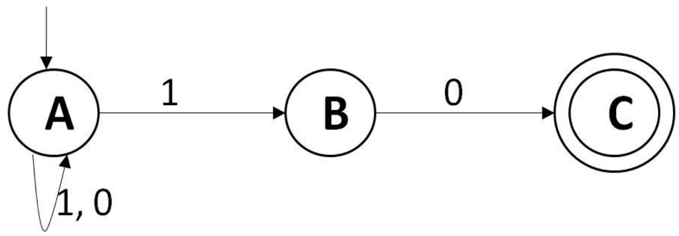


b) Prove that the NFA accepts the language described in (a) using induction on the length of the input.

We use mutual induction on the three states and their properties. The induction hypothesis is given by $H(n)$ = for all strings $w$ of length $\leq n$, the following three statements are true.

1) $A \in \hat{\delta}(A,w)$ ;
2) $B \in \hat{\delta}(A,w) \equiv w$ ends with a 1 ; and
3) $C \in \hat{\delta}(A,w) \equiv w$ ends with 10.

To prove these statements, we will have to consider how the NFA can reach each state. Also note that if the machine is to accept a string $x$, then $\hat{\delta}(A, x)$ must contain C as C is the only accepting state. Thus, proof of the third clause in the induction hypothesis, for $w$ of arbitrary length, will prove that the NFA accepts only those strings ending in 10.

**Basis:** H(0)

Since $|w| \leq 0$, we know that $w=\varepsilon$. Because A is the start state of the NFA, and there are no ε-transitions out of A, $\hat{\delta}(A, \varepsilon) = \{A\}$. Hence, statement 1 holds. Statement 2 also holds, because ε does not end in 1, and $B \notin \hat{\delta}(A, \varepsilon)$, so both sides of statement 2 are false. Similarly, $C \notin \hat{\delta}(A, \varepsilon)$, and ε does not end in 10, so Statement 3 holds, because both sides are false.

**Induction**: $H(n)$ □ $H(n + 1)$

Assume that $w=x\,a$, where $a$ is a symbol which is either 0 or 1, and $|x|= n$. Thus, the induction hypothesis holds for $x$ and our task is to prove it for $w$, since $|w|= n+1$.

1. $A \in \hat{\delta}(A, xa)$. The induction hypothesis clause 1 tells us that $A \in \hat{\delta}(A,x)$. $a$ is either 0 or 1, and transitions on both 0 and 1 go from A to A. Thus statement 1 is proved.
2. If $a= 1$, the rhs of statement 2 is true. Using statement 1, we know that $A \in \hat{\delta}(A, x)$. Since there is a transition on 1 from A to B, we know that $B \in \hat{\delta}(A, xa)$, and the lhs is also true.
   If $a= 0$, the rhs of statement 2 is false. Thus we have to show that $B \notin \hat{\delta}(A,xa)$. This is clearly the case, since the only transition into B is labeled with a 1.
3. We prove this equivalence in two parts.
   [⇐] Suppose $w$ ends in 10. So if $w=xa$, $a=0$ and $x$ ends in 1. By statement 2 applied to $x$, $B \in \hat{\delta}(A, x)$. Since $\delta(B, 0) = \{C\}$, $\hat{\delta}(A, w) = \hat{\delta}(A, x0) = \delta(\hat{\delta}(A, x), 0)$ contains C.
   [⇒] Suppose $\hat{\delta}(A, w)$ contains C. From the diagram, it is evident that the only way to get to C is for $w$ to be of the form $x0$, where $B \in \hat{\delta}(A, x)$. By statement 2 applied to x, we know that x ends in 1. Thus $w$ ends in 10.
   Thus statement 3 is proved.

Notice that we needed only the third statement in the induction hypothesis to prove that the NFA accepts only those strings ending in 10. However, in the proof of statement 3 (for $w$ of length n+1) we needed to assume statement 2 (for $w$ of length n). Moreover, in the proof of statement 2, we needed to assume statement 1. This is why we needed all three parts of the induction hypothesis.