Sample Proof – CFG Pumping Lemma

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November 5, 2010

Use the pumping lemma to prove that the following language is not context free.

$L = \{0^n 1^n 0^n 1^n \mid n \geq 0\}$

Proof. Assume that L is context free. Then by the pumping lemma for context free languages, there must be a pumping length p such that if s is a string in the language with magnitude greater than p, then s satisfies the conditions of the pumping lemma.

Let $s = 0^p 1^p 0^p 1^p$. Clearly $|s| \geq p$, as required by the pumping lemma. Now, according to the pumping lemma, $s = uvxyz$ with $|vxy| \leq p$. This means, there are three cases that describe vxy.

1. vxy is comprised of all 0s and is contained entirely within either the first or second string of 0s. Since $|vy| > 0$, then either v or y must contain at least one 0. Now consider uv^0xy^0z. This forces either the first or the second string of 0s to have at least one fewer 0s than the other. Thus $uv^0xy^0z \notin L$ which is a contradiction of the pumping lemma.

2. vxy is comprised of all 1s and is contained entirely within either the first or the second string of 1s. By the same reasoning in 1., we can see that a contradiction derived.

3. vxy is comprised of a mix of 0s and 1s. This really describes two cases, where vxy is a string of 0s followed by a string of 1s or vxy is a string of 1s followed by a string of 0s. We take the first case to be representative. In this case vxy either straddles the first 0-1 division or it straddles the

second 0-1 division. Again, because $|vxy| \leq p$, it follows that pumping either up or down will only affect the substrings immediately adjacent to the division that is straddled. The other two substrings will be unaffected. Thus the length of the straddled substrings will be changed by pumping while the length of the other two will not be. Thus the result of pumping will result in a string that is not in the language, and a contradiction is again derived.

Since for every case, s cannot be pumped, we have a contradiction with the pumping lemma. Therefore our original assumption was false and L is not context free. $\square$