spanning tree will do for this.

- But now suppose that we want to minimize the total span of all the bridges as well … How should we proceed?

Minimum spanning trees:

- To take account of the distances between the islands, we need to use a labeled, or weighted graph.



- A *minimum spanning tree* (MST) is a spanning tree that minimizes the total of the weights on its edges.

- Not all spanning trees have this property.

The MST problem:

- Suppose that we have a connected, undirected graph G=(V,E), with a numerical weighting w(u,v) for each edge (u,v).

  Problem: Find an acyclic subset $T \subseteq E$ that connects all of the vertices in V, and minimizes:

  $$\Sigma \{w(u,v) \mid (u,v) \in T\}$$

  Solution: We will look for an algorithm of the form:

```
E_T ← empty set of edges
while (E_T is not a spanning tree)
    add an edge to E_T
```

- At each stage we will ensure that E_T is a subset of a MST.
- Obviously true when we start … the trick is to ensure that the invariant is preserved when we add an element ...

Greedy Choice

- Whenever we add an edge, let’s make the Greedy choice:
  - add the edge with the lowest weight that does not form a cycle
  - Edges that do form a cycle are not needed in the spanning tree
- Does making the Greedy choice ever add an edge that we don't need?

A key result:

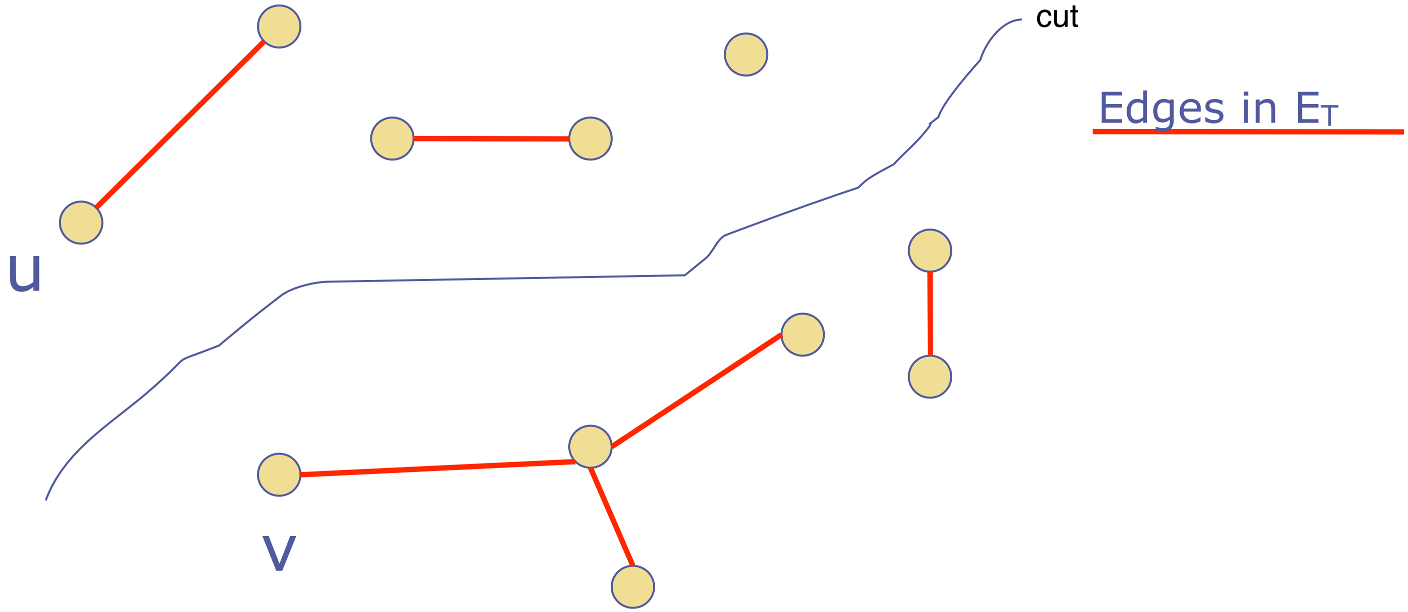
Suppose that we partition V into two sets (a "*cut*"), and that none of the edges in E_T crosses between the two sets (the cut "*respects*" E_T).

Suppose also that (u,v) is an edge that crosses between the two halves, and that no other edge that crosses has lower weight — (u,v) is a "*light edge*".

Claim: $E_T \cup \{(u,v)\}$ is a subset of a minimum spanning tree: (u,v) is "*safe*" for E_T.

Proof:

Proof:



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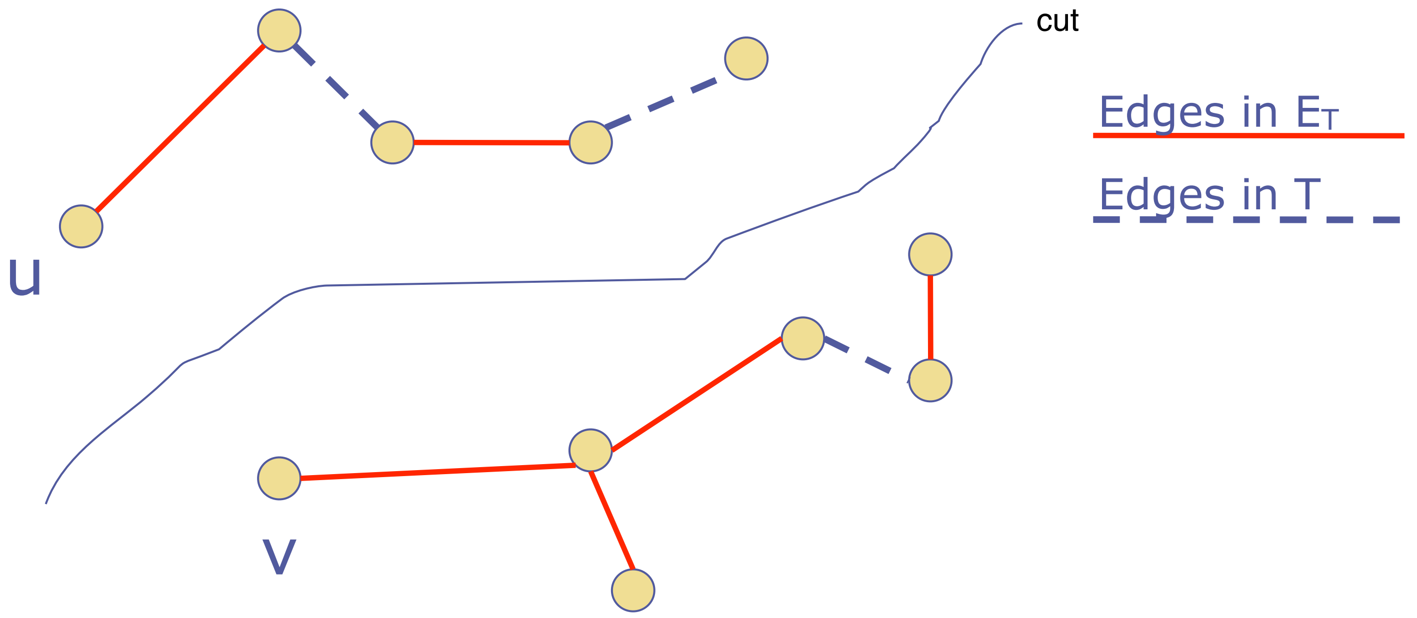

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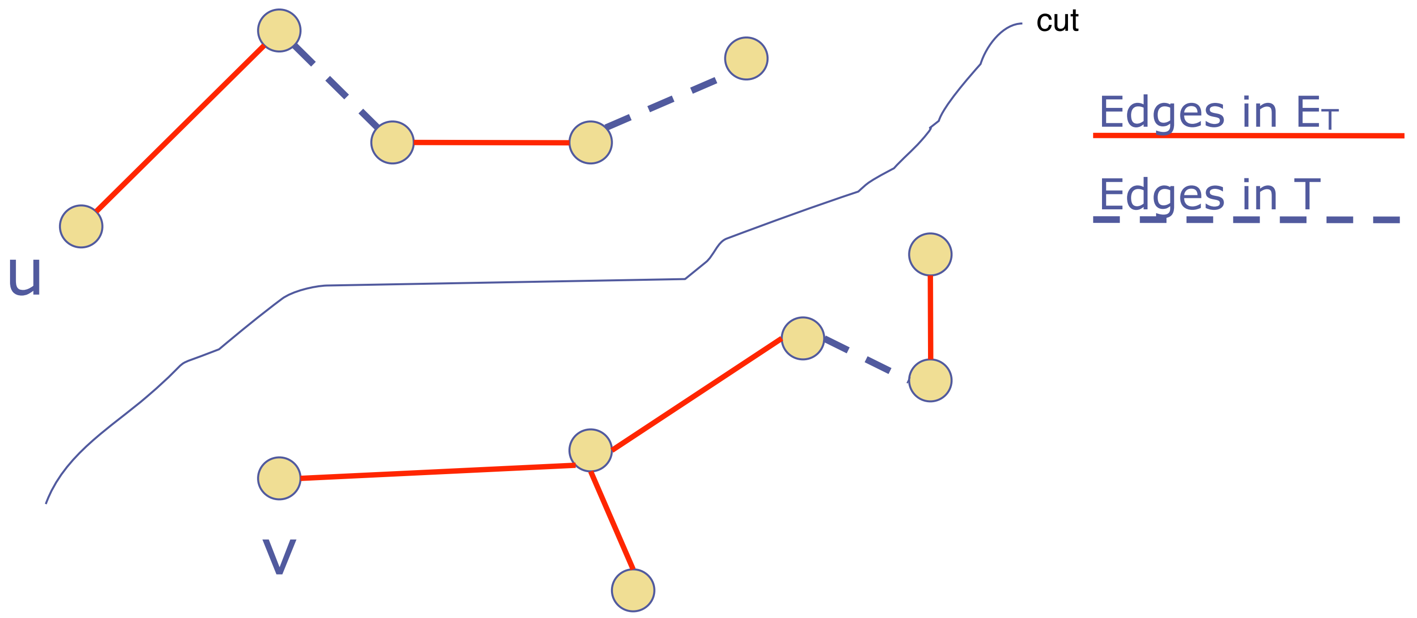
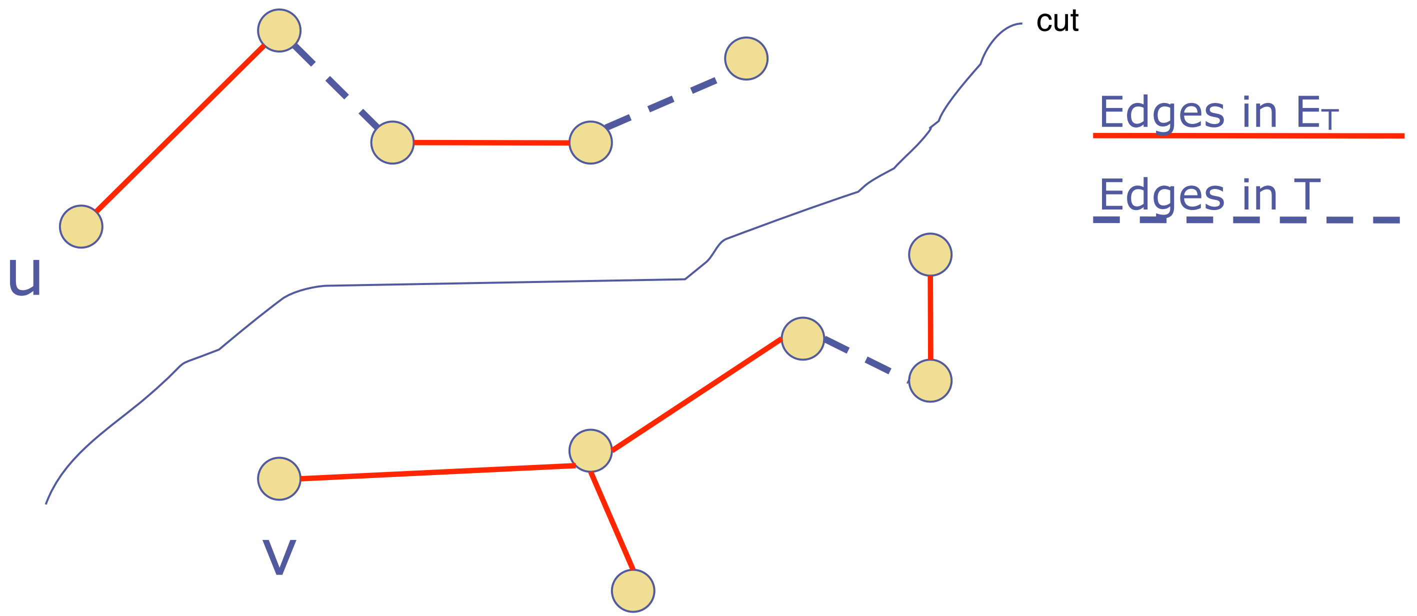


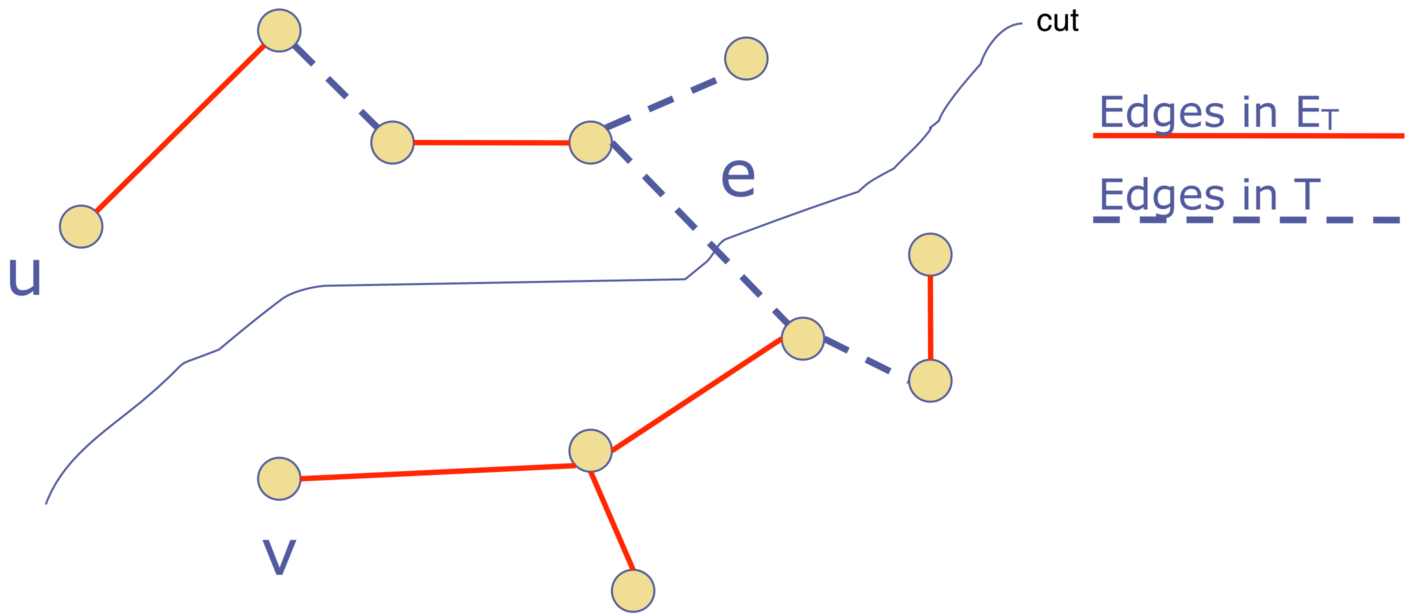
- E_T is a subset of some minimum spanning tree T.

Proof:



- E_T is a subset of some minimum spanning tree T.
- Because u and v are on opposite sides, there is an edge e in T that crosses the cut.

Proof:



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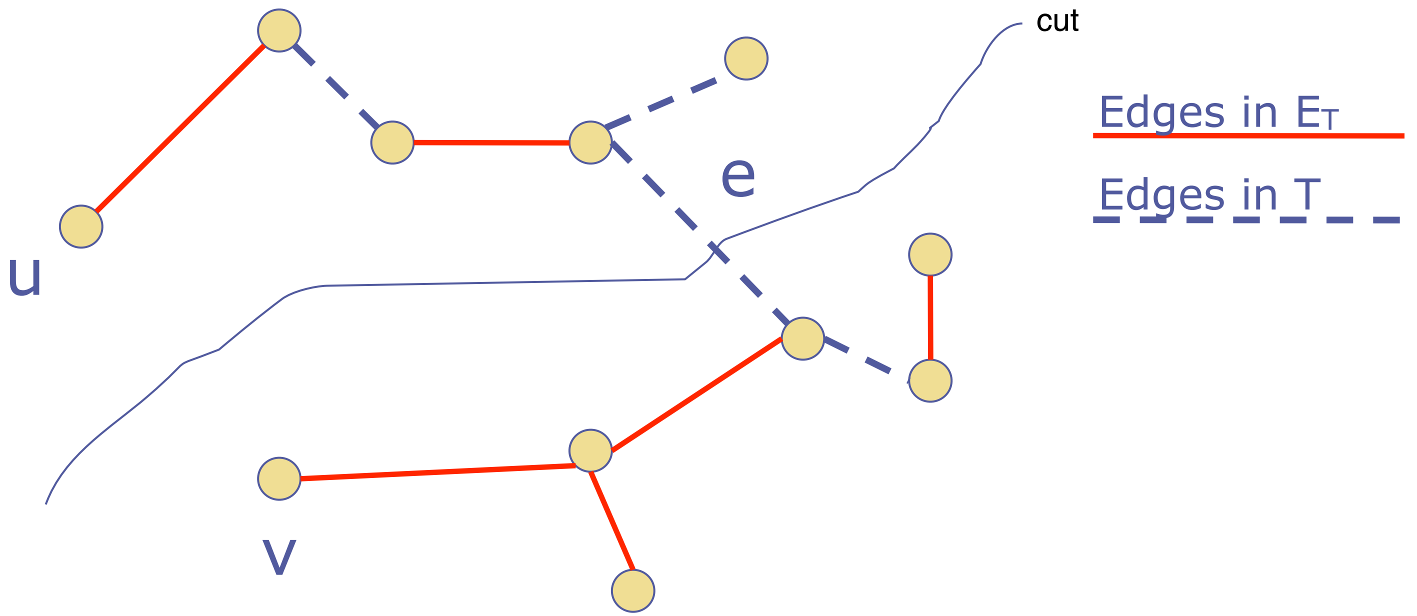

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- By assumption weight of (u,v) ≤ the weight of e.

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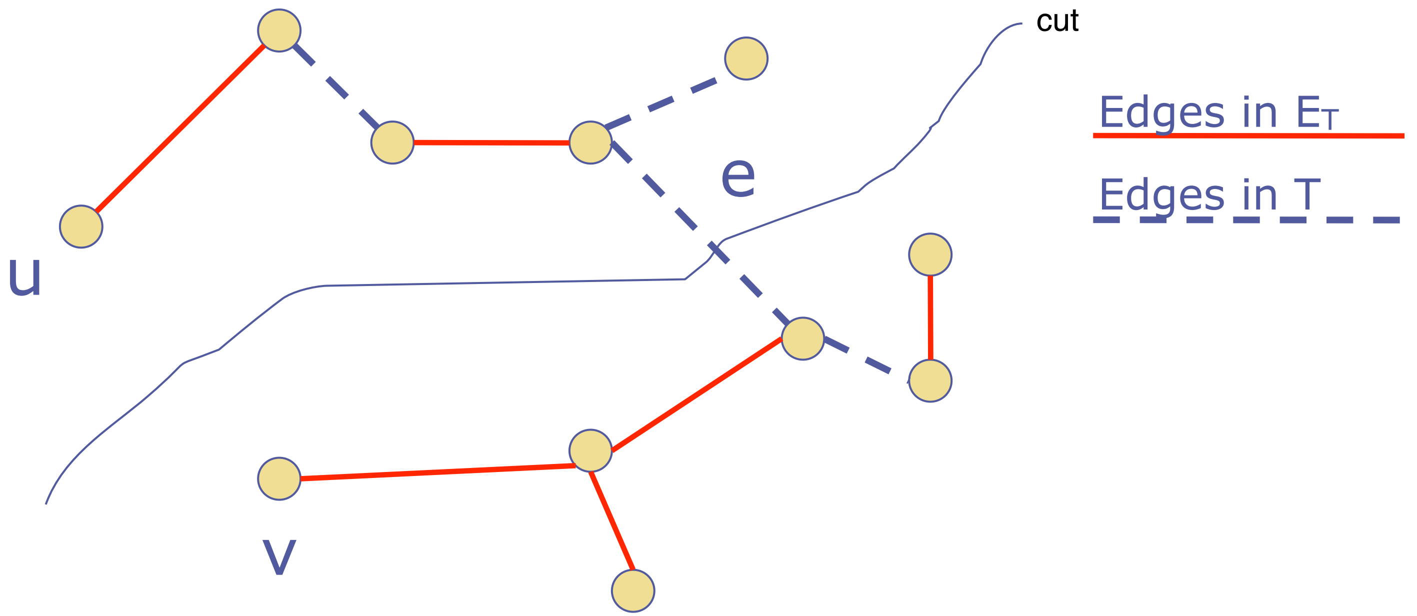

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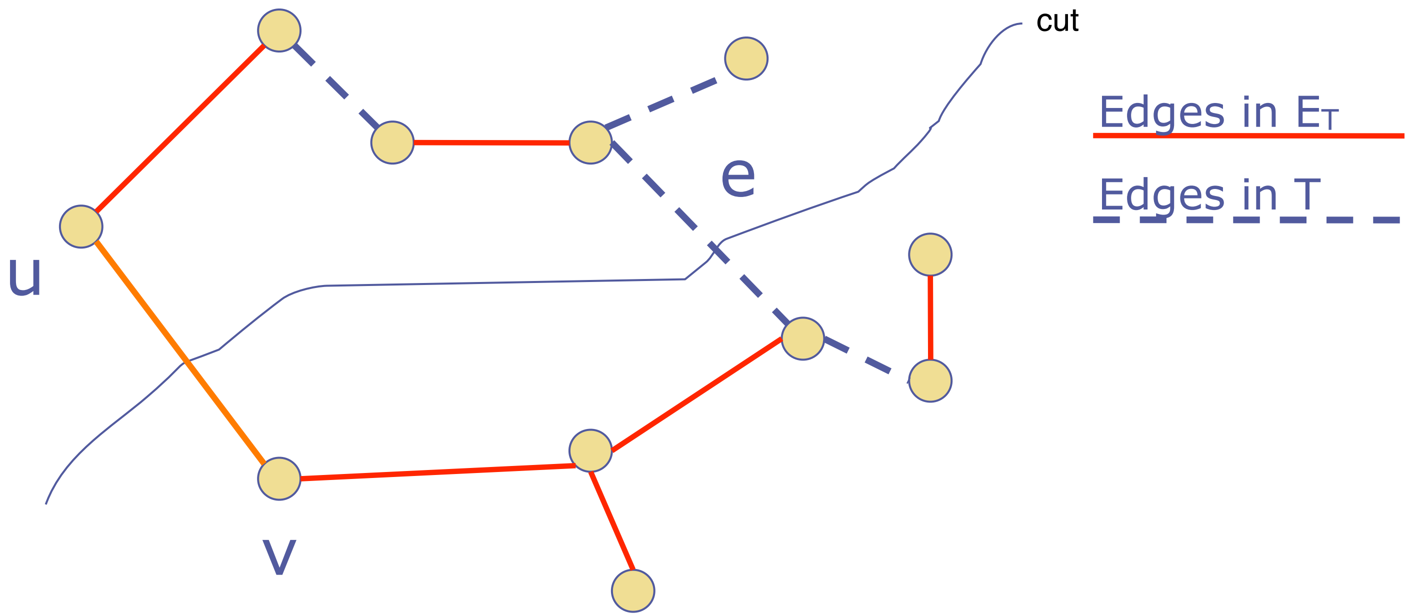

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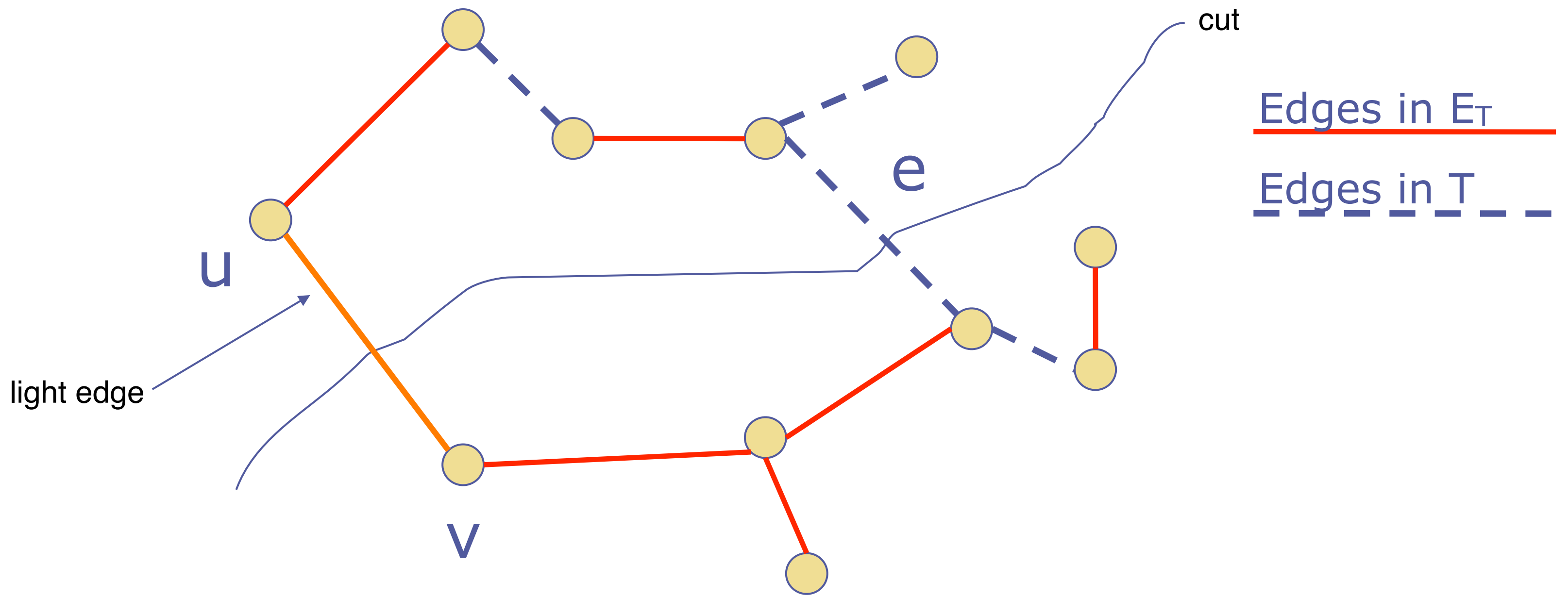

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Corollary:

- Suppose that:
  - C is a connected component in the forest (V, E_T);
  - (u,v) is a *light edge* connecting C to some other component in G.
- Then (u,v) is safe for E_T.

- Follows directly by using a cut to separate the vertices in C from the vertices outside.
- Requiring C to be a connected component of (V, E_T) ensures that no edge in E_T crosses the cut.

Kruskal's algorithm:

- Given a connected graph G=(V, E):

```
E_T ← empty set of edges
for each v in V
   make a singleton set {v}

sort the edges of E by nondecreasing weight

for each edge (u,v) in E
   if S_u ≠ S_v, then
      replace S_u and S_v with S_u ∪ S_v
      add (u,v) to E_T
```

- Complexity is O(|E| log |E|).
  - (With our simple union-find, more like $O(|E|^2)$)

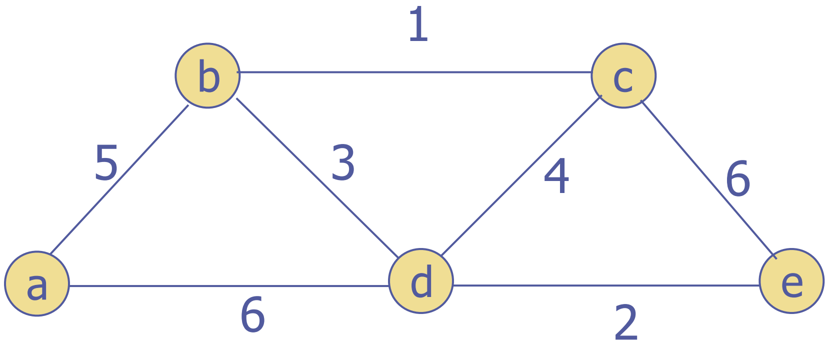
How does this work?

- Suppose that C and D are the two connected components in the forest (V,E_T) that are connected by an edge (u,v).

- Then (u,v) must have the least weight of any edge between C and D (otherwise C and D would have already been connected).

Your turn!

- Apply Kruskal's algorithm to this graph:



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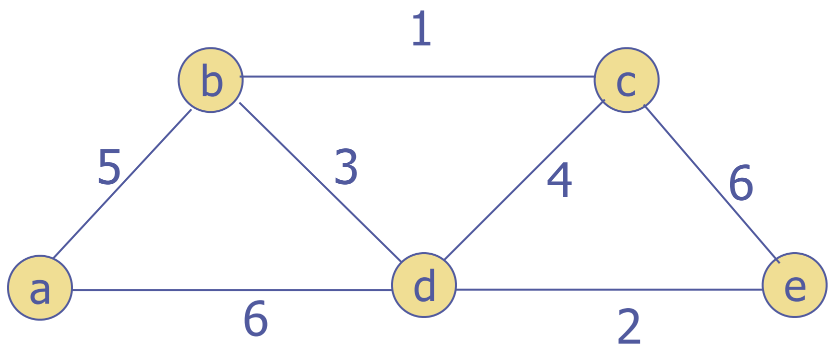

Tree edges	List of edges (sorted by weight)
	bc_1 de_2 bd_3 cd_4 ab_5 ad_6 ce_6

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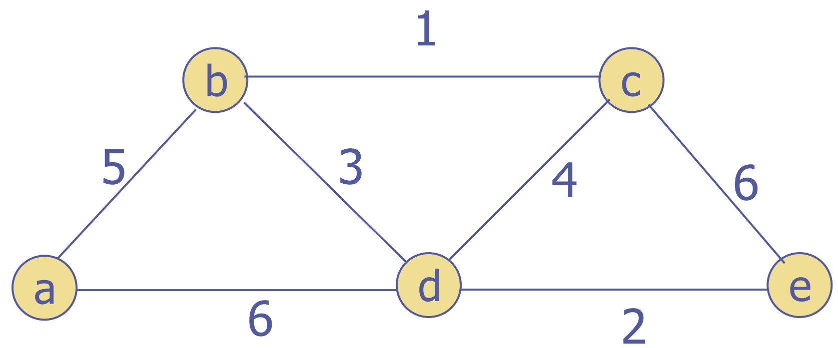

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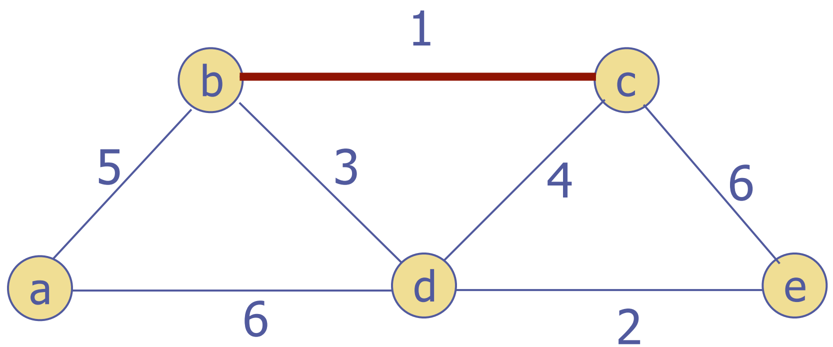

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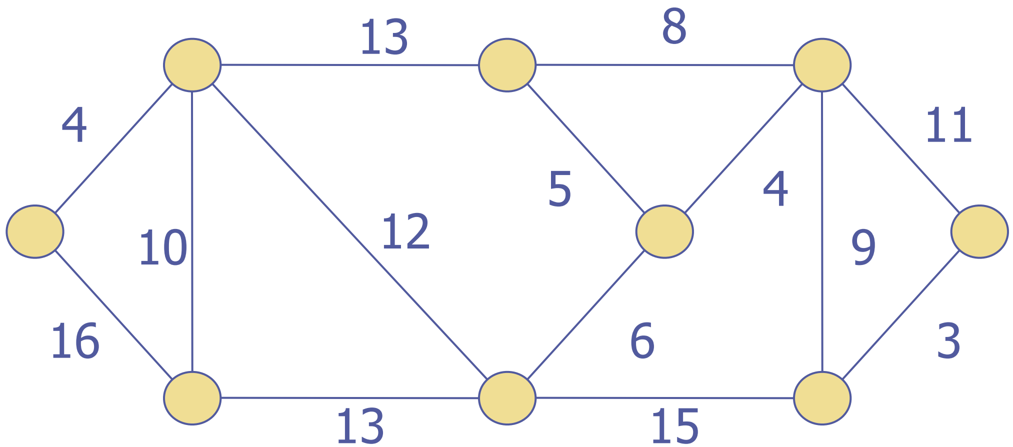

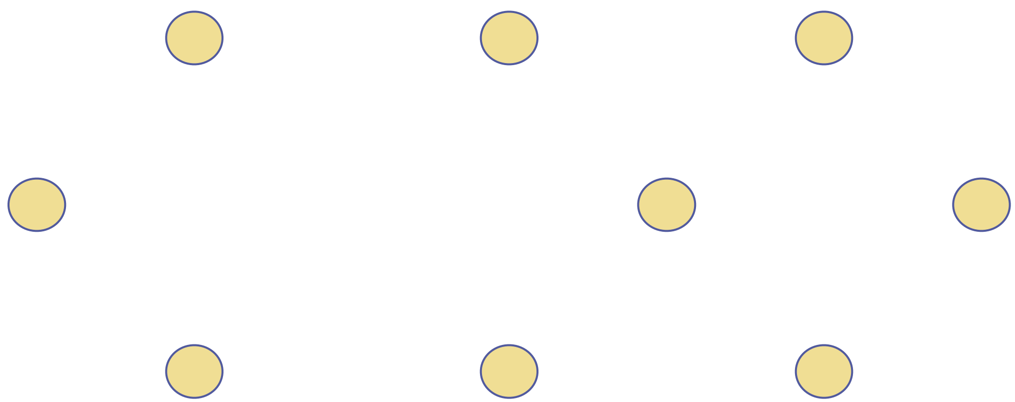

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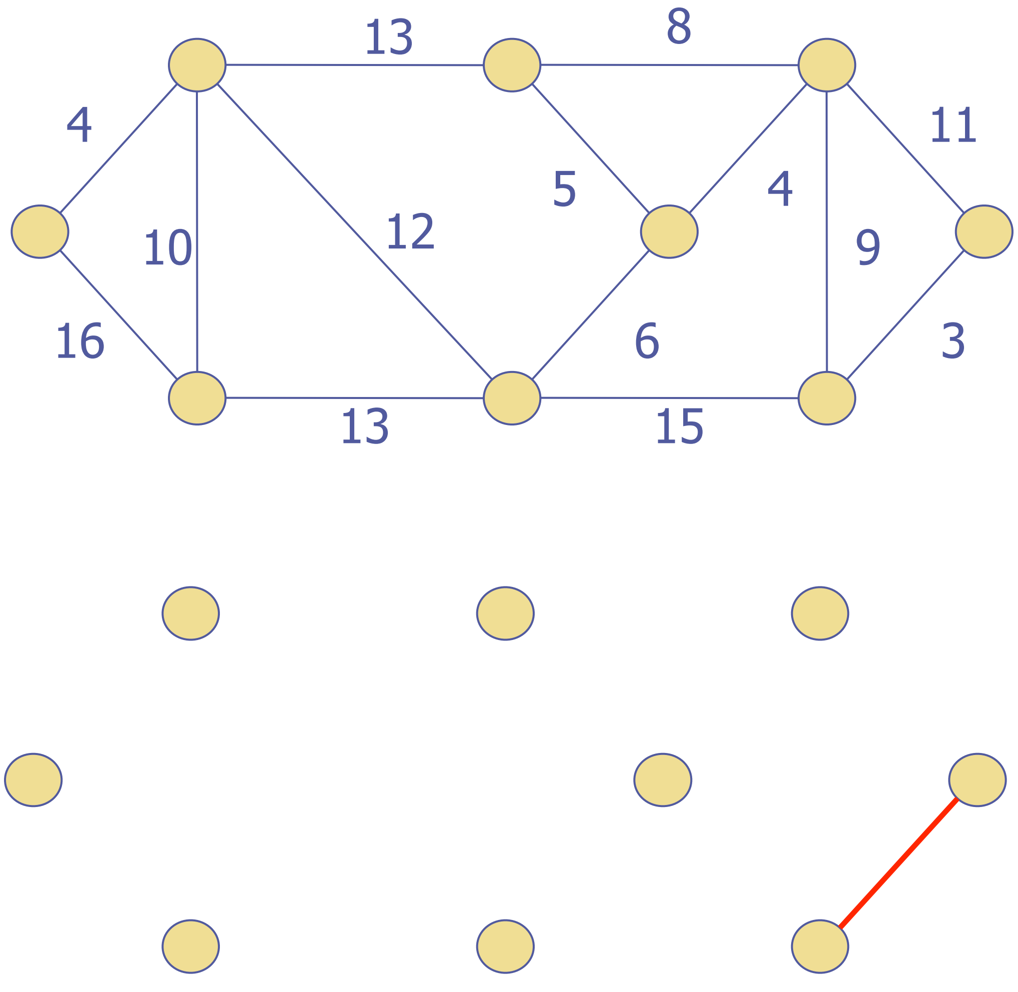

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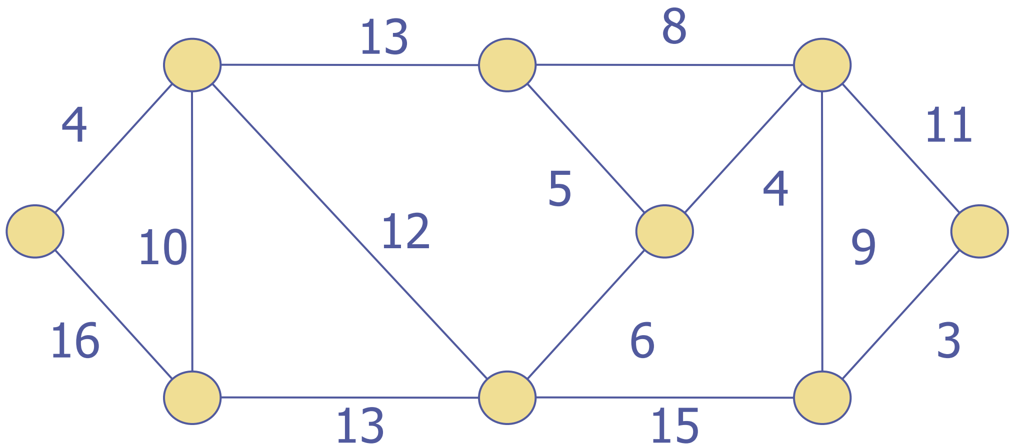

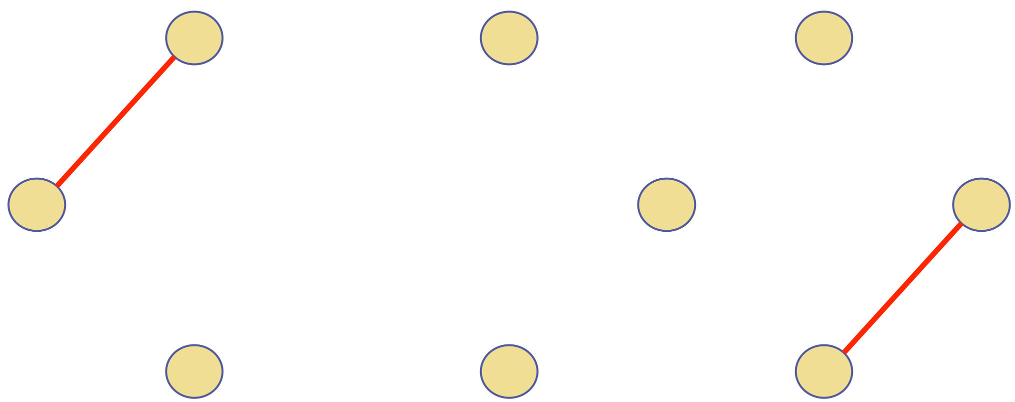

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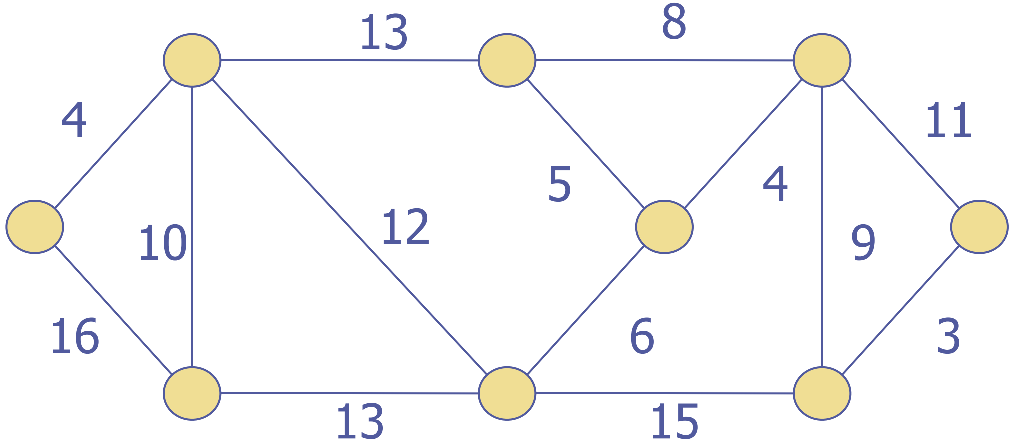

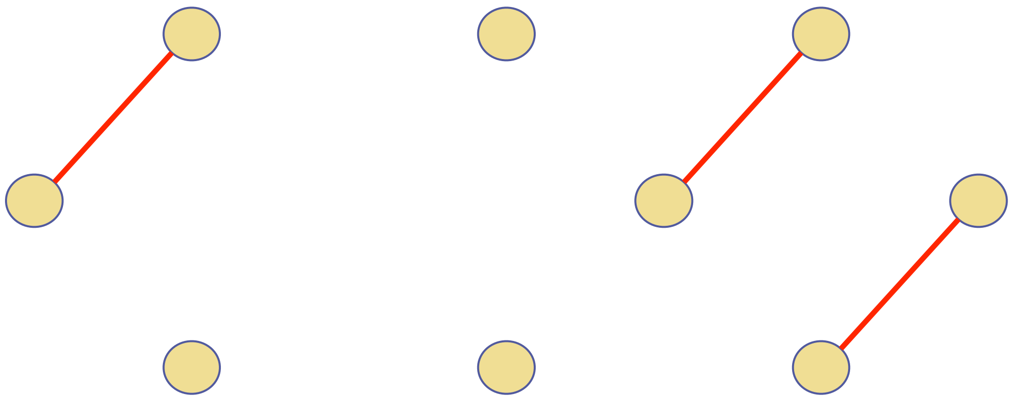

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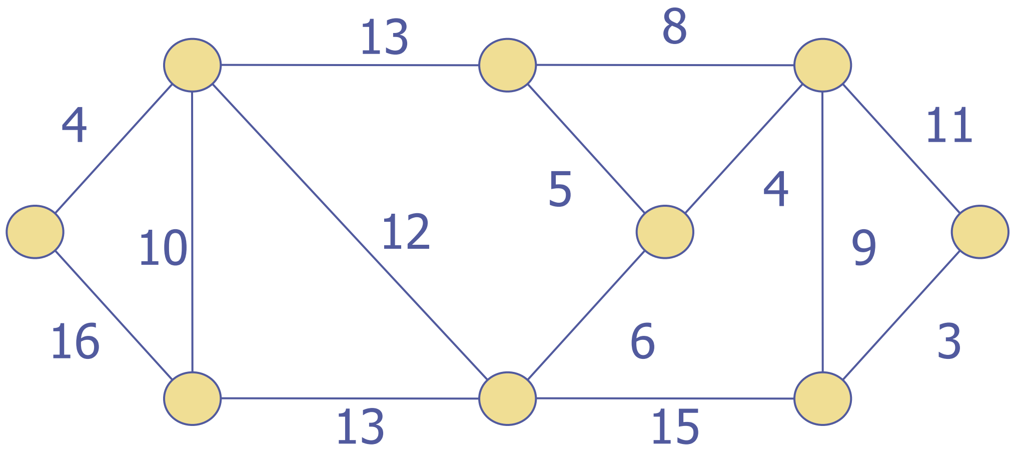

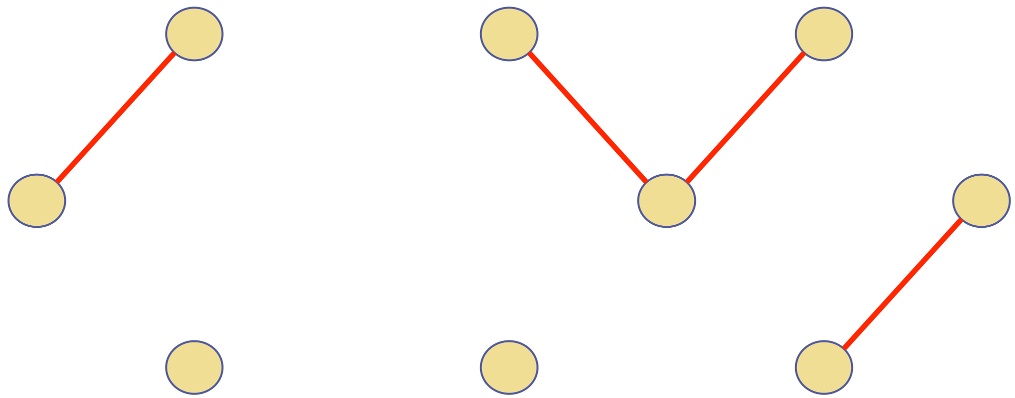

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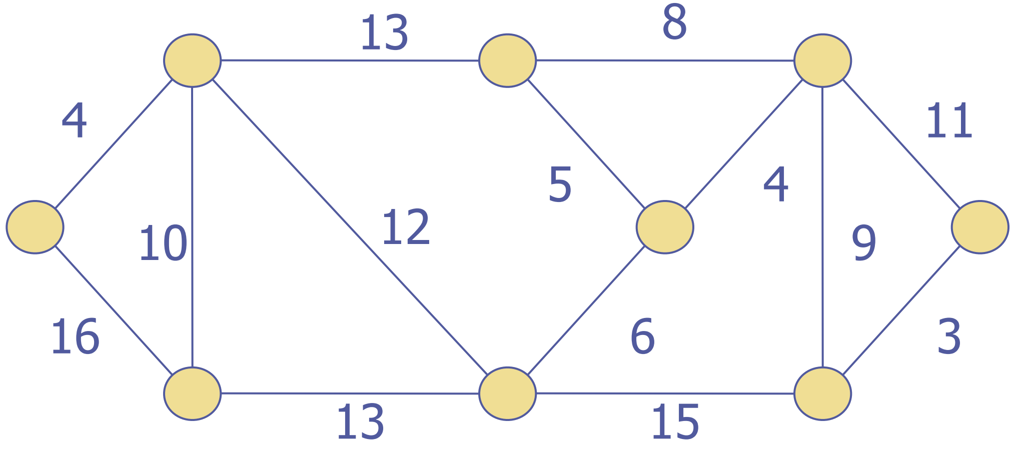

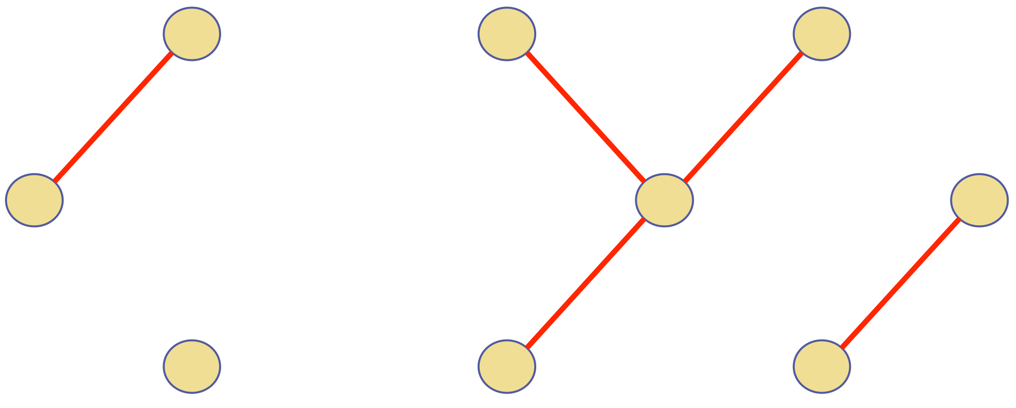

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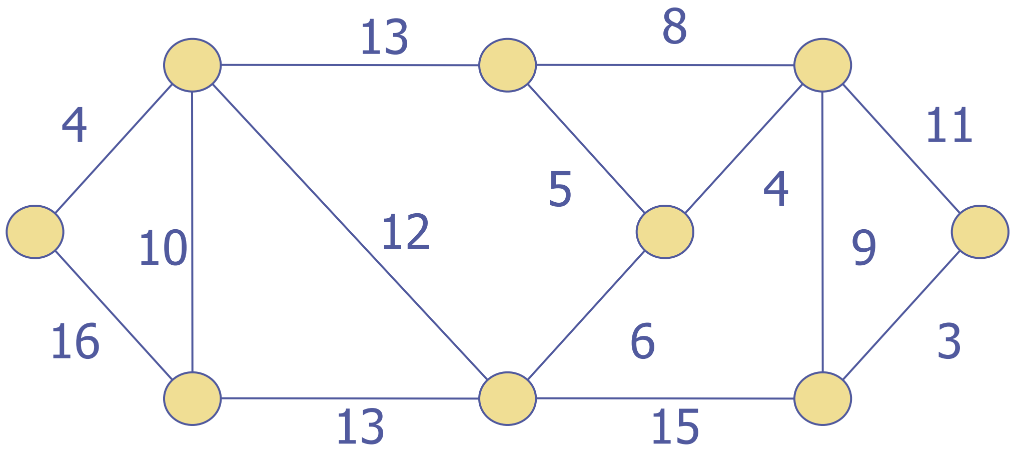

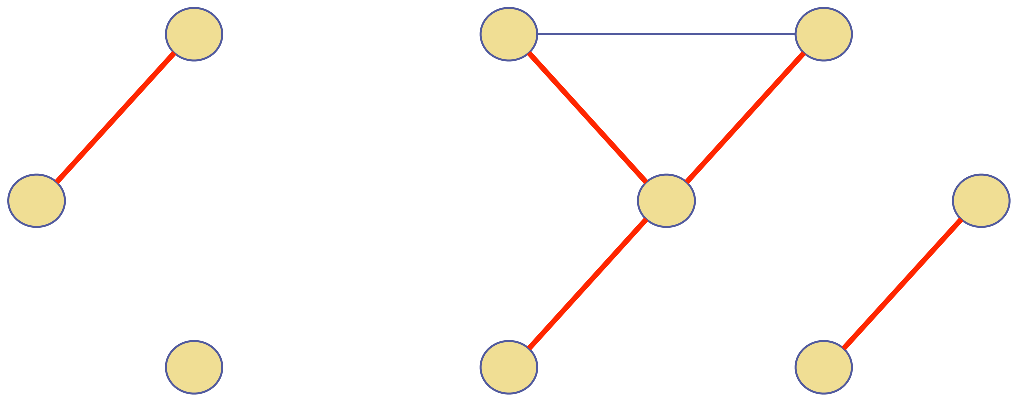

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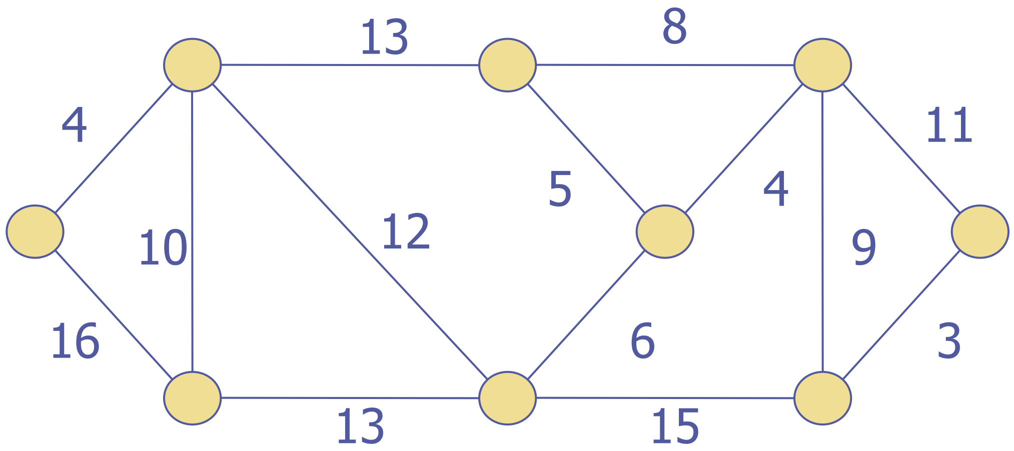

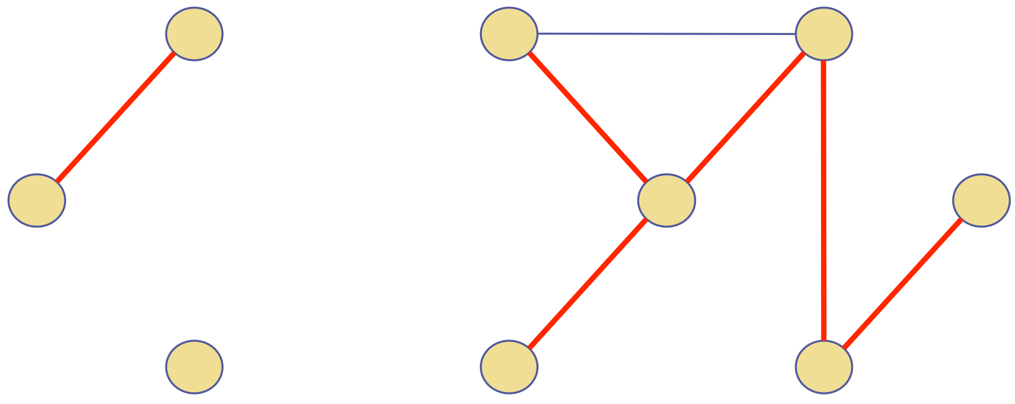

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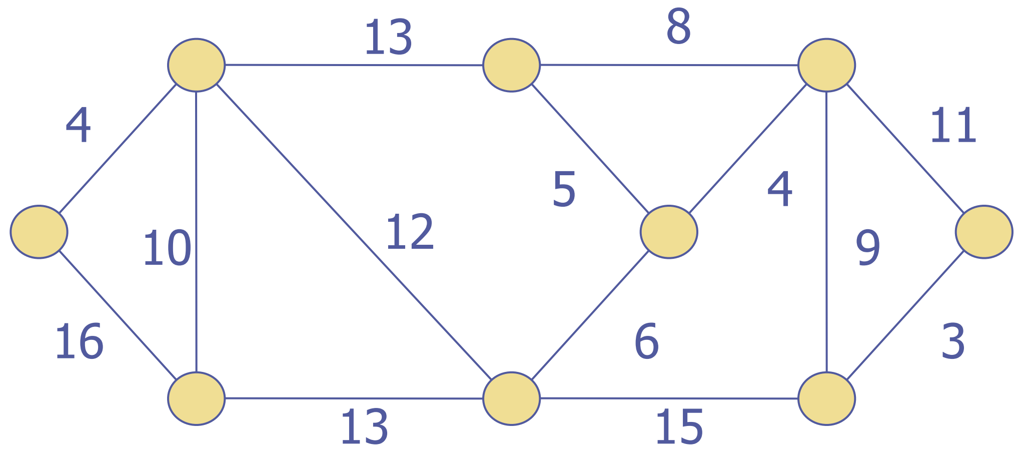

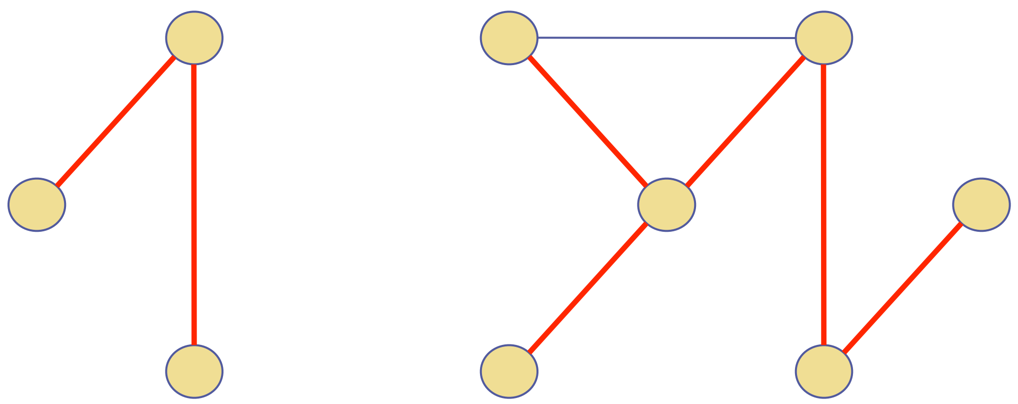

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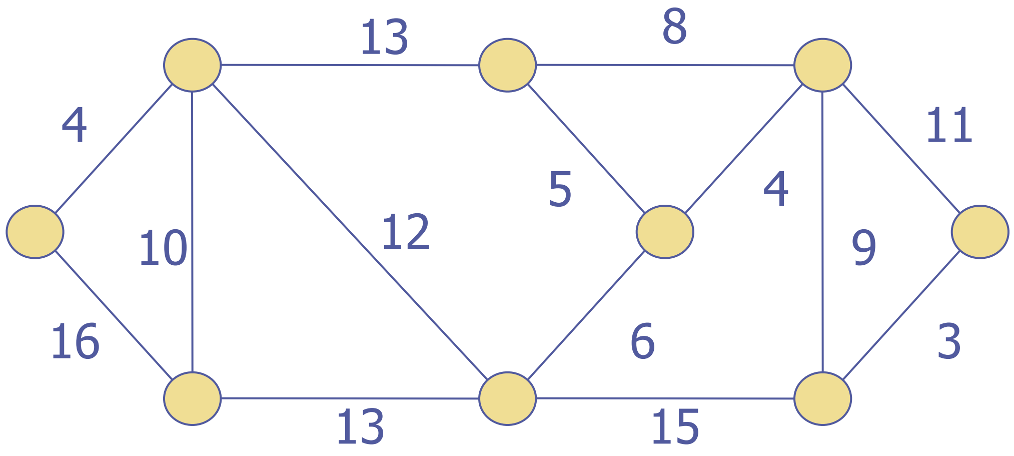

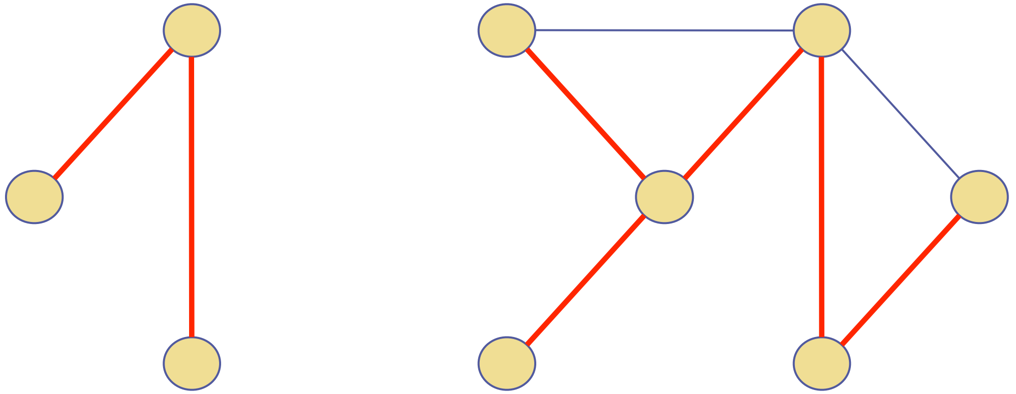

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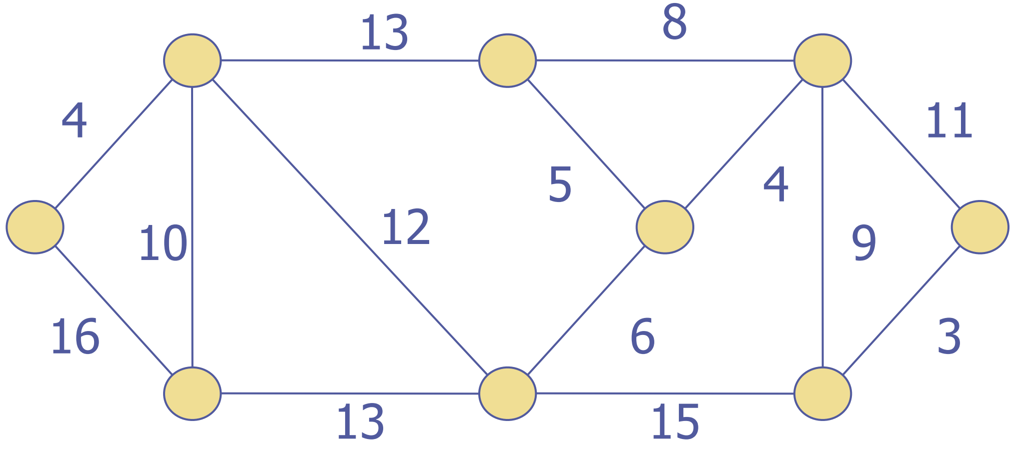

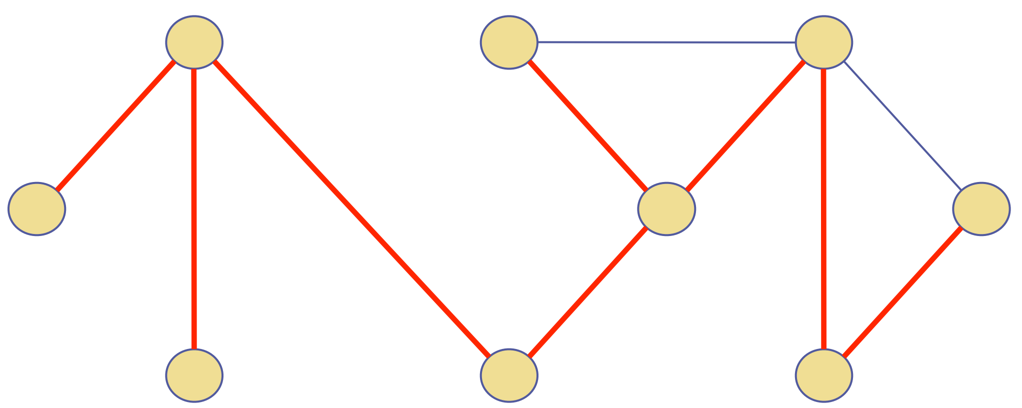

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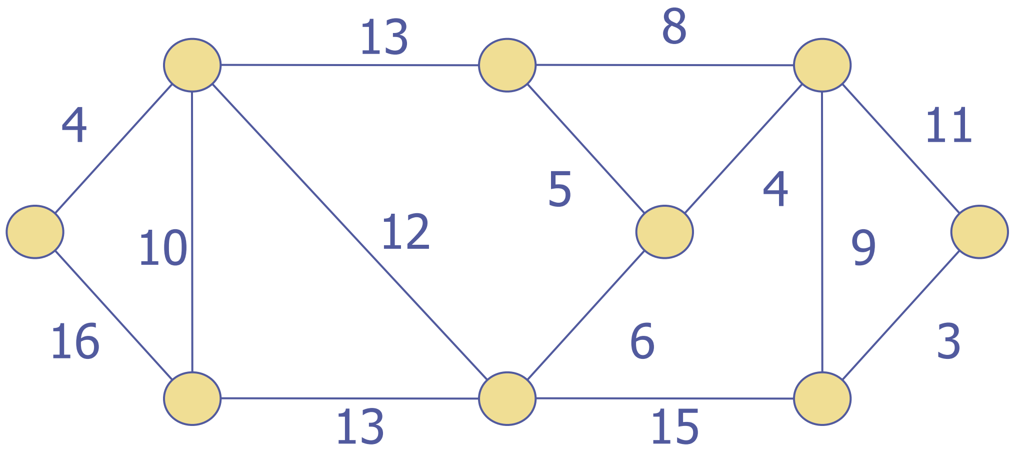

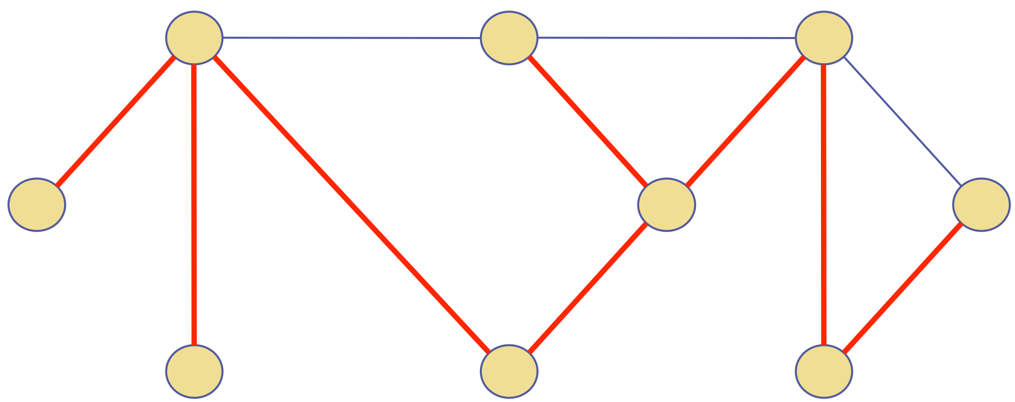

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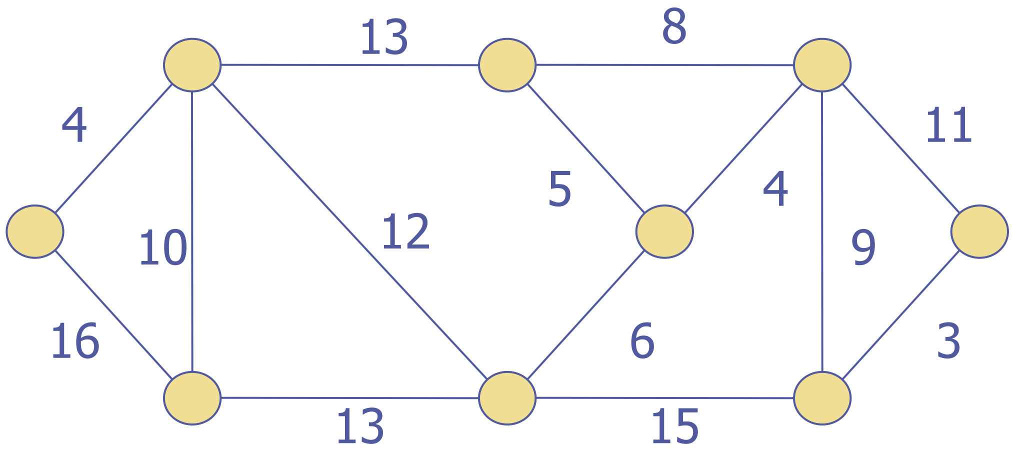

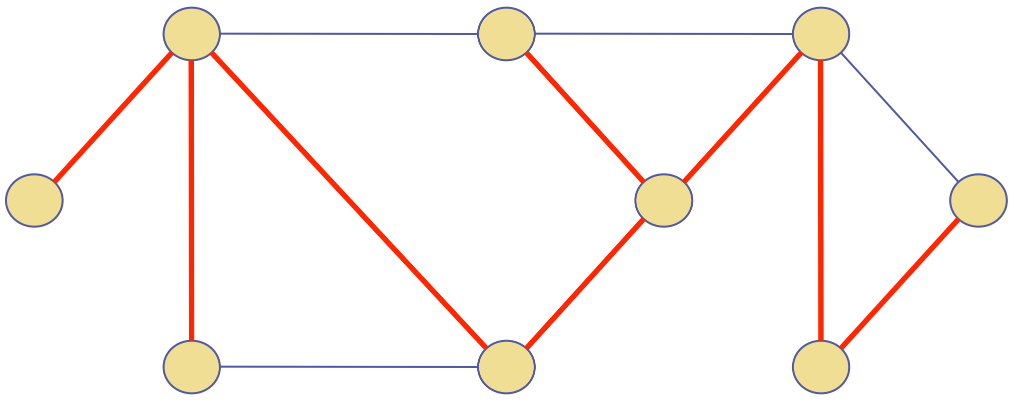

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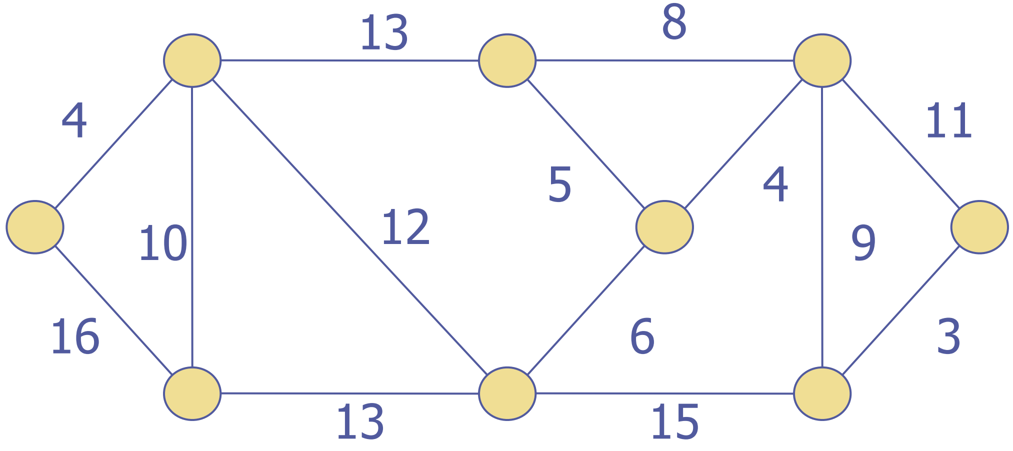

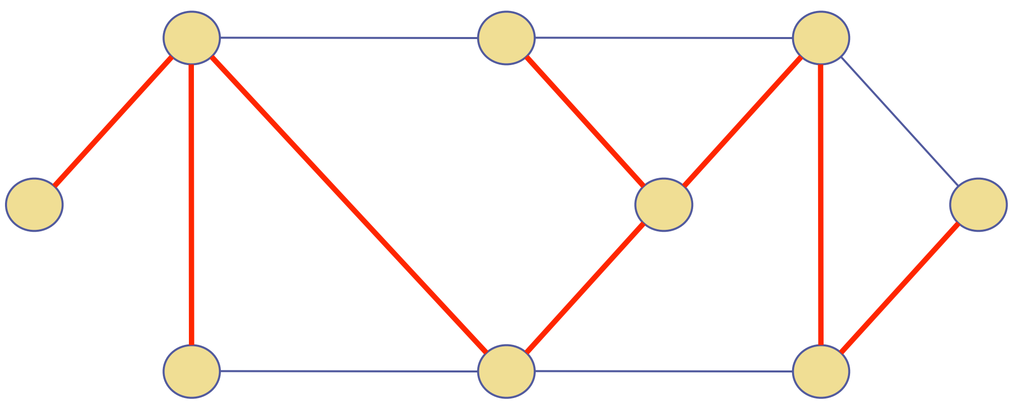

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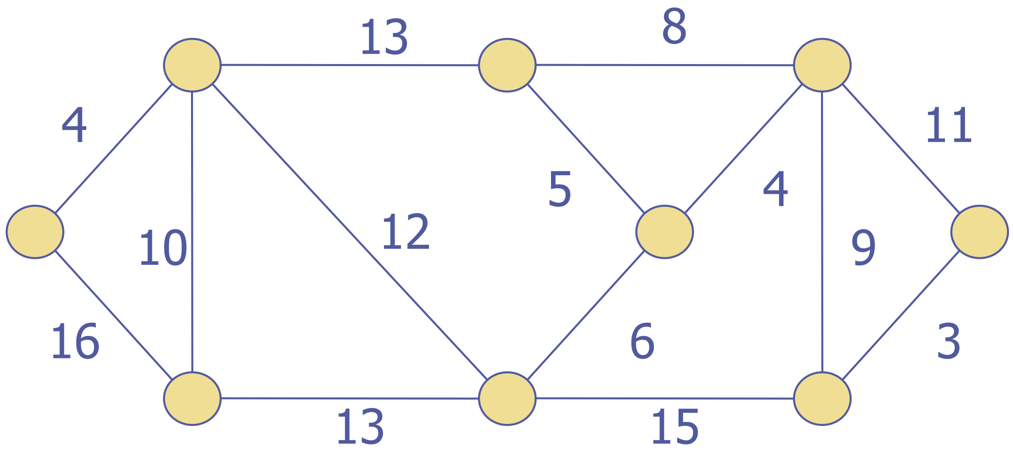

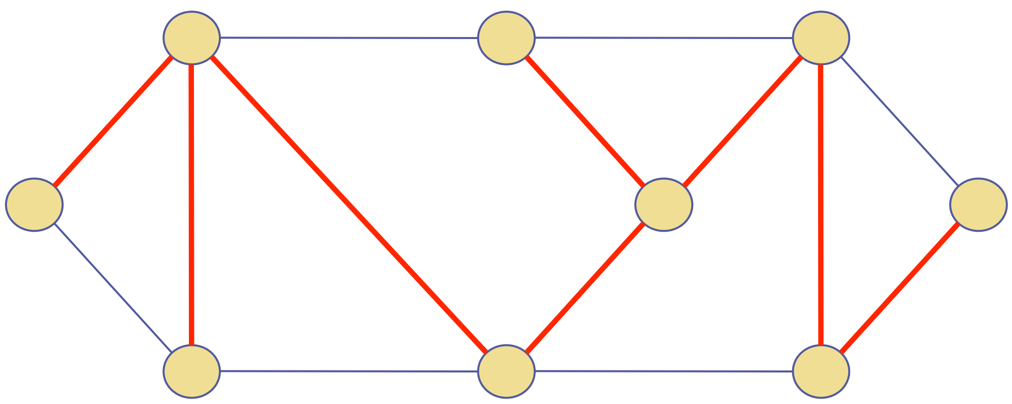

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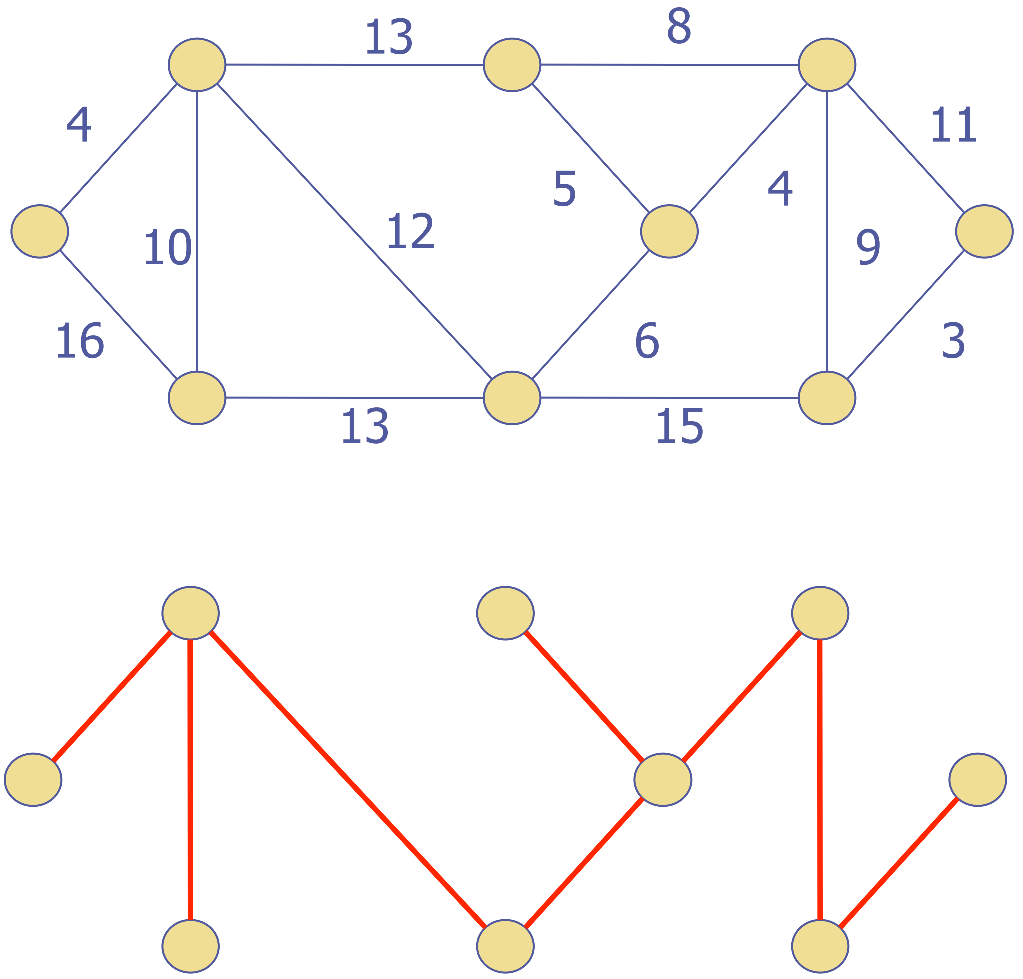

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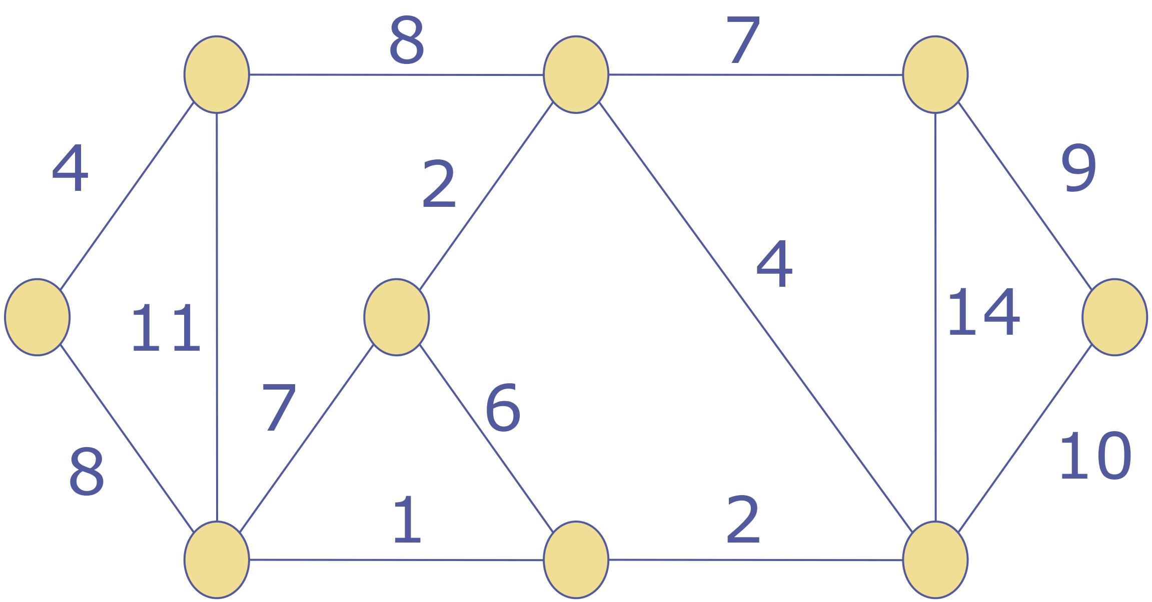

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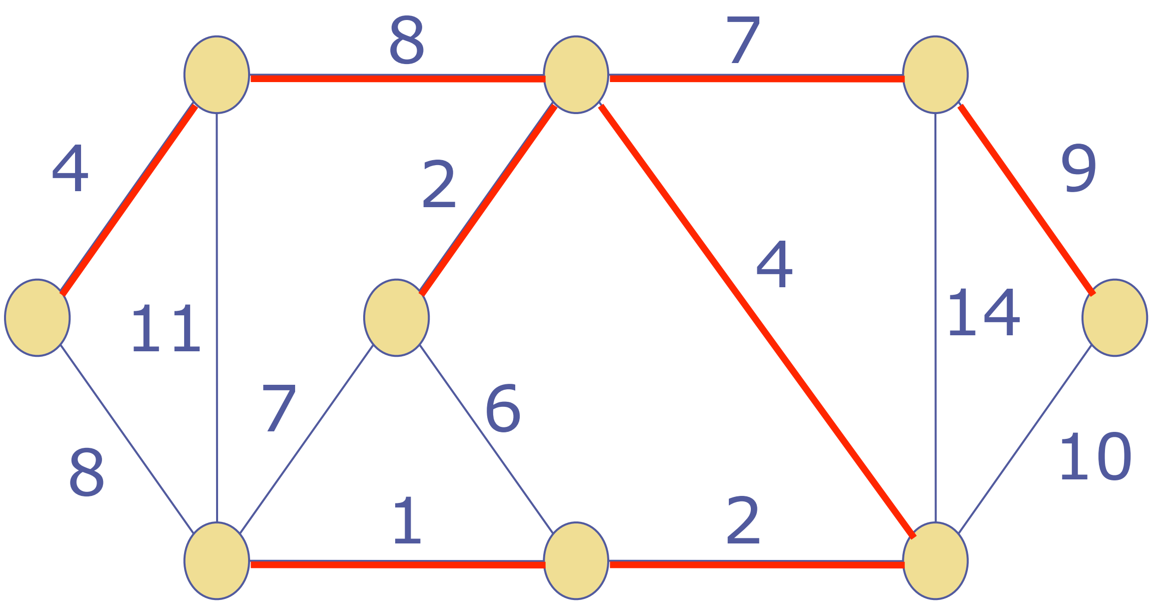

Prim’s Algorithm

Prim's algorithm:

- In Kruskal's algorithm, E_T is a forest whose components are combined as the algorithm runs until just one component remains.
- Suppose instead that we start with an arbitrary vertex r, and then add edges while ensuring that E_T is just a single tree at each stage.
- This is the essence of Prim's algorithm:

```
let VT ← {r}, ET ← empty set
while (VT ≠ V)
    find a light edge (u,v) for some u∈VT and v∉VT
    add v to VT; add (u,v) to ET
(V, ET) is the required MST
```

Why does this work?

- V_T cuts V into two pieces: V_T and $(V - V_T)$;
- The edges that we add to E_T are light edges across the cut;
- Hence, they are safe to add.

Choosing the edges:

- Store all vertices that are <u>not</u> in (V_T, E_T) in a priority queue Q with an extractMin operation.

- If u is a vertex in Q, what's key[u] (the value that determines u's position in Q)?
  - key[u] = minimum weight of edge from u into V_T
  - if no such edge exists, key[u] = ∞.

- We maintain information about the parent (in (V_T, E_T)) of each vertex v in an array parent[].
  - E_T is kept implicitly as $\{(v, parent[v]) \mid v \in V - Q - \{r\}\}$.

- The input is the graph $G=(V, E)$, and a root $r \in V$.

```
for each v in V
   key[v]    ← ∞;
   parent[v] ← null;
key[r] ← 0;
add all vertices in V to the queue Q.

while (Q is nonempty) {
   u ← extractMin(Q);
   for each vertex v that is adjacent to u {
      if v ∈ Q and weight(u,v) < key[v] {
         parent[v] ← u;
         key[v]    ← weight(u,v);
      }
   }
}
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How can this test be implemented in O(1)?

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        }
    }
}
```

How can this test be implemented in O(1)?

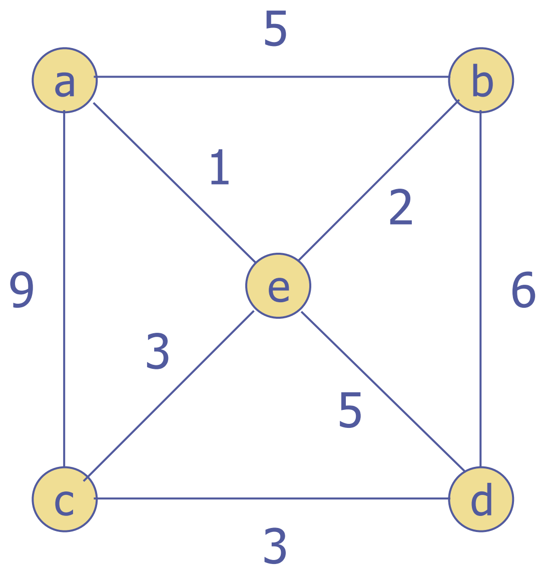
When we change a key, we will also need to readjust the priority queue

Complexity:

- Assuming a binary heap …
  - Initialization takes $O(|V|)$ time.
  - Main loop is executed $|V|$ times, and each extractMin takes $O(\log |V|)$.
  - The body of the inner loop is executed a total of $O(|E|)$ times; each adjustment of the queue takes $O(\log |V|)$ time.
- Overall complexity: $O((|V|+|E|) \log |V|) = O(|E| \log |V|)$.

Your turn!

- Apply Prim's algorithm to this graph:



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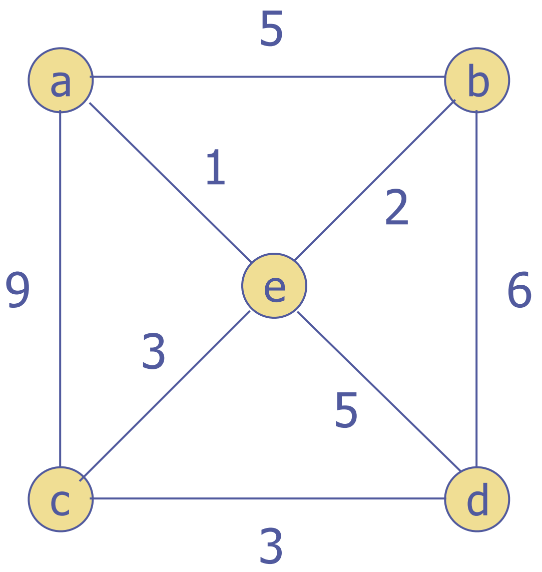

Tree vertices	Priority Queue of remaining vertices

Your turn!

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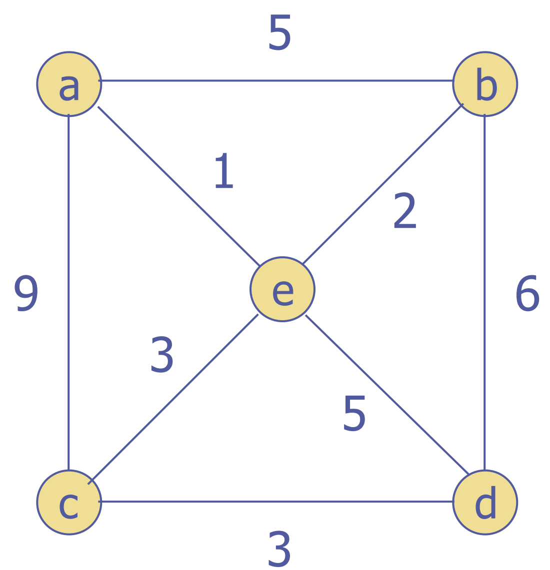

Tree vertices	Priority Queue of remaining vertices
a(−, −)	

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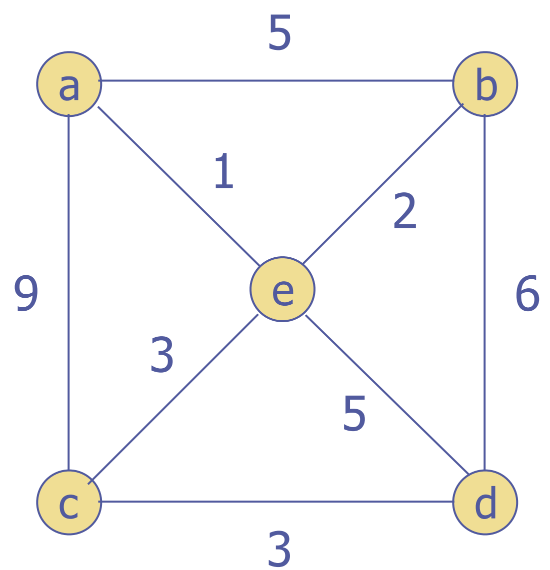

parent

Tree vertices	Priority Queue of remaining vertices
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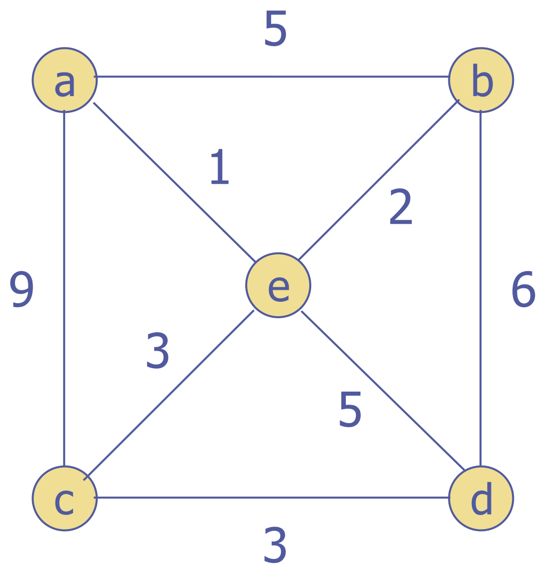
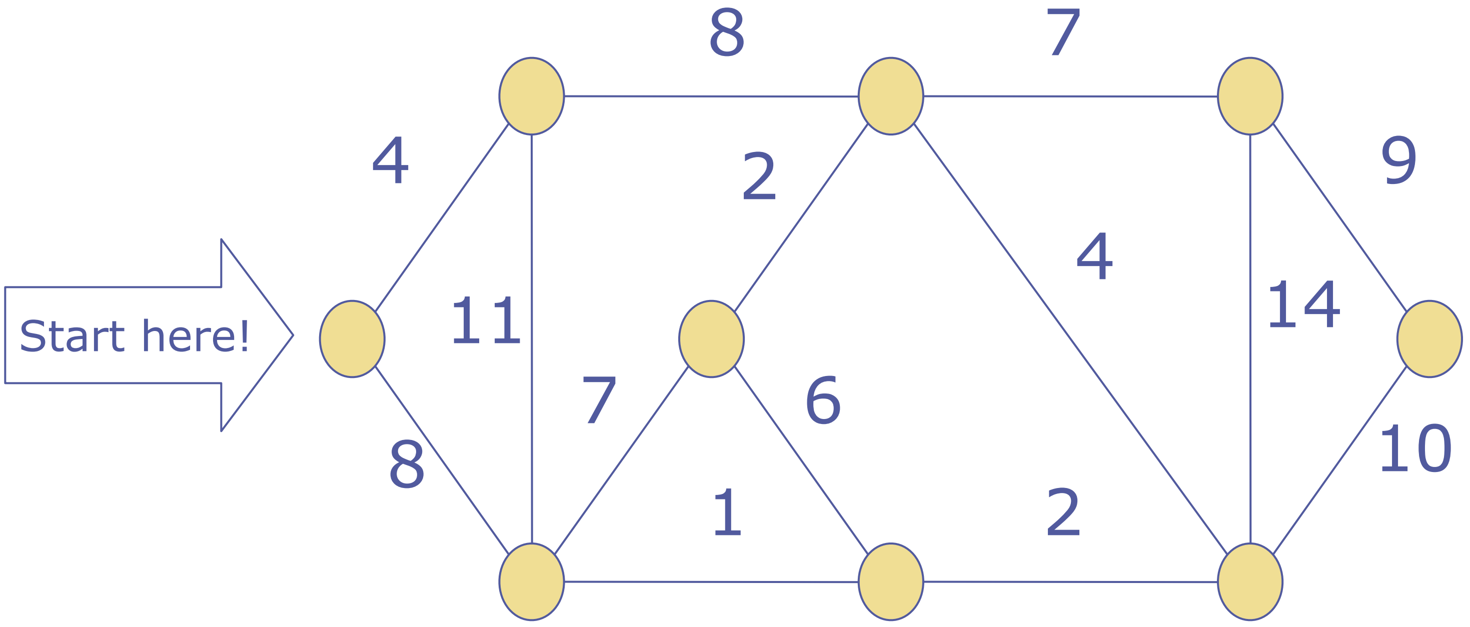


weight of edge

parent

Tree vertices	Priority Queue of remaining vertices
a(−, −)	

Apply Prim's Algorithm



Try it out!