Example Proof using the Pumping Lemma for Regular Languages

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Prove that the language
\[ E = \{ w \in (01)^* | w \text{ has an equal number of 0s and 1s} \} \]
is not regular.

Proof

We prove the required result by contradiction. So, we assume that \( E \) is regular. Then, by the pumping lemma, there is a pumping length \( p \) such that all strings \( s \) in \( E \) of length \( p \) or more can be written as \( s = xyz \) where

1. \( y \neq \Lambda \)
2. \( |xy| \leq p \), and
3. \( xy^iz \in E \), for all \( i \geq 0 \)

Consider the string \( s = 0^p1^p \). Clearly, \( p \in E \) and \( |s| \geq p \), so we should be able to find a decomposition of \( s \) into \( xyz \) that meets conditions 1–3 above.

How about \( x = z = \Lambda \), \( y = 0^p1^p \)? This meets conditions 1 and 3. But no, it fails to meet condition 2. If \( |xy| \leq p \), then \( xy \) must contain just 0s and no 1s. Hence, \( y \) must contain just 0s and no 1s. So, if \( s = xyz \in E \), it follows that \( xz \notin E \), since \( xz \) has fewer 0s than \( xyz \) but the same number of 1s.

Thus, we have found a string in \( E \) that cannot be pumped, which contradicts the assumption that \( E \) is regular.

\[ \square \]

Note that we get to choose a string \( s \) to suit our purposes. If, instead, we had chosen \( (01)^p \), then we would not have been able to complete the proof. Why not? Because that particular string can be pumped. But this is not a problem: the lemma says that all strings can be pumped, so all that we need do is find one string that cannot be pumped, and we have the contradiction that we are looking for.
What is the minimum pumping length for the language \( L = 0001^* \)?

The minimum pumping length for a language \( L \) is the smallest \( p \) such that all strings of length \( p \) or more can be pumped.

In \( L = 0001^* \):

1. \( 000 \) can’t be pumped (because \( 0000 \) is not in \( L \))
2. \( 0001 \) can be pumped: put \( x = 000 \), \( y = 1 \), \( z = \Lambda \).

So the minimum pumping length is 4.

How many states would you expect to find in a DFA recognizing \( L \)?