CS311—Computational Structures

# Properties of Regular Languages

Lecture 5

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(with material from Andrew P Black and Tim Sheard)

# Closure properties

1. The union of two regular languages is regular
2. The concatenation of two regular languages is regular
3. The Kleene-closure (*) of a regular language is regular
4. The complement of a regular language is regular
5. The intersection of two regular languages is regular

Which of these properties do we have to prove?

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- Query: Would this proof work if we started with an NFA instead?

# Proof of Closure under Intersection

- Recall the product construction by which we proved that given DFA's $M_1$ and $M_2$, we can always construct a machine M that recognizes $L(M_1) \cup L(M_2)$.
  - In that construction, each state of M corresponds to a pair of states $(q_1,q_2)$, with $q_1 \in Q_1$ and $q_2 \in Q_2$.
  - The final states of M are those for which **either** $q_1 \in F_1$ **or** $q_2 \in F_2$ (or both)
- To build a machine that recognizes $L(M_1) \cap L(M_2)$, we just make the final states be those for which **both** $q_1 \in F_1$ **and** $q_2 \in F_2$

# Another, quicker proof

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$$L \cap M = \overline{\overline{L} \cup \overline{M}}$$

# Other Closure Properties

- The regular languages stay closed under a remarkable variety of operations
  - Difference
  - Reversal (see IALC)
  - Shuffle
  - DROP-OUT(L) =
    $\{xz \mid xyz \in L, \text{ where } x,z \in \Sigma^*, y \in \Sigma\}$
  - $A/B = \{w \mid wx \in A \text{ for some } x \in B\}$ when A is a regular language and B is **any** language

# Limits of finite state machines

- Consider the language
  $L = \{0^k 1^k \mid k=0,1,2, \ldots\}$
- Is this language regular?
- If so, there is some DFA that recognizes it
- Intuitively, this should not be possible
  - such a machine would have to keep track of an **arbitrarily large** number k
  - but D**F**As only have a finite number of states!

# Long strings need loops

- But some DFA's certainly can recognize arbitrarily long strings
- How? By entering some state(s) more than once.
  - i.e. by going into a **loop**

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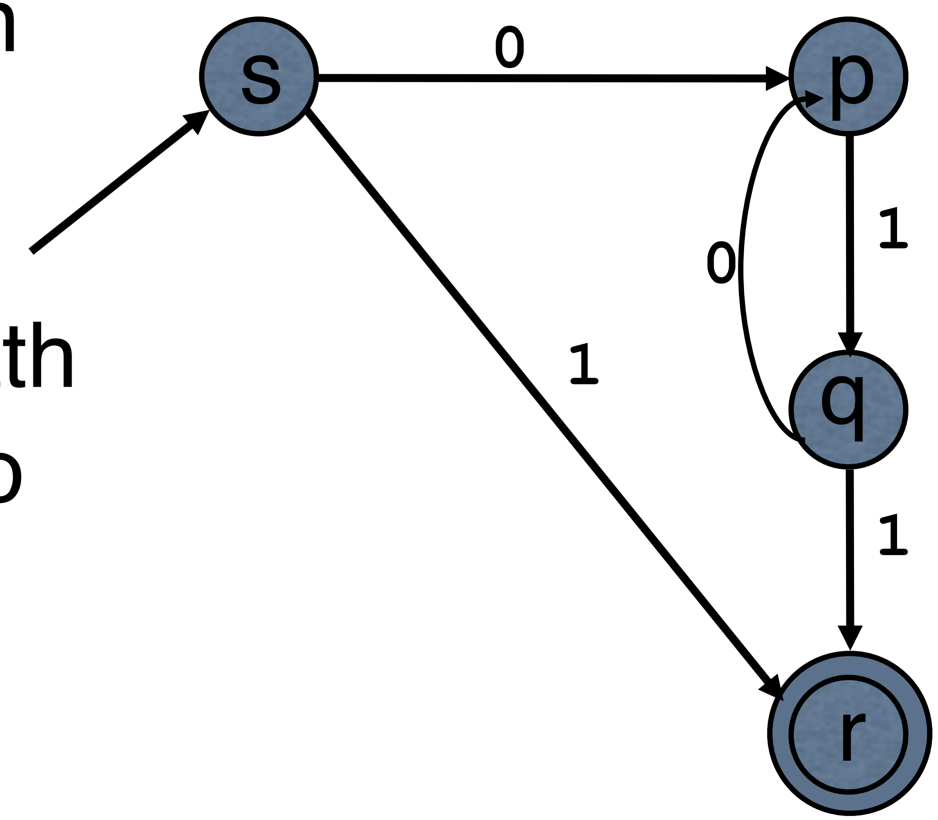
# Consequences of loops

Consider this DFA. The input string `01011` is accepted after an execution that goes through the state sequence $s \rightarrow p \rightarrow q \rightarrow p \rightarrow q \rightarrow r$. This path contains a loop (corresponding to the substring `01`) that starts and ends at p.




There are two simple ways of modifying this path without changing its beginning and ending states:

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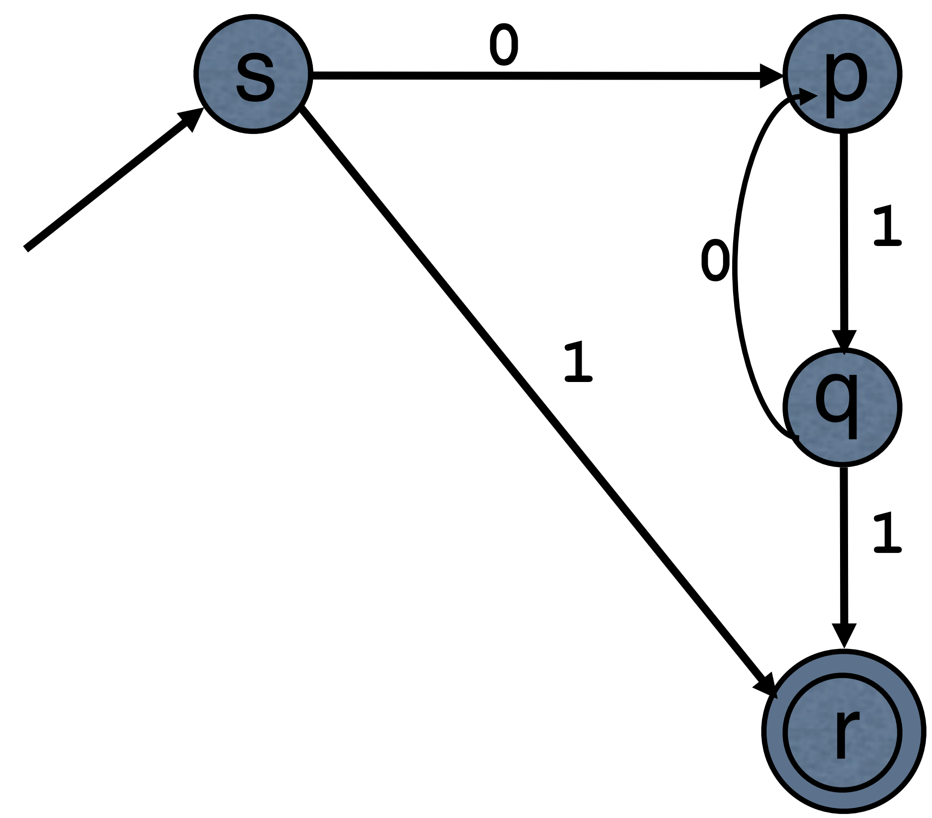
(1) delete the loop from the path, showing that `011` is accepted

(2) instead of going around the loop once, do it several times, showing that `0101011,010101011,...` are accepted




In general, we see that all strings of the form $\mathtt{0(10)}^{i}\mathtt{11}$ (where $i \geq 0$) are accepted.

# Long paths *must* contain a loop

- Suppose a DFA has $n$ states but it accepts a string of length $\geq n$
  - In so doing, it visits at least $n+1$ states
  - Therefore it must visit some state twice
    - This is a consequence of the **pigeonhole principle**
- Thus, every path of length $n$ or longer must contain a loop

# Loops make "pumps"

- Suppose $L$ is a regular language, and `w=xuy` is a string in $L$, where `u` is non-empty.
- We say that `u` is a **pump** in `w` if all strings $\texttt{xu}^i\texttt{y}$ ($i \geq 0$) belong to $L$.
  - So, `xy`, `xuy`, `xuuy`, `xuuuy`, … are all in $L$
    - When we increase $i$ we're "pumping up"
    - When we decrease $i$ to 0 we're "pumping down"

# The Pumping Lemma

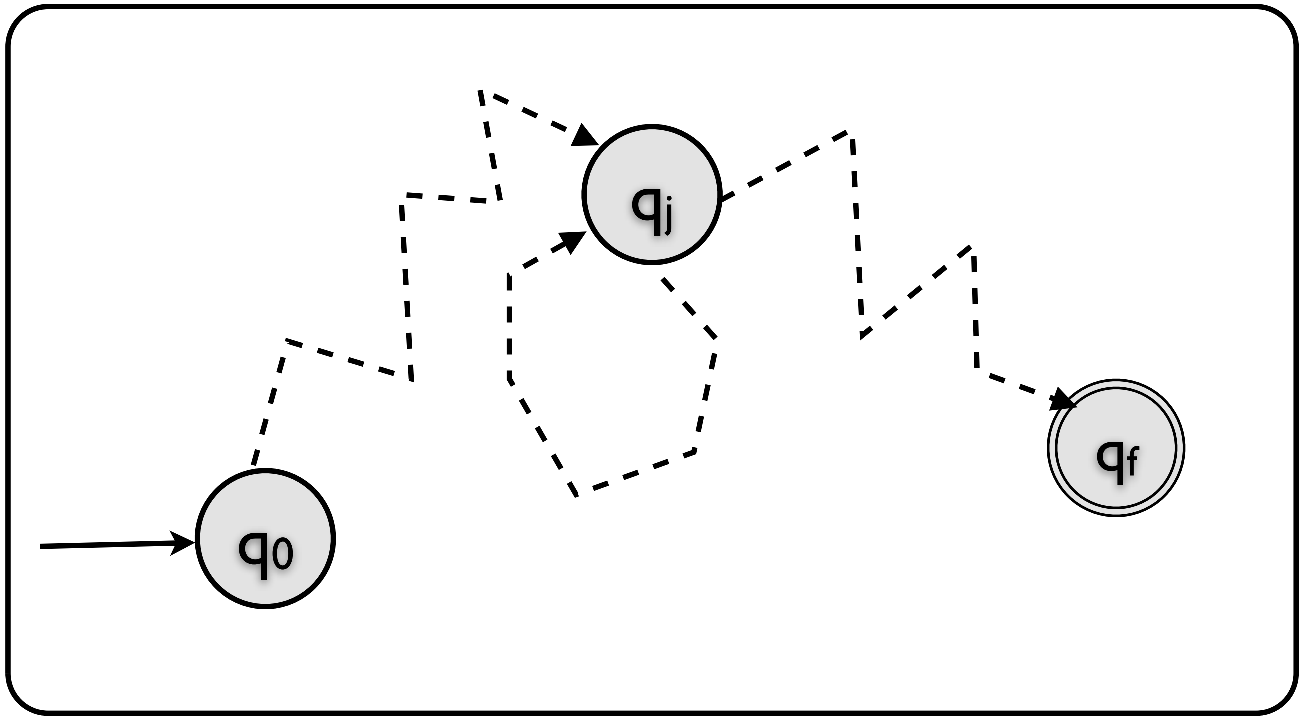
- For every regular language L, there is a number $n$ (called the **pumping length** of L) such that every string in L of length at least $n$ contains a "pump."
  - In fact, it contains a pump in the first $n$ symbols.
  - In practice it doesn't matter exactly what $n$ is — just that it exists.
- **Formally**: If L is regular, then $\exists n$ such that if $w \in L$ and $|w| \geq n$ then we can write w as xyz where:

1. $xy^iz \in L$ for every $i \geq 0$ (y is a "pump")
2. $y \neq \varepsilon$ (the "pump" is non-empty)
3. $|xy| \leq n$ (it appears in the first $n$ symbols)

# Proof Idea

- Let M be a DFA that recognizes L
- Choose $n$ to be the number of states in M
- Choose any w $\in$ L such that $|w| \geq n$
  - What if there is none?
- What sequence of transitions does M make to accept w?

M




- What happens when M accepts w?
  - it starts in the start state $q_0$
  - it moves through a series of $|w|$ other states,
  - ending in a final state, say $q_f$
- Since $|w| \geq n$, this path must have at least one repeated state, and hence a loop
  - say the first state to be entered twice is $q_j$

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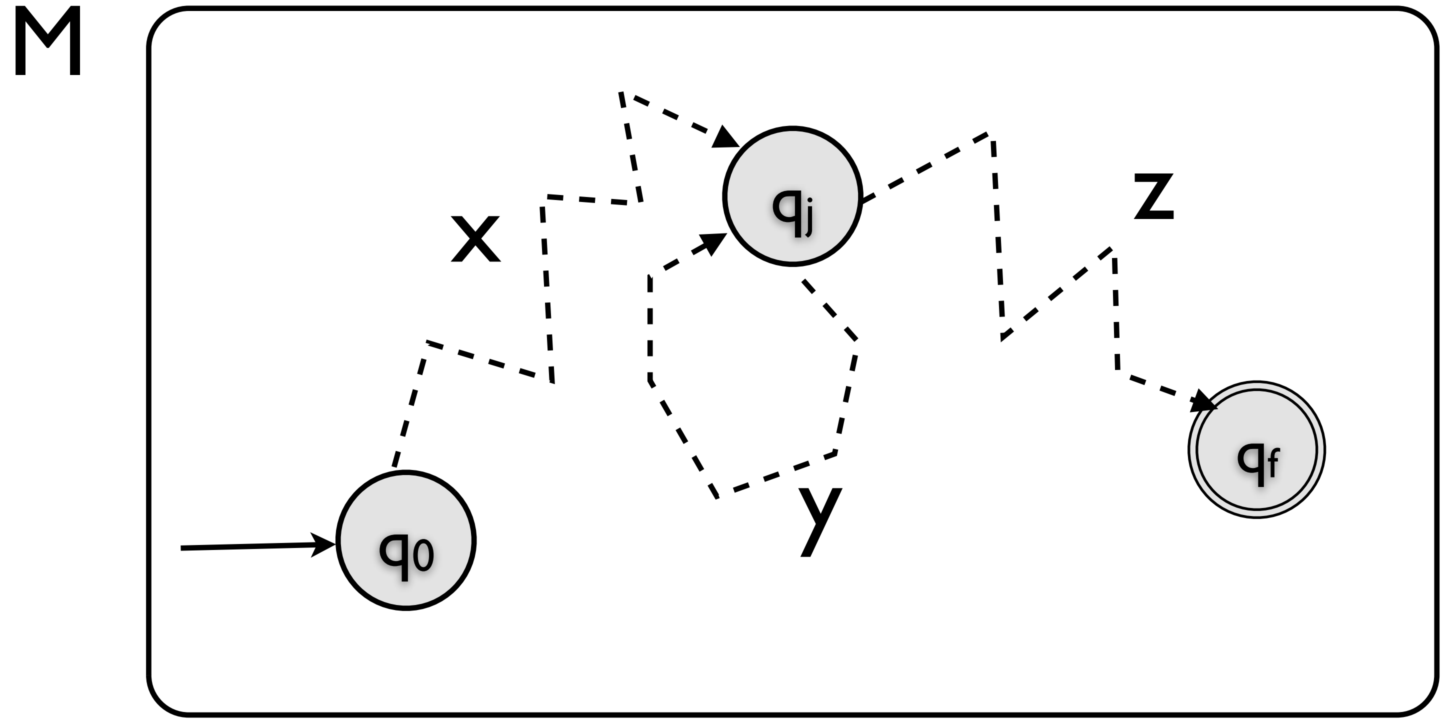


- Label the pieces of w as in the diagram
- Then xz, xyz, xyyz, xyyyz, etc. are all in L
  - $|y| > 0$ or else there would be no loop
  - $|xy| \leq n$, or else we could have found a repeated state sooner

# Using the Pumping Lemma

- The Pumping Lemma says that regular languages follow a very restrictive pattern
  - If L is regular, any sufficiently long string in L can be "pumped" to produce many other strings in L
- We can use it to show a language is **not** regular by showing that it **doesn't** follow the pattern
  - We exhibit an arbitrarily long string in L which, when "pumped," produces some string **not** in L

# Informal proof example 1

- Let's argue that $L = \{0^k 1^k \mid k=0,1,2, \ldots\}$ is not regular.
- Why? Because there is an arbitrarily long string in L that, when pumped, produces a string not in L.
- In fact, that's true of **every** string in L:
  - Consider $0^n1^n$ for any n and suppose it has a "pump"
  - If the pump is all 0's, pumping will change the number of 0's but not the number of 1's, so result is not in L.
  - If the pump is all 1's...(similarly)...result is not in L.
  - If the pump is of the form $0^+1^+$, pumping up produces a string not of the form $0^*1^*$, so result is not in L.

# Formalizing Example 1

We prove that $L = \{0^k1^k \mid k=0,1,2, \ldots\}$ is not regular.

- **Assume** L is regular. We'll show that this leads to a contradiction.
- Let the pumping length of L be n.
- Take $w=0^n1^n$. Then certainly $|w| \geq n$.
- So, by the pumping lemma, we can write w as xyz with
  - $xy^iz \in L$ for $i \geq 0$, $|y| > 0$, and $|xy| \leq n$.
- There are three possibilities for y:

1. $y = 0^m$ for some $m > 0$. But then, taking $i = 2$, $xyyz = 0^{n+m}1^n \in L$.
2. $y = 1^m$ for some $m > 0$ ...(by similar argument)... $0^n1^{m+n} \in L$.
3. $y = 0^q1^r$ for some $q,r > 0$. But then, taking $i = 2$, $xyyz = 0^n1^r0^q1^n \in L$.

- But each of these cases leads to a contradiction with the definition of L. Hence our **assumption** that L is regular must have been wrong. So L is **not** regular. QED.

# Shortening Example 1

We prove that $L = \{0^k 1^k \mid k=0,1,2, \ldots\}$ is not regular.

- **Assume** L is regular. We'll show that this leads to a contradiction.
- Let the pumping length of L be n.
- Take $w=0^n1^n$. Then certainly $|w| \geq n$.
- So, by the pumping lemma, we can write w as xyz with
  - $xy^iz \in L$ for $i \geq 0$, $|y| > 0$, and $|xy| \leq n$.
- Since $|xy|$:
  - $y = 0^m$ for some $m > 0$. But then, taking $i = 2$, $xyyz = 0^{n+m}1^n \in L$.
- But this leads to a contradiction with the definition of L. Hence, our **assumption** that L is regular must have been wrong. So L is **not** regular. QED.

# Quick Logic Review

- Suppose we know that $A \Rightarrow B$.
- The **contrapositive** statement is $\neg B \Rightarrow \neg A$.
  - The contrapositive of a true fact is always automatically true too: $A \Rightarrow B \equiv \neg B \Rightarrow \neg A$
  - In proof by contradiction, we show that $A \wedge \neg B \Rightarrow$ falsehood, and conclude $A \Rightarrow B$.
- Also, recall how negation interacts with quantification:
  - $\neg(\forall x.\ P(x)) \Leftrightarrow \exists x. \neg P(x)$
  - $\neg(\exists x.\ P(x)) \Leftrightarrow \forall x. \neg P(x)$

# Pumping lemma contrapositive

**Pumping Lemma** says:
($L$ is regular)
$\Rightarrow$
$(\exists n,$
$\forall w \in L$ where $|w| \geq n,$
$\exists x, y, z$ where $w = xyz$ and $y \neq \epsilon$ and $|xy| \leq n,$
$\forall i \geq 0, xy^i z \in L).$

**Contrapositive** says:
$\neg( \exists n,$
$\forall w \in L$ where $|w| \geq n,$
$\exists x, y, z$ where $w = xyz$ and $y \neq \epsilon$ and $|xy| \leq n,$
$\forall i \geq 0, xy^i z \in L)$
$\Rightarrow$
$\neg$($L$ is regular).

# Contrapositive, rewritten

**Contrapositive** says:
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$\forall w \in L$ where $|w| \geq n,$
$\exists x, y, z$ where $w = xyz$ and $y \neq \epsilon$ and $|xy| \leq n,$
$\forall i \geq 0, xy^i z \in L)$
$\Rightarrow$
$\neg(L$ is regular$).$

Equivalently, **Contrapositive** says:
$(\forall n,$
$\exists w \in L$ where $|w| \geq n,$
$\forall x, y, z$ where $w = xyz$ and $y \neq \epsilon$ and $|xy| \leq n,$
$\exists i \geq 0, xy^i z \notin L)$
$\Rightarrow$
$(L$ is not regular$).$

# Example 2

Show that

**$L = \{w \in \{0,1\}^* \mid$ w contains an equal number of 0s and 1s$\}$**

is not regular.

We apply the contrapositive of the Pumping Lemma:

- For any n, choose $w = 0^n1^n$. Then $|w| \geq n$.
- For any x,y,z where w = xyz, $|y| > 0$ and $|xy| \leq n$, it must be the case that $y = 0^m$ for some $m > 0$. (Why?)
- Now choose i = 2. Then $xy^iz = xyyz = 0^{n+m}1^n$ which is not in L

Hence L is not regular.

- Note that in this proof, choice of w matters!

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**L = {w $\in$ {0,1}* | w contains an equal number of 0s and 1s}**

is not regular.

Let M = 0*1*, a regular language

- Then $L \cap M = \{ 0^n 1^n \mid n \geq 0\}$
- If L were regular, then $L \cap M$ would be regular. But it's not, so neither is L.

# It’s a game!

- We pick a language L to prove non-regular
- Our opponent picks n, but doesn’t tell us what it is
- We give w of length ≥ n (w can depend on n)
  - This is a key move! It requires *skill and ingenuity:* we must find w that will work for us in the last move no matter how our opponent plays
- Our opponent factors w into xyz, obeying only the constraints $|y| > 0$ and $|xy| \leq n$.
- We show that for some i, $xy^iz$ is not in L.
  - Sometimes picking i also requires cleverness.

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# Example 3

Show that $L = \{ uu \mid u \in \{a,b\}^* \}$ is not regular.

- Suppose it were and let n be its pumping length.
- Then we choose $w=a^n b a^n b$, which clearly has length greater than n.
- The opponent has to divide w into xyz, where $|xy| \leq n$ and $|y| > 0$. But then y must have the form $a^m$ for some $m > 0$.
- We choose $i = 0$. That has the effect of dropping m *a*s. So $a^{n-m} b a^n b$ must be in L. But it isn't, so we "win": L is not regular.
- Question: if we choose $w=a^n a^n$, the opponent has a chance to win. How?

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# Example 4

- We claim that $L = \{1^p \mid p \text{ is prime}\}$ isn't regular
- Suppose it were, with pumping length n.
- We choose $w = 1^p$ for any $p \geq n+2$.
  - Such a p must exist, because there are an infinity of primes
- The opponent picks $x=1^q$, $y=1^r$, $z = 1^s$ where $r > 0$, $q+r \leq n$, and $q+r+s = p$. (Note that the opponent has no choice here.)
- We cleverly pick $i = p-r$. Then $xy^{p-r}z = 1^m \in L$, where $m = q+(p-r)r + s$. So m must be prime.
- But $m = (q+s) + (p-r)r = (p-r) + (p-r)r = (r+1)(p-r)$
  - Moreover $r+1 > 1$ (why?) and $p-r > 1$ (why?)
  - So m is the product of two integers each > 1, and therefore not prime!

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# Some Important DFA Facts

## — that we won't study

- There is an algorithm to convert any DFA M to a **minimum-state** DFA recognizing L(M).
- The minimum-state DFA is **unique** up to renaming of states.
- There is thus an algorithm to determine whether two DFA's recognize the same language.