Proof of a Property of Regular Expressions

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8th April, 2008

In class last Thursday (3rd April, 2008) we defined the formal language of Regular Expressions, and then gave the terms of the language meaning as regular languages. To recap, we defined a meaning function $\mathcal{M}[\![\ ]\!]$ that maps regular expressions over an alphabet A to regular languages as follows.

$$\begin{aligned}
\mathcal{M}[\![\Lambda]\!] &=_{\text{def}} \Lambda & (1)\\
\mathcal{M}[\![\varnothing]\!] &=_{\text{def}} \varnothing & (2)\\
\forall\, a \in A.\quad \mathcal{M}[\![a]\!] &=_{\text{def}} \{a\} & (3)\\
\forall\, R, S \in \text{RE}.\quad \mathcal{M}[\![R+S]\!] &=_{\text{def}} \mathcal{M}[\![R]\!] \ \cup\ \mathcal{M}[\![S]\!] & (4)\\
\forall\, R, S \in \text{RE}.\quad \mathcal{M}[\![RS]\!] &=_{\text{def}} \mathcal{M}[\![R]\!] \ \cdot\ \mathcal{M}[\![S]\!] & (5)\\
\forall\, R \in \text{RE}.\quad \mathcal{M}[\![R^*]\!] &=_{\text{def}} \mathcal{M}[\![R]\!]^\star & (6)
\end{aligned}$$

Note that this is an inductive definition of a function. Rules 1–3 are the base cases, and rules 4–6 are the induction rules. There is one rule for each clause in the definition of Regular Expressions.

Then we proved the commutative property of + (Lecture 2 Slide 9) by appealing to this definition. Next we tried to prove the first part of property 11.1.7 from Hein (p. 638):

$$(R+S)^* = (R^* + S^*)^*$$

but got stuck. This property is also a theorem, and it can also be proved as a consequence of the meaning we gave to Regular Expressions.

Theorem 1 (A property of Regular Expressions)

$$(R+S)^* = (R^* + S^*)^*$$

Proof

$$\begin{aligned}
\mathcal{M}[\![(R+S)^*]\!] &= \mathcal{M}[\![(R+S)]\!]^\star && \text{(definition of } \mathcal{M}[\![]\!] \text{ (6))}\\
&= (\mathcal{M}[\![R]\!] \ \cup\ \mathcal{M}[\![S]\!])^\star && \text{(definition of } \mathcal{M}[\![]\!] \text{ (4))}\\
&= (\mathcal{M}[\![R]\!]^\star \ \cup\ \mathcal{M}[\![S]\!]^\star)^\star && \text{(property of } {}^\star \text{ (Hein property 1.16 \textbf{d} (p. 45))}\\
&= (\mathcal{M}[\![R^*]\!] \ \cup\ \mathcal{M}[\![S^*]\!])^\star && \text{(definition of } \mathcal{M}[\![]\!] \text{ (6))}\\
&= (\mathcal{M}[\![R^* + S^*]\!])^\star && \text{(definition of } \mathcal{M}[\![]\!] \text{ (4))}\\
&= \mathcal{M}[\![(R^* + S^*)^*]\!] && \text{(definition of } \mathcal{M}[\![]\!] \text{ (6))}
\end{aligned}$$

Q.E.D.

In other words, the theorem says that the required property of Regular Expressions follows from the corresponding property of the closure operation *:

$$(L^\star \ \cup\ M^\star)^\star = (L \ \cup\ M)^\star$$

This may seem like cheating, since Hein does not actually prove property 1.16 **d**. So, to make sure that we are on firm ground, let's prove this property.

Theorem 2 (Hein's Closure Property 1.16 d)

$$(L^\star \ \cup \ M^\star)^\star = (L \ \cup \ M)^\star$$

Proof

To prove the required equality, we prove the two inequalities

$$(L^\star \ \cup \ M^\star)^\star \ \subseteq \ (L \ \cup \ M)^\star \tag{7}$$
$$(L^\star \ \cup \ M^\star)^\star \ \supseteq \ (L \ \cup \ M)^\star \tag{8}$$

This suffices to prove our result, because $\subseteq$ is antisymmetric.

Proof of equation 7

$$\begin{array}{rcll} R & \subseteq & R \ \cup \ S & \text{(property of } \cup \text{)} & (9) \\ R^\star & \subseteq & (R \ \cup \ S)^\star & \text{(definition of } {}^\star\text{)} & (10) \\ S^\star & \subseteq & (R \ \cup \ S)^\star & \text{(line 10, interchanging } R \text{ and } S\text{)} & (11) \\ R^\star \ \cup \ S^\star & \subseteq & (R \ \cup \ S)^\star & \text{(union of lines 10 and 11)} & (12) \\ (R^\star \ \cup \ S^\star)^\star & \subseteq & ((R \ \cup \ S)^\star)^\star & \text{(apply } {}^\star \text{ to line 12)} & (13) \\ (R^\star \ \cup \ S^\star)^\star & \subseteq & (R \ \cup \ S)^\star & (L^\star = (L^\star)^\star\text{, Hein property 1.16 } \mathbf{c}) & (14) \end{array}$$

Proof of equation 8

$$\begin{array}{rcll} L & \subseteq & L^\star & \text{(definition of } {}^\star\text{)} & (15) \\ M & \subseteq & M^\star & \text{(definition of } {}^\star\text{)} & (16) \\ L \ \cup \ M & \subseteq & L^\star \ \cup \ M^\star & \text{(union of equations 15 and 16)} & (17) \\ (L \ \cup \ M)^\star & \subseteq & (L^\star \ \cup \ M^\star)^\star & \text{(apply } {}^\star \text{ to line 17)} & (18) \end{array}$$

Combining 7 and 8, we have our result.

Q.E.D.