

CS 311 Homework 3

due 16:30, Thursday, 14th October 2010

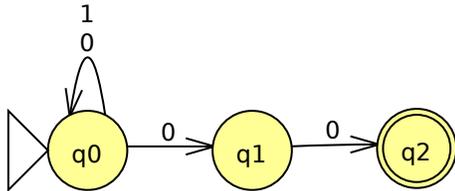
Homework must be submitted on paper, in class.

Question 1. [15 pts.; 5 pts. each]

Draw state diagrams for NFAs recognizing the following languages:

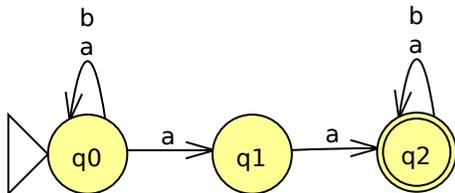
a. $L = \{w \mid w \in \{0, 1\}^* \text{ and } w \text{ ends with } 00\}$.

Answer.



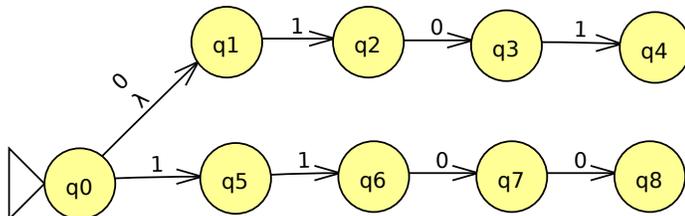
b. L is the language represented by the regular expression $(a + b)^*aa(a + b)^*$.

Answer.



c. $L = \{0101, 101, 1100\}$.

Answer.

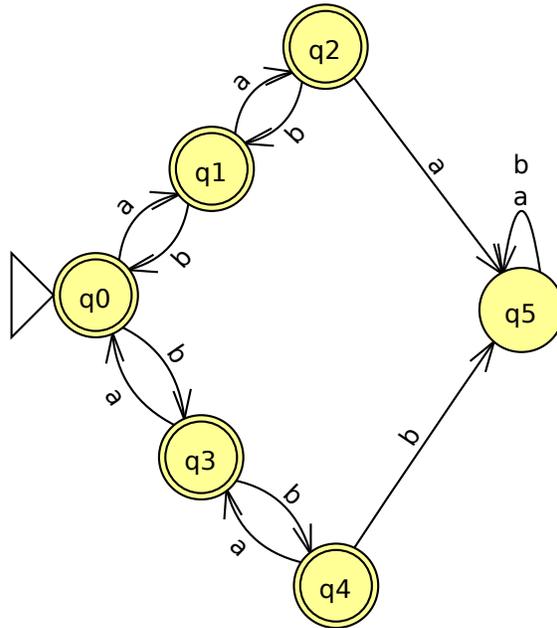


Question 2. [30 pts.; *a*, *b* 5 pts.; *c* 20 pts.]

$L^{\mathcal{R}}$ is the language of strings which are the reverse of the strings in the language L . For $L = \{w \mid w \in \{a, b\}^* \text{ and in each initial substring of } w, |\text{number of as} - \text{the number of bs}| \leq 2\}$ complete the following exercises:

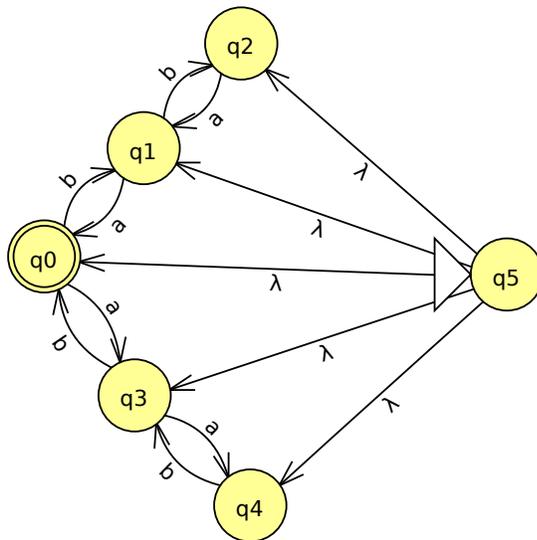
- a.* Construct the diagram of a DFA which accepts strings in the language L .

Answer.



b. Construct the diagram of an NFA which accepts strings in the language L^R .

Answer.



c. Convert the NFA to a DFA. Be sure to provide all five components of the DFA, show the transition table for δ , and provide a diagram. You may relabel the states for

convenience, if you like. If you do, be sure to show the intermediate steps where relabeling takes place.

Answer.

Using the lazy construction method, we do ε -closure on state $q5$ to obtain the initial state $q0q1q2q3q4q5$ and then build the transition table from there. Note the relabeling in the same table.

δ		a		b	
$q0q1q2q3q4q5$	A	$q0q1q3q4$	B	$q0q1q2q3$	C
$q0q1q3q4$	B	$q0q3q4$	D	$q0q1q2q3$	C
$q0q1q2q3$	C	$q0q1q3q4$	B	$q0q1q2$	F
$q0q3q4$	D	$q3q4$	G	$q0q1q3$	H
$q0q1q2$	F	$q0q1q3$	H	$q1q2$	I
$q3q4$	G	$q4$	L	$q0q3$	J
$q0q1q3$	H	$q0q3q4$	D	$q0q1q2$	F
$q1q2$	I	$q0q1$	K	$q2$	M
$q0q3$	J	$q3q4$	G	$q0q1$	K
$q0q1$	K	$q0q3$	J	$q1q2$	I
$q4$	L	\emptyset	Q	$q3$	N
$q2$	M	$q1$	O	\emptyset	Q
$q3$	N	$q4$	L	$q0$	P
$q1$	O	$q0$	P	$q2$	M
$q0$	P	$q3$	N	$q1$	O
\emptyset	Q	\emptyset	Q	\emptyset	Q

We use all states that contain $q0$ as accepting states giving

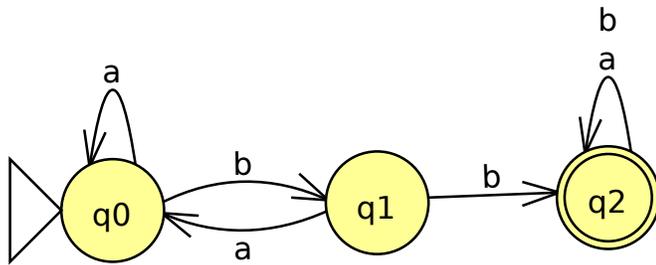
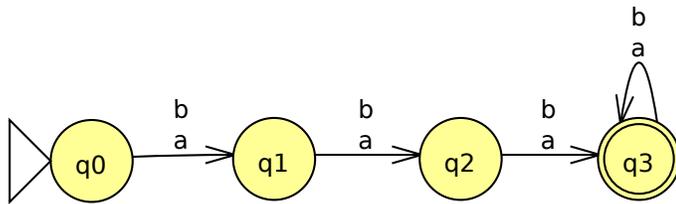
$$F = \{A, B, C, D, F, H, J, K, P\}.$$

Where δ is as defined in the table, we have

$$M = \langle \{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q\}, \{a, b\}, \delta, A, \{A, B, C, D, F, H, J, K, P\} \rangle$$

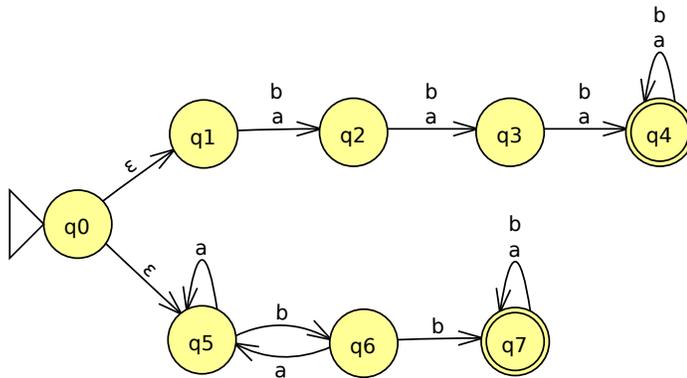
Here is the diagram. Look how pretty it is!

Answer.



b. Construct the diagram of an NFA which accepts strings in the language $L_1 \cup L_2$.

Answer.



c. Convert the NFA to a DFA, in the same way as in Question 2c.

Answer.

We define a new initial state based on the ϵ -closure of the initial state q_0 giving $q_0q_1q_5$. Now we lazily construct the transition table.

δ		a		b	
$q_0q_1q_5$	A	q_2q_5	B	q_2q_6	C
q_2q_5	B	q_3q_5	D	q_3q_6	E
q_2q_6	C	q_3q_5	D	q_3q_7	F
q_3q_5	D	q_4q_5	G	q_4q_6	H
q_3q_6	E	q_4q_5	G	q_4q_7	I
q_3q_7	F	q_4q_7	I	q_4q_7	I
q_4q_5	G	q_4q_5	G	q_4q_6	H
q_4q_6	H	q_4q_5	G	q_4q_7	I
q_4q_7	I	q_4q_7	I	q_4q_7	I

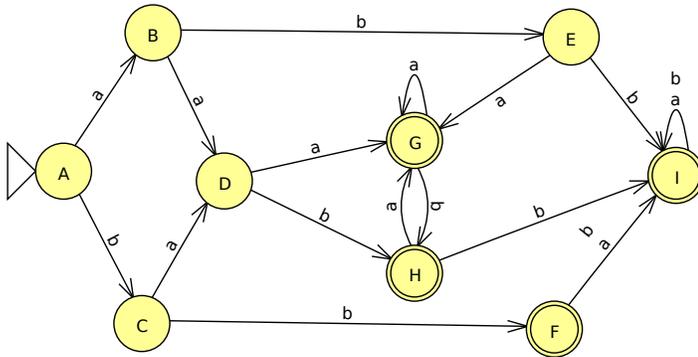
The accepting states are those states that are accepted by either one machine or the other. That would be all the states that include q_4 or q_7 . Thus we have

$$F = \{F, G, H, I\}$$

and we can define the entire machine with δ from the transition table above.

$$M = \langle \{A, B, C, D, E, F, G, H, I\}, \{a, b\}, \delta, A, \{F, G, H, I\} \rangle$$

And the pretty picture, with no minimization...



- d. Define a new set of accepting states that will make the DFA accept strings in the language $L_1 \cap L_2$. Explain briefly the difference between this new set of accepting states and the set defined in part c. of this question.

Answer.

The set of accepting states for the union of two DFAs is simply the set of all states in the union which contain one of the accepting states from the original machines. Formally this is denoted

$$F = \{F_1 \times Q_2 \cup Q_1 \times F_2\}$$

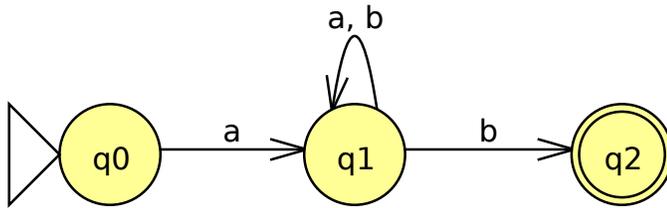
For the intersection of two DFAs we want the accepting states to be those states which include accepting states from *both* machines. Formally this would be

$$F = \{F_1 \times F_2\}$$

In the case of this specific problem the new set of accepting states would be $F = \{I\}$.

Question 4. [25 pts; a 7 pts; b 18 pts.]

- a. Describe the language accepted by the following NFA:



Answer.

This NFA accepts all strings over $\{a, b\}^*$ that begin with a and end with b . This can be described by the regular expression $a(a + b)^*b$.

- b. Prove that this NFA accepts the language that you described in part (a). In your proof, use induction on the length of the input. Be sure to state your induction hypothesis explicitly. *Hint:* You will use what Hopcroft et al. call “mutual induction” (§1.4.4) (*end of hint*).

Answer.

We will complete this proof using mutual induction, where a set of conditions is created to describe the action of the machine. This set of conditions will demonstrate that the machine behaves as described above.

First, the three conditions.

- (a) $q0 \in \hat{\delta}(q0, w) \Leftrightarrow w = \varepsilon$.
- (b) $q1 \in \hat{\delta}(q0, w) \Leftrightarrow w$ begins with **a**
- (c) $q2 \in \hat{\delta}(q0, w) \Leftrightarrow w$ begins with **a** and ends with **b**

Our task is to verify that these conditions hold for *all* input and that the appropriate input leads to acceptance.

Base Case For $|w| = 0$, we know the only string with magnitude 0 is ε , so $w = \varepsilon$.

- (a) Because $q0$ is the initial state of the machine, and there are no ε -transitions from $q0$, then $q0 \in \hat{\delta}(q0, w)$ so the left hand side holds. Since $w = \varepsilon$, the right hand side holds as well and the biconditional is true.

- (b) Since there are no ε -transitions from the initial state to $q1$, it follows that $q1 \notin \hat{\delta}(q0, w)$ and the left hand side is false. It is clear that w does not start with **a** and so the right hand side is false as well. Therefore the biconditional is true.
- (c) By the same reasoning as (b), it follows that condition (c) is also true.

Inductive Hypothesis Assume that for $|w| = n$, all three conditions hold.

Inductive Step We now prove that the three conditions hold for $|w| = n + 1$. Let $w = xz$ where $|x| = n$ and $z \in \Sigma$. Thus $|w| = |x| + 1 = n + 1$.

- (a) There are no transitions into $q0$, so for any w such that $|w| \geq 1$, $q0 \notin \hat{\delta}(q0, w)$ and the left hand side is false. Since clearly $w \neq \varepsilon$, the right hand side is also false and the biconditional is true.
- (b) Consider the prefix string x . By the inductive hypothesis, all the conditions hold for x , which means that condition (b) holds for x . This means there are two cases to consider. The first, x starts with **a**, means that $q1 \in \hat{\delta}(q0, x)$. Since there is a transition from $q1$ to $q1$ on both **a** or **b**, it does not matter what symbol z is, it follows that $q1 \in \hat{\delta}(q0, w)$. In this case, no matter what z is, x will still start with **a** and thus w will start with **a**. Thus the biconditional will always be true. The second case, x starts with **b**, means that $q1 \notin \hat{\delta}(q0, x)$. Further, because there is no transition out of $q0$ on a **b**, the machine is “stuck”. Thus it does not matter what symbol z is, it will always be the case that $q1 \notin \hat{\delta}(q0, w)$. Since x starts with **b** it follows that no matter what z is, w will also start with **b**. Thus both sides of the biconditional will always be false and the biconditional itself will always be true.
- (c) The only transition into $q2$ is on a **b** from $q1$. So any case which does not ensure $q1 \in \hat{\delta}(q0, x)$ implies that $q2 \notin \hat{\delta}(q0, w)$ and from condition (b) we know that if x does not start with **a** then $q1 \notin \hat{\delta}(q0, x)$ and it follows that $q2 \notin \hat{\delta}(q0, w)$. Thus in this situation, the biconditional will always be true. Now consider the situation where $q1 \in \hat{\delta}(q0, x)$. By the inductive hypothesis and (b), we know this implies that x starts with **a**. Here we have two cases: $z = \mathbf{a}$ and $z = \mathbf{b}$. In the first case, $z = \mathbf{a}$, it follows that w ends with **a**, and not **b**. Since there is no transition from $q1$ to $q2$ on an **a**, then we can conclude that $q2 \notin \hat{\delta}(q0, w)$. Likewise, if $q2 \notin \hat{\delta}(q0, w)$ then we can surmise that the transition from $q1$ to $q2$ on a **b** was not taken and therefore, w does not end in **b**. Thus the biconditional is true in both directions. In the second case, $z = \mathbf{b}$, it follows that w ends in **b**. Since we know $q1 \in \hat{\delta}(q0, x)$ and there is a transition from $q1$ to $q2$ on **b**, it follows that $q2 \in \hat{\delta}(q0, w)$. Likewise, if $q2 \in \hat{\delta}(q0, w)$, since the only transition into $q2$ is from $q1$ on a **b**, then it follows that w ends with **b**. Again the biconditional is true in both directions.

Since in condition (c), it is required to pass through $q1$ to get to $q2$, and $q2$ is the only accepting state of the machine, it follows that $q1 \in \hat{\delta}(q0, x)$ is necessary for acceptance,

and thus w must start with **a**. Again, by condition (c), for $q_2 \in \hat{\delta}(q_0, w)$ to be true, then w must end with **b**. Thus we show that acceptance by this machine only occurs for strings that start with **a** and end with **b**.

I don't hold that this proof is completely bullet-proof. If there are any holes, please feel free to point them out, and I'll make corrections. It is tricky – there are a lot of cases to consider all the way through. But hopefully, this proof illustrates the process reasonably well.