CS 350 Algorithms and Complexity

Winter 2019

Lecture 3: Analyzing Non-Recursive Algorithms

Andrew P. Black

Department of Computer Science
Portland State University
Analysis of time efficiency

✧ Time efficiency is analyzed by determining the number of repetitions of the “basic operation”

✧ Almost always depends on the size of the input

✧ “Basic operation”: the operation that contributes most towards the running time of the algorithm

\[ T(n) \approx c_{op} \times C(n) \]
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\[ T(n) \approx c_{op} \times C(n) \]

- run time
- number of times basic op is executed
- cost of basic op: constant
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Best-case, average-case, worst-case

- For some algorithms, efficiency depends on the input:
  - **Worst case:** \( C_{\text{worst}}(n) \) – maximum over inputs of size \( n \)
  - **Best case:** \( C_{\text{best}}(n) \) – minimum over inputs of size \( n \)
  - **Average case:** \( C_{\text{avg}}(n) \) – “average” over inputs of size \( n \)

  - Number of times the basic operation will be executed on typical input
    - *Not* the average of worst and best case
  - Expected number of basic operations under some assumption about the probability distribution of all possible inputs
Discuss:

\textbf{ALGORITHM} \textit{UniqueElements}(A[0..n - 1])

// Determines whether all the elements in a given array are distinct
// Input: An array A[0..n - 1]
// Output: Returns “true” if all the elements in A are distinct
// and “false” otherwise

\begin{verbatim}
for i \leftarrow 0 \text{ to } n - 2 \text{ do }
   for j \leftarrow i + 1 \text{ to } n - 1 \text{ do }
return true
\end{verbatim}

\begin{itemize}
  \item What’s the best case, and its running time?
  \begin{itemize}
    \item A. constant \hspace{1em} O(1)
    \item B. linear \hspace{1em} O(n)
    \item C. quadratic \hspace{1em} O(n^2)
  \end{itemize}
\end{itemize}
Discuss:

**ALGORITHM**  \textit{UniqueElements}(A[0..n-1])

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✧ What’s the worst case, and its running time?

A. constant — \( O(1) \)
B. linear — \( O(n) \)
C. quadratic — \( O(n^2) \)
Discuss:

**ALGORITHM**  \( \text{UniqueElements}(A[0..n - 1]) \)

// Determines whether all the elements in a given array are distinct
// Input: An array \( A[0..n - 1] \)
// Output: Returns "true" if all the elements in \( A \) are distinct
// and "false" otherwise

\[
\text{for } i \leftarrow 0 \text{ to } n - 2 \text{ do}
\]
\[
\text{for } j \leftarrow i + 1 \text{ to } n - 1 \text{ do}
\]
\[
\text{if } A[i] = A[j] \text{ return false}
\]

return true

✦ What’s the average case, and its running time?

A. constant — \( O(1) \)
B. linear — \( O(n) \)
C. quadratic — \( O(n^2) \)
General Plan for Analysis of non-recursive algorithms

1. Decide on parameter $n$ indicating input size

2. Identify algorithm’s basic operation

3. Determine worst, average, and best cases for input of size $n$

4. Set up a sum for the number of times the basic operation is executed

5. Simplify the sum using standard formulae and rules (see Levitin Appendix A)
“Basic Operation”

ALGORITHM  MaxElement(A[0..n − 1])

//Determines the value of the largest element in a given array
//Input: An array A[0..n − 1] of real numbers
//Output: The value of the largest element in A
maxval ← A[0]
for i ← 1 to n − 1 do
    if A[i] > maxval
        maxval ← A[i]
return maxval

Why choose > as the basic operation?

- Why not  i ← i + 1?
- Or [ ]?
Same Algorithm:

ALGORITHM MaxElement (A: List)
    // Determines the value of the largest element in the list A
    // Input: a list A of real numbers
    // Output: the value of the largest element of A
    maxval ← A.first
    for each in A do
        if each > maxval
            maxval ← each
    return maxval

Why choose > as the basic operation?

- Why not $i ← i + 1$?
- Or $[ ]$?
From Algorithm to Formula

✧ We want a formula for the # of basic ops
✧ Basic op will normally be in inner loop
✧ Bounds of for loop become bounds of summation
✧ e.g. for i ← l .. h do:

3 basic operations

\[ \sum_{i=l}^{h} 3 \]
Works for nested loops too

**ALGORITHM**  *UniqueElements(A[0..n − 1])*

// Determines whether all the elements in a given array are distinct
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Works for nested loops too

**ALGORITHM**  \textit{UniqueElements}(A[0..n - 1])

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        \textbf{if} \ A[i] = A[j] \ \textbf{return} \ false
    \textbf{return} \ true

\[ \sum_{i=0}^{n-2} \binom{n}{i} \]
Works for nested loops too

**Algorithm**  
*UniqueElements*(*A*[0..*n* − 1])

//Determines whether all the elements in a given array are distinct
//Input: An array *A*[0..*n* − 1]
//Output: Returns “true” if all the elements in *A* are distinct
// and “false” otherwise

for *i* ← 0 to *n* − 2 do
  for *j* ← *i* + 1 to *n* − 1 do
    if *A*[i] = *A*[j] return false
return true

\[
\sum_{i=0}^{n-2} \left( \sum_{j=i+1}^{n-1} 1 \right)
\]
Useful Summation Formulae

$\sum_{1 \leq i \leq u} 1 = $ $u - l + 1$

In particular, $\sum_{1 \leq i \leq n} 1 = n$

$\sum_{1 \leq i \leq n} i = $ $\frac{n(n+1)}{2}$

$\sum_{1 \leq i \leq n} i^2 = $ $\frac{n(n+1)(2n+1)}{6}$

$\sum_{0 \leq i \leq n} a^i = $ $\frac{a^{n+1} - 1}{a - 1}$

In particular, $\sum_{0 \leq i \leq n} 2^i = 2^{n+1} - 1$

$\sum (a_i \pm b_i) = $ $\sum c a_i = $ $\sum_{l \leq i \leq u} a_i = $
Useful Summation Formulae

\[ \sum_{l \leq i \leq u} 1 = 1 + 1 + \ldots + 1 = u - l + 1 \]

In particular, \( \sum_{1 \leq i \leq n} 1 = n \)

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\[ \sum_{1 \leq i \leq n} i^2 = \]

\[ \sum_{0 \leq i \leq n} a^i = \]

In particular, \( \sum_{0 \leq i \leq n} 2^i = 2 \cdot (2^n - 1) \)

\[ \sum (a_i \pm b_i) = \]
\[ \sum_{l \leq i \leq u} a_i = \]

\[ \sum c \cdot a_i = \]
Useful Summation Formulae

\[ \sum_{l \leq i \leq u} 1 = 1 + 1 + \ldots + 1 = u - l + 1 \]

In particular, \( \sum_{1 \leq i \leq n} 1 = n - 1 + 1 = n \in \Theta(n) \)

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\[ \sum_{0 \leq i \leq n} a^i = \]

In particular, \( \sum_{0 \leq i \leq n} 2^i = 2 \)

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\[ \sum_{1 \leq i \leq n} i = 1 + 2 + \ldots + n = \frac{n(n+1)}{2} \approx \frac{n^2}{2} \in \Theta(n^2) \]

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In particular, \( \sum_{0 \leq i \leq n} 2^i = 2 \)

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In particular, \( \sum_{0 \leq i \leq n} 2^i = 2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1 \in \Theta(2^n) \)

\[ \sum (a_i \pm b_i) = \sum a_i \pm \sum b_i \]

\[ \sum_{l \leq i \leq u} a_i = \sum_{l \leq i \leq m} a_i + \sum_{m+1 \leq i \leq u} a_i \]

\[ \sum c \, a_i = c \sum a_i \]
Where do the Summation formulae come from?

✧ Answer: mathematics.

✧ Example:

The Euler–Mascheroni constant $\gamma$ is defined as:

$$
\gamma = \lim_{n \to \infty} \left( \sum_{i=1}^{n} \frac{1}{i} - \ln n \right)
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✧ Answer: mathematics.

✧ Example:

The Euler–Mascheroni constant $\gamma$ is defined as:

$$\gamma = \lim_{n \to \infty} \left( \sum_{i=1}^{n} \frac{1}{i} - \ln n \right)$$
Where do the Summation formulae come from?

✧ **Answer:** mathematics.

✧ **Example:**

The Euler–Mascheroni constant $\gamma$ is defined as:

\[
\gamma = \lim_{n \to \infty} \left( \sum_{i=1}^{n} \frac{1}{i} - \ln n \right)
\]
What does Levitin’s \( \approx \) mean?

✧ “becomes almost equal to as \( n \to \infty \)”

✧ So formula 8

\[
\lim_{n \to \infty} \left( \sum_{i=1}^{n} \lg i - n \lg n \right) = 0
\]
Example: Counting Binary Digits

ALGORITHM  Binary(n)

//Input: A positive decimal integer n
//Output: The number of binary digits in n’s binary representation

count ← 1
while n > 1 do
    count ← count + 1
    n ← ⌊n/2⌋
return count
Example: Counting Binary Digits

**ALGORITHM**  \( \text{Binary}(n) \)

//Input: A positive decimal integer \( n \)
//Output: The number of binary digits in \( n \)'s binary representation

\[ \text{count} \leftarrow 1 \]

**while** \( n > 1 \) **do**

\[ \text{count} \leftarrow \text{count} + 1 \]

\[ n \leftarrow \lfloor n/2 \rfloor \]

**return** \( \text{count} \)

✧ How many times is the basic operation executed?
Example: Counting Binary Digits

**Algorithm**  
*Binary(n)*

// Input: A positive decimal integer n
// Output: The number of binary digits in n’s binary representation

`count ← 1`

**while** `n > 1` **do**

`count ← count + 1`

`n ← [n/2]`

**return** `count`

✧ How many times is the basic operation executed?

✧ Why is this algorithm harder to analyze than the earlier examples?
Ex 2.3, Problem 1

Working with a partner:

1. Compute the following sums.

   a. \( 1 + 3 + 5 + 7 + \ldots + 999 \)

   b. \( 2 + 4 + 8 + 16 + \ldots + 1024 \)

   c. \( \sum_{i=3}^{n+1} 1 \)

   d. \( \sum_{i=3}^{n+1} i \)

   e. \( \sum_{i=0}^{n-1} i(i + 1) \)

   f. \( \sum_{j=1}^{n} 3^{j+1} \)

   g. \( \sum_{i=1}^{n} \sum_{j=1}^{n} ij \)

   h. \( \sum_{i=1}^{n} 1/i(i + 1) \)
Ex 2.3, Problem 2

2. Find the order of growth of the following sums.

a. \[\sum_{i=0}^{n-1} (i^2 + 1)^2\]

b. \[\sum_{i=2}^{n-1} \lg i^2\]

c. \[\sum_{i=1}^{n} (i + 1)2^{i-1}\]

d. \[\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} (i + j)\]

Use the \(\Theta(g(n))\) notation with the simplest function \(g(n)\) possible.
Ex 2.3, Problem 3

3. The sample variance of \( n \) measurements \( x_1, x_2, \ldots, x_n \) can be computed as

\[
\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n - 1}
\]

where \( \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \)

or

\[
\frac{\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2/n}{n - 1}
\]

Find and compare the number of divisions, multiplications, and additions/subtractions (additions and subtractions are usually bunched together) that are required for computing the variance according to each of these formulas.
Ex 2.3, Problem 4

4. Consider the following algorithm.

Algorithm Mystery($n$)
//Input: A nonnegative integer $n$
$S \leftarrow 0$
for $i \leftarrow 1$ to $n$ do
  $S \leftarrow S + i \times i$
return $S$
Ex 2.3, Problem 4

4. Consider the following algorithm.

Algorithm Mystery\((n)\)

//Input: A nonnegative integer \(n\)
\(S \leftarrow 0\)
for \(i \leftarrow 1\) to \(n\) do
\(S \leftarrow S + i \times i\)
return \(S\)

What does this algorithm compute?

A. \(n^2\)
B. \(\sum_{i=1}^{n} i\)
C. \(\sum_{i=1}^{n} i^2\)
D. \(\sum_{i=1}^{n} 2i\)
Ex 2.3, Problem 4

4. Consider the following algorithm.

**Algorithm Mystery** \( n \)
// Input: A nonnegative integer \( n \)
\( S \leftarrow 0 \)
\( \text{for } i \leftarrow 1 \text{ to } n \text{ do } \)
\( \qquad S \leftarrow S + i \times i \)
\( \text{return } S \)
Ex 2.3, Problem 4

4. Consider the following algorithm.

Algorithm Mystery(n)
//Input: A nonnegative integer n
S ← 0
for i ← 1 to n do
   S ← S + i * i
return S

What is the basic operation?

A. multiplication
B. addition
C. assignment
D. squaring
Ex 2.3, Problem 4

4. Consider the following algorithm.

Algorithm Mystery(n)
//Input: A nonnegative integer n
S ← 0
for i ← 1 to n do
    S ← S + i * i
return S

How many times is the basic operation executed?

A. once
B. n times
C. lg n times
D. none of the above
Ex 2.3, Problem 4

4. Consider the following algorithm.

```
Algorithm Mystery(n)
//Input: A nonnegative integer n
S ← 0
for i ← 1 to n do
   S ← S + i * i
return S
```

What is the efficiency class of this algorithm? [b is # of bits needed to represent n]

A. Θ(1)

B. Θ(n)

C. Θ(b)

D. Θ(2^b)
e. Suggest an improvement or a better algorithm altogether and indicate its efficiency class. If you cannot do it, try to prove that, in fact, it cannot be done.
Problem 5 — Group work

5. Consider the following algorithm.

Algorithm Secret(A[0..n − 1])
//Input: An array A[0..n − 1] of n real numbers
minval ← A[0];  maxval ← A[0]
for i ← 1 to n − 1 do
    if A[i] < minval
        minval ← A[i]
    if A[i] > maxval
        maxval ← A[i]
return maxval − minval

a. What does this algorithm compute?
b. What is its basic operation?
c. How many times is the basic operation executed?
d. What is the efficiency class of this algorithm?
e. Suggest an improvement or a better algorithm altogether and indicate its efficiency class.
Ex 2.3, Problem 9

Prove the formula

\[ \sum_{i=1}^{n} i = 1 + 2 + ... + n = \frac{n(n + 1)}{2} \]

either by mathematical induction or by following the insight of a 10-year old schoolboy named Karl Friedrich Gauss (1777–1855) who grew up to become one of the greatest mathematicians of all times.
Ex 2.3, Problem 11

Algorithm $GE(A[0..n-1, 0..n])$

//Input: An $n$-by-$n + 1$ matrix $A[0..n-1, 0..n]$ of real numbers

for $i \leftarrow 0$ to $n - 2$ do
    for $j \leftarrow i + 1$ to $n - 1$ do
        for $k \leftarrow i$ to $n$ do

a. Find the time efficiency class of this algorithm
b. What glaring inefficiency does this code contain, and how can it be eliminated?
c. Estimate the reduction in run time.
Problem 11: von Neumann neighborhood

How many one-by-one squares are generated by the algorithm that starts with a single square, and on each of its $n$ iterations adds new squares around the outside. How many one-by-one squares are generated on the $n^{th}$ iteration? Here are the neighborhoods for $n = 0, 1, \text{ and } 2$. 

$n = 0$

$n = 1$

$n = 2$