Lists ...
Why Study Lists?

- Lists are a heavily used data structure in many functional programs
- Special syntax is provided to make programming with lists more convenient
- Lists are a special case / an example of:
  - An algebraic datatype
  - A parameterized datatype
  - A monad
What is a List?

- An ordered collection (multiset) of values
  - \([1,2,3,4], [4,3,2,1], [1,1,2,2,3,3,4,4]\) are distinct lists of integers

- A list of type \([T]\) contains zero or more elements of type \(T\)
  - \([\text{True}, \text{False}] :: [\text{Bool}]\)
  - \([1,2,3] :: [\text{Integer}]\)
  - \(['a', 'b', 'c'] :: [\text{Char}]\)
  - \([[]],[1],[1,2],[1,2,3]] :: [[\text{Integer}]]\)

- All elements have the same type:
  - \([\text{True}, 2, 'c']\) is not a valid list
Naming Convention:

We often use a simple naming convention:

- If a typical value in a list is called \( x \), then a typical list of such values might be called \( xs \) (i.e., the plural of \( x \))

- ... and a list of lists of values called \( x \) might be called \( xss \)

- A simple convention, minimal clutter, and a useful mnemonic
How do you Make a List?

- The empty list, `[]`, which has type `[a]` for any (element) type `a`

- Enumerations: `[e₁, e₂, e₃, e₄]`

- Arithmetic Sequences:
  - `[elem₁ .. elem₃]`
  - `[elem₁, elem₂ .. elem₃]`
  - Only works for certain element types: integers, booleans, characters, ...
  - (omit last element to specify an “infinite list”)


... continued:

- Using list **comprehensions**:  
  - `[ 2*x+1 | x <- [1,3,7,11] ]`

- Using **constructor functions**:  
  - `[]` and `(:)` ("nil" and "cons")

- Using **prelude/library functions**:  
  - ...

Prelude Functions:

(++) :: [a] -> [a] -> [a]
reverse :: [a] -> [a]
take :: Int -> [a] -> [a]
drop :: Int -> [a] -> [a]
takeWhile :: (a -> Bool) -> [a] -> [a]
dropWhile :: (a -> Bool) -> [a] -> [a]
zip :: [a] -> [b] -> [(a,b)]
replicate :: Int -> a -> [a]
iterate :: (a -> a) -> a -> [a]
repeat :: a -> [a]

...
map:

- map :: (a -> b) -> [a] -> [b]
- map f xs produces a new list by applying the function f to each element in the list xs
- map (1+) [1,2,3] = [2,3,4]
- map even [1,2,3] = [False, True, False]
- map id xs = xs, for any list xs
- We can also think of map as a function that turns functions of type (a -> b) into list transformers of type ([a] -> [b])
filter:

- `filter :: (a -> Bool) -> [a] -> [a]`
- `filter even [1..10] = [2,4,6,8,10]`
- `filter (<5) [1..100] = [1,2,3,4]`
- `filter (<5) [100,99..1] = [4,3,2,1]`

We can think of `filter` as mapping predicates/functions of type `(a -> Bool)`, to list transformers of type `[a] -> [a]`
... Tests ...
Testing:

- Testing can confirm expectations about how things work
- Conversely, testing can set expectations about how things should work
- It can be dangerous to generalize from tests
  “Testing can be used to show the presence of bugs, but never to show their absence” [Edsger Dijkstra, 1969]
- But testing does help us to find & avoid:
  - Bugs in the things we build
  - Bugs in the claims we make about those things
Making Tests Executable:

test1 = filter even [1..10] == [2,4,6,8,10]
test2 = filter (<5) [1..100] == [1,2,3,4]
test3 = filter (<5) [100,99..1] == [4,3,2,1]
Making Tests Executable:

test1 = filter even [1..10] == [2,4,6,8,10]
test2 = filter (<5) [1..100] == [1,2,3,4]
test3 = filter (<5) [100,99..1] == [4,3,2,1]

tests = test1 && test2 && test3
Making Tests Executable:

test1 = filter even [1..10] == [2,4,6,8,10]
test2 = filter (<5) [1..100] == [1,2,3,4]
test3 = filter (<5) [100,99..1] == [4,3,2,1]

tests = and [test1, test2, test3]
Making Tests Executable:

test1 = filter even [1..10] == [2,4,6,8,10]
test2 = filter (<5) [1..100] == [1,2,3,4]
test3 = filter (<5) [100,99..1] == [4,3,2,1]

tests = and [test1, test2, test3]

and :: [Bool] -> Bool
and [] = True
and (b:bs) = b && and bs
Issues:

- Want to see results for all tests
- Text to identify individual tests (especially useful when a test fails)
- Summary statistics
- Handle more complex behavior (e.g., testing code that performs I/O actions)
- Support tests for code that is supposed to fail (e.g., raise an exception)
Enter HUnit:

- A library for unit testing
- Written in Haskell
  (Or from http://hackage.haskell.org)
- Built-in to recent versions of Hugs and GHC
- Just “import Test.HUnit” and you’re ready!
Defining Tests:

import Test.HUnit

test1 = TestCase (assertEqual
    "filter even [1..10]"
    (filter even [1..10])
    [2,4,6,8,10])

test2 = ...

test3 = ...

tests = TestList [test1, test2, test3]
Running Tests:

Main> runTestTT tests
Cases: 3  Tried: 3  Errors: 0  Failures: 0

Main>
Detecting Faults:

import Test.HUnit

test1 = testCase (assertEqual "filter even [1..10]" (filter even [1..10]) [2,4,6,9,10])
test2 = ...
test3 = ...
tests = TestList [test1, test2, test3]
Using HUnit:

Main> runTestTT tests
### Failure in: 0
filter even [1..10]
expected: [2,4,6,8,10]
but got: [2,4,6,9,10]
Cases: 3  Tried: 3  Errors: 0  Failures: 1

Main>
Labeling Tests:

... 

tests = TestLabel "filter tests"

$ TestList [test1, test2, test3]
Using HUnit:

Main> runTestTT tests
### Failure in: filter tests:0
filter even [1..10]
expected: [2,4,6,8,10]
but got: [2,4,6,9,10]
Cases: 3  Tried: 3  Errors: 0  Failures: 1

Main>
The Test and Assertion Types:

data Test = TestCase Assertion
  | TestList [Test]
  | TestLabel String Test

runTestTT :: Test -> IO Counts

assertFailure :: String -> Assertion
assertBool :: String -> Bool -> Assertion
assertEqual :: (Eq a, Show a) => String -> a -> a -> Assertion
Problems:

- Finding and running tests is a manual process (easily skipped/overlooked)
- Can be hard to trim tests from distributed code
- Can’t solve the halting problem 😊
Example: merge

Let’s develop a `merge` function for combining two sorted lists into a single sorted list:

```haskell
merge :: [Int] -> [Int] -> [Int]
merge = undefined
```

What about test cases?
Merge Tests:

- **Simple examples:**
  
  ```
  merge [1,5,9] [2,3,6,10] == [1,2,3,5,6,9,10]
  ```

- **One or both arguments empty:**
  
  ```
  merge [] [1,2,3] == [1,2,3]
  merge [1,2,3] [] == [1,2,3]
  ```

- **Duplicate elements:**
  
  ```
  merge [2] [1,2,3] == [1,2,3]
  merge [1,2,3] [2] == [1,2,3]
  ```
Capturing the Tests:

mergeTests
  = TestLabel "merge tests"
$ TestList [simpleTests, emptyTests, dupTests]

simpleTests
  = TestLabel "simple tests"
$ TestCase (assertEqual "merge [1,5,9] [2,3,6,10]"
  (merge [1,5,9] [2,3,6,10])
  [1,2,3,5,6,9,10])

emptyTests
  = ...
Capturing the Tests:

Main> runTestTT mergeTests
Cases: 6 Tried: 0 Errors: 0 Failures: 0
Program error: Prelude.undefined

Main>
Reining the Definition (1):

Let’s provide a little more definition for `merge`:

```
merge :: [Int] -> [Int] -> [Int]
merge xs ys = []
```

What happens to the test cases now?
Back to the Tests:

Main> runTestTT mergeTests
### Failure in: merge tests:0:simple tests
merge [1,5,9] [2,3,6,10]
expected: []
  but got: [1,2,3,5,6,9,10]
...
Cases: 6  Tried: 6  Errors: 0  Failures: 5

Main>
Refining the Definition (2):

Let’s provide a little more definition for `merge`:

\[
\text{merge} \quad :: \quad [\text{Int}] \to [\text{Int}] \to [\text{Int}]
\]

\[
\text{merge } \mathbf{xs} \mathbf{ys} = \mathbf{xs}
\]

What happens to the test cases now?
Back to the Tests:

Main> runTestTT mergeTests
### Failure in: merge tests:0:simple tests
merge [1,5,9] [2,3,6,10]
expected: [1,5,9]
  but got: [1,2,3,5,6,9,10]
### Failure in: merge tests:2:duplicate elements:0
merge [2] [1,2,3]
expected: [2]
  but got: [1,2,3]
Cases: 6  Tried: 6  Errors: 0  Failures: 2

Main>
Refining the Definition (3):

Use type information to break the definition down into multiple cases:

\[
\text{merge} :: [\text{Int}] -> [\text{Int}] -> [\text{Int}]
\]

\[
\text{merge } [] \ ys = ys
\]

\[
\text{merge } (x:x:\text{xs}) \ ys = ys
\]
Refining the Definition (4):

Repeat ...

merge :: [Int] -> [Int] -> [Int]
merge [] ys = ys
merge (x:xs) [] = x:xs
merge (x:xs) (y:ys)  
                         = x:xs
Refining the Definition (5):

Use guards to split into cases:

```
merge :: [Int] -> [Int] -> [Int]
merge [] ys = ys
merge (x:xs) [] = x:xs
merge (x:xs) (y:ys)
    | x<y       = x : merge xs (y:ys)
    | otherwise = y : merge (x:xs) ys
```
Back to the Tests:

Main> runTestTT mergeTests
### Failure in: merge tests:2:duplicate elements:0
merge [2] [1,2,3]
expected: [1,2,2,3]
  but got: [1,2,3]
### Failure in: merge tests:2:duplicate elements:1
merge [1,2,3] [2]
expected: [1,2,2,3]
  but got: [1,2,3]
Cases: 6  Tried: 6  Errors: 0  Failures: 2

Main>
Refining the Definition (6):

Use another guards to add another case:

merge :: [Int] -> [Int] -> [Int]
merge [] ys = ys
merge (x:xs) [] = x:xs
merge (x:xs) (y:ys)
  | x<y     = x : merge xs (y:ys)
  | y<x     = y : merge (x:xs) ys
  | x==y    = x : merge xs ys
Back to the Tests:

Main> runTestTT mergeTests
Cases: 6  Tried: 6  Errors: 0  Failures: 0

Main>
Modifying the Definition:

Suppose we decide to modify the definition:

\[ \text{merge} : \mathbb{[Int]} \rightarrow \mathbb{[Int]} \rightarrow \mathbb{[Int]} \]
\[ \text{merge} \ (x:xs) \ (y:ys) \]
\[ \quad | \ x < y \quad = \ x : \text{merge} \ xs \ (y:ys) \]
\[ \quad | \ y < x \quad = \ y : \text{merge} \ (x:xs) \ ys \]
\[ \quad | \ x == y \quad = \ x : \text{merge} \ xs \ ys \]
\[ \text{merge} \ xs \ y s \quad = \ xs ++ \ ys \]

Is this still a valid definition?
Back to the Tests:

Main> runTestTT mergeTests
Cases: 6  Tried: 6  Errors: 0  Failures: 0

Main>
Lessons Learned:

- Writing tests (even before we’ve written the code we want to test) can expose key details / design decisions
- A library like HUnit can help to (partially) automate the process
- Development alternates between coding and testing
- Bugs are expensive, running tests is cheap
- Good tests can last a long time; continuing use as code evolves
... and Laws
Lawful Programming:

How can we give useful information about a function without necessarily having to give all the details of its definition?

Informal description:

“map applies its first argument to every element in its second argument ...”

Type signature:

\[ \text{map :: (a -> b) -> [a] -> [b]} \]

Laws:

- Normally in the form of equalities between expressions ...
Algebra of Lists:

- 

\((+++)\) is associative with unit \([\,]\)

\[
\text{xs} +++ \text{(ys} +++ \text{zs}) = (\text{xs} +++ \text{ys}) +++ \text{zs} \\
[\,] +++ \text{xs} = \text{xs} = \text{xs} +++ [\,]
\]

- map preserves identities, distributes over composition and concatenation:

\[
\text{map id} = \text{id} \\
\text{map (f} \cdot \text{g}) = \text{map f} \cdot \text{map g} \\
\text{map f (xs} +++ \text{ys}) = \text{map f xs} +++ \text{map f ys}
\]
... continued:

- **filter distributes over concatenation**
  \[
  \text{filter } p \ (xs \ ++ \ ys) = \text{filter } p \ xs \ ++ \ \text{filter } p \ ys
  \]

- **Filters and maps:**
  \[
  \text{filter } p \ . \ \text{map } f = \ \text{map } f \ . \ \text{filter } (p \ . \ f)
  \]

- **Composing filters:**
  \[
  \text{filter } p \ . \ \text{filter } q = \ \text{filter } r
  \]
  
  where \( r \ x = q \ x \ \&\& \ p \ x \)
Aside: Lambda Notation

The syntax $\text{vars} \rightarrow \text{expr}$ denotes a function that takes arguments $\text{vars}$ and returns the corresponding value of $\text{expr}$.

Referred to as a lambda expression after the corresponding construct in $\lambda$-calculus.

Examples:

- $(\lambda x \rightarrow x + 1)$
- $\text{filter p . filter q} = \text{filter} (\lambda x \rightarrow q \ x \ && \ p \ x)$
- $(\lambda x \rightarrow 1 + 2*x)$
- $(\lambda x \ y \rightarrow (x + y) \ * \ (x - y))$
Laws Describe Interactions:

- A lot of laws describe how one operator interacts with another

Example: interactions with reverse:
  - reverse . map f = map f . reverse
  - reverse . filter p = filter p . reverse
  - reverse (xs ++ ys) = reverse ys ++ reverse xs
  - reverse . reverse = reverse

Caution: stating a law doesn’t make it true! (e.g., the last two laws for reverse ...)
Uses for Laws:

Laws can be used:
- To capture/document deep intuitions about program behavior
- To support reasoning about program behavior
- To optimize or transform programs (either by hand, or in a compiler)
- As properties to be tested
- As properties to be proved
Laws for Merge:

What laws might we formulate for merge?

- If \(xs\) and \(ys\) are sorted, then \(\text{merge } xs \ ys\) is sorted
- \(\text{merge } (\text{sort } xs) \ (\text{sort } ys)\) should be sorted
- \(\text{merge } xs \ ys == \text{merge } ys \ xs\)
- \(\text{merge } xs \ (\text{merge } ys \ zs) == \text{merge } (\text{merge } xs \ ys) \ zs\)
- \(\text{merge } [] \ ys == ys\) and \(\text{merge } xs \ [] == xs\)
- \(\text{merge } xs \ xs == xs\)
- \(\text{length } (\text{merge } xs \ ys) <= \text{length } xs + \text{length } ys\)
- \(xs\) is a subset/subsequence of \(\text{merge } xs \ ys\)
From Laws to Functions:

mergeProp1 :: [Int] -> [Int] -> Bool
mergeProp1 xs ys = sorted xs ==> sorted ys ==> sorted (merge xs ys)

(==> ) :: Bool -> Bool -> Bool
x ==> y = not x || y

sorted :: [Int] -> Bool
sorted xs = and [ x <= y | (x,y) <- zip xs (tail xs) ]
Testing mergeProp1:

Main> mergeProp1 [1,4,7] [2,4,6]
True
Main> mergeProp1 [1,4,7] [2,4,1]
True
Main> sorted [1,4,7]
True
Main> sorted [2,4,1]
False

**Question:** to test **merge**, I wrote more code ...

If I don’t trust my programming skills, why am I writing even more (untrustworthy) code?
Formulate More Tests!

```haskell
import List(sort)

sortSorts :: [Int] -> Bool
sortSorts xs = sorted (sort xs)

sortedEmpty :: Bool
sortedEmpty = sorted []

sortIdempotent :: [Int] -> Bool
sortIdempotent xs = sort (sort xs) == sort xs
```
More Laws to Functions:

mergePreservesOrder :: [Int] -> [Int] -> Bool
mergePreservesOrder xs ys
    = sorted (merge (sort xs) (sort ys))

mergeCommutes :: [Int] -> [Int] -> Bool
mergeCommutes xs ys
    = merge us vs == merge vs us
    where us = sort xs
          vs = sort ys

etc...
Testing mergeProp1:

Main> mergeCommutes [1,4,7] [2,4,6]
True
Main> mergeCommutes [1,4,7] [2,4,1]
True
Main> mergePreservesOrder [1,4,7] [2,4,6]
True
Main> mergePreservesOrder [1,4,7] [2,4,1]
True
Main>
Automated Testing:

- Of course, we can run as many individual test cases as we like:
  - Pick a test case
  - Execute the program
  - Compare actual result with expected result

- Wouldn’t it be nice if the environment could help us to go directly from properties to tests?

- Wouldn’t it be nice if the environment could run the tests for us automatically too?
QuickCheck:

This is a job for QuickCheck!

“QuickCheck: A Lightweight Tool for Random Testing of Haskell Programs” by Koen Claessen and John Hughes, Chalmers University, Sweden. (Published at ICFP 2000)

In Hugs: import Test.QuickCheck
Understand Before you Code:

- Haskell programmers write types first ...  
  - ... type checking might find bugs.

- Extreme programmers write tests first ...  
  - ... running the tests might find bugs.

- Very few programmers write laws first ...  
  - ... because nothing encourages or rewards them for writing laws.
Wanted! Reward!

In the short-term, programmers won’t see any reward for writing laws ...

... so they won’t write them.

If programmers can derive some benefit from writing laws, then perhaps they will do it ...
Lawful Programming:

reverse :: [a] -> [a]
reverse xs = ...

{- reverse satisfies the following:
  reverse (xs ++ ys)
  ==
  reverse ys ++ reverse xs
-}
Lawful Programming:

```haskell
reverse :: [a] -> [a]
reverse xs = ...

prop_RevApp xs ys
    = reverse (xs++ys)
    ==
    reverse ys ++ reverse xs
```
Running QuickCheck:

Prelude> :load reverse.hs

Main> reverse [1,2,3]
[3,2,1]

Main> quickCheck prop_RevApp

✓, passed 100 tests

Main>
Not All Laws are True:

Main> quickCheck (\b -> b == not b)
Falsifiable, after 0 tests:
True

Sometimes this points to a bug in the law.
Type-Checked Laws:

- Laws are type checked as part of the main program source text.

  prop_RevApp :: [Int] -> [Int] -> Bool

- If the laws and the code are inconsistent, then an error will be detected!
The Testable Class:

quickCheck :: Testable a => a -> IO a

instance Testable Bool where ...

instance (Arbitrary a, Show a, Testable b) => Testable (a -> b) where ...

Indicates an ability to generate arbitrary values of type a.
The Testable Class:

quickCheck :: Testable a => a -> IO a

instance Testable Bool where ...

instance (Arbitrary a, Show a, Testable b) => Testable (a -> b) where ...

Indicates an ability to display arguments for counter examples
Generating Arbitrary Values:

```haskell
class Arbitrary a where
  arbitrary :: Gen a

instance Arbitrary ()
instance Arbitrary Bool
instance Arbitrary Int
instance Arbitrary Integer
instance Arbitrary Float
instance Arbitrary Double
instance (Arbitrary a, Arbitrary b) => Arbitrary (a,b)
instance Arbitrary a => Arbitrary [a]
```

arbitrary is a generator of random values
Quantified or Parameterized?

Main> quickCheck prop_revApp
OK, passed 100 tests.

Main> quickCheck (prop_revApp [1,2,3])
OK, passed 100 tests.

Main>

If you don’t give a specific value for an argument, quickCheck will generate arbitrary (i.e. random) values for you.
QuickCheck-ing merge:

Main> quickCheck mergeCommutes
OK, passed 100 tests.

Main> quickCheck mergePreservesOrder
OK, passed 100 tests.

Main>

So far, so good ...
Continued ...

mergeProp1 :: [Int] -> [Int] -> Bool
mergeProp1 xs ys = sorted xs ==> 
                sorted ys ==> 
                sorted (merge xs ys)

What happens?
Main> quickCheck mergeProp1
Falsifiable, after 7 tests:
[-1,-5,5,4,3,-5]
[-6,2,6,-6,0]
Huh?

Main>
What went wrong?

Main> sorted [-1,-5,5,4,3,-5]
False
Main> sorted [5,-6,2,6,-6,0]
False
Main> sorted (merge [-1,-5,5,4,3,-5] [5,-6,2,6,-6,0])
False
Main> False ====> False ====> False
False
Main> False ====> (False ====> False)
True
Main>
A Fix! (in fact, infix)

\[
\text{infixr} \Rightarrow
\]

\[
(\Rightarrow) :: \text{Bool} \to \text{Bool} \to \text{Bool}
\]

\[
x \Rightarrow y = \text{not } x \text{ } || \text{ } y
\]

What happens?

Main> quickCheck mergeProp1
OK, passed 100 tests.

Main>

Hooray!!!
Are we Happy Now?

\[
\text{mergeProp1} :: [\text{Int}] \rightarrow [\text{Int}] \rightarrow \text{Bool}
\]

\[
\text{mergeProp1 \hspace{2em} xs \hspace{2em} ys} = \text{sorted} \hspace{2em} xs \implies \\
\hspace{2em} \text{sorted} \hspace{2em} ys \implies \\
\hspace{2em} \text{sorted} \hspace{2em} \text{merge} \hspace{1em} xs \hspace{1em} ys
\]

100 tests passed!

But how many of them were trivial (i.e., one or both arguments unsorted)?
Understanding Test Results:

Use the **collect combinator**:

```haskell
mergeProp1sorted xs ys = collect (sorted xs, sorted ys) (mergeProp1 xs ys)
```

**Testing:**

Main> quickCheck mergeProp1sorted
OK, passed 100 tests.
45% (False,False).
25% (True,True).
20% (True,False).
10% (False,True).

Main>
Understanding Test Results:

Or use the classify combinator:

```haskell
mergeProp1long xs ys
    = classify (length xs > 10) "long"
    $ classify (length xs <= 5) "short"
    $ mergeProp1 xs ys
```

Testing:

```haskell
Main> quickCheck mergeProp1long
OK, passed 100 tests.
49% short.
29% long.
```

Main>
Understanding ==> :

- The real (==>) operator is not a standard “implies” function of type `Bool -> Bool -> Bool`

- When we test a property `p ==> q`, QuickCheck will try to find 100 test cases for which `p` is true, and will test `q` in each of those 100 cases

- If it tries 1000 candidates without finding enough solutions, then it will give up:
  ```haskell
  Main> quickCheck (\b -> (b == not b) ==> b)
  Arguments exhausted after 0 tests.
  Main>
  ```

- QuickCheck can be configured to use different numbers of tests/attempt
Writing Custom Generators:

Instead of generating random values and selecting only some, we can try to generate the ones we want directly:

```haskell
sortedList :: Gen [Int]
sortedList = do ns <- arbitrary
                return (sort ns)
```
More Examples:

Now we can use QuickCheck’s `forall` combinator to define:

```haskell
prop_mergePreservesOrder = forall sortedList $ \xs ->
forall sortedList $ \ys ->
  sorted (merge xs ys)
```

```haskell
prop_mergeCommutes = forall sortedList $ \xs ->
forall sortedList $ \ys ->
  merge xs ys == merge ys xs
```

```haskell
prop_mergeIdempotent = forall sortedList $ \xs ->
  merge xs xs == xs
```
Lessons Learned:

- QuickCheck is a useful and lightweight tool that encourages and rewards the lawful programmer!

- There is a script that automatically runs quickCheck on all of the properties in a file that have names of the form prop XXX

- Interpreting test results may require some care ...

- “Good” (random) test data can be hard to find ...