Sample solution for Pierce 8.3.4.

THEOREM [PRESERVATION]: If $\mathtt{t} : \mathtt{T}$ and $\mathtt{t} \to \mathtt{t}'$ then $\mathtt{t}' : \mathtt{T}$.

Proof by induction on the derivation of $\mathtt{t} \to \mathtt{t}'$.

Case E-IFTRUE: $\quad \mathtt{t} = \mathtt{if\ true\ then\ t_2\ else\ t_3} \quad \mathtt{t}' = \mathtt{t_2}$

Applying the Inversion lemma (8.2.2) on $\mathtt{t} : \mathtt{T}$, we have $\mathtt{t_2} : \mathtt{T}$, i.e. $\mathtt{t}' : \mathtt{T}$, as required.

Case E-IFFALSE: $\quad \mathtt{t} = \mathtt{if\ false\ then\ t_2\ else\ t_3} \quad \mathtt{t}' = \mathtt{t_3}$

Similar.

Case E-IF: $\quad \mathtt{t} = \mathtt{if\ t_1\ then\ t_2\ else\ t_3} \quad \mathtt{t}' = \mathtt{if\ t_1'\ then\ t_2\ else\ t_3} \quad \mathtt{t_1} \to \mathtt{t_1'}$

By inversion on $\mathtt{t} : \mathtt{T}$, we have $\mathtt{t_1} : \mathtt{Bool}$, $\mathtt{t_2} : \mathtt{T}$, and $\mathtt{t_3} : \mathtt{T}$. By induction on the subderivation for $\mathtt{t_1} \to \mathtt{t_1'}$, we have $\mathtt{t_1'} : \mathtt{Bool}$. So by T-IF, we can conclude $\mathtt{if\ t_1'\ then\ t_2\ else\ t_3} : \mathtt{T}$, as required.

Case E-SUCC: $\quad \mathtt{t} = \mathtt{succ\ t_1} \quad \mathtt{t}' = \mathtt{succ\ t_1'} \quad \mathtt{t_1} \to \mathtt{t_1'}$

By inversion, we have $\mathtt{T} = \mathtt{Nat}$ and $\mathtt{t_1} : \mathtt{Nat}$. By induction, $\mathtt{t_1'} : \mathtt{Nat}$. So by T-SUCC, $\mathtt{succ\ t_1'} : \mathtt{T}$, as required.

Case E-PREDZERO: $\quad \mathtt{t} = \mathtt{pred\ 0} \quad \mathtt{t}' = \mathtt{0}$

By inversion, $\mathtt{T} = \mathtt{Nat}$, so by T-ZERO we have $\mathtt{t}' : \mathtt{T}$.

Case E-PREDSUCC: $\quad \mathtt{t} = \mathtt{pred\ (succ\ nv_1)} \quad \mathtt{t}' = \mathtt{nv_1}$

By (repeated) inversion, $\mathtt{T} = \mathtt{Nat}$ and $\mathtt{nv_1} : \mathtt{Nat}$, so $\mathtt{t}' : \mathtt{T}$.

Case E-PRED: $\quad \mathtt{t} = \mathtt{pred\ t_1} \quad \mathtt{t}' = \mathtt{pred\ t_1'} \quad \mathtt{t_1} \to \mathtt{t_1'}$

By inversion, $\mathtt{T} = \mathtt{Nat}$ and $\mathtt{t_1} : \mathtt{Nat}$. By induction, $\mathtt{t_1'} : \mathtt{Nat}$. So by T-PRED, $\mathtt{pred\ t_1'} : \mathtt{T}$ as required.

Case E-ISZEROZERO: $\quad \mathtt{t} = \mathtt{iszero\ 0} \quad \mathtt{t}' = \mathtt{true}$

By inversion, $\mathtt{T} = \mathtt{Bool}$, so by T-TRUE we have $\mathtt{t}' : \mathtt{T}$.

Case E-ISZEROSUCC: $\quad \mathtt{t} = \mathtt{iszero\ (succ\ nv_1)} \quad \mathtt{t}' = \mathtt{false}$

Similar.

Case E-ISZERO: $\quad \mathtt{t} = \mathtt{iszero\ t_1} \quad \mathtt{t}' = \mathtt{iszero\ t_1'} \quad \mathtt{t_1} \to \mathtt{t_1'}$

By inversion, $\mathtt{T} = \mathtt{Bool}$ and $\mathtt{t_1} : \mathtt{Nat}$. By induction $\mathtt{t_1'} : \mathtt{Nat}$. So by T-ISZERO, $\mathtt{iszero\ t_1'} : \mathtt{Bool}$ as required. □

Sample solution for Pierce 9.2.2.

1. Let $\Gamma = \texttt{f}\!:\!\texttt{Bool}\!\to\!\texttt{Bool}$.

$$\dfrac{\dfrac{\texttt{f}\!:\!\texttt{Bool}\!\to\!\texttt{Bool} \in \Gamma}{\Gamma \vdash \texttt{f} : \texttt{Bool}\!\to\!\texttt{Bool}}\ \textsc{T-Var} \qquad \dfrac{\dfrac{}{\Gamma \vdash \texttt{false} : \texttt{Bool}}\ \textsc{T-False} \quad \dfrac{}{\Gamma \vdash \texttt{true} : \texttt{Bool}}\ \textsc{T-True} \quad \dfrac{}{\Gamma \vdash \texttt{false} : \texttt{Bool}}\ \textsc{T-False}}{\Gamma \vdash \texttt{if false then true else false} : \texttt{Bool}}\ \textsc{T-If}}{\Gamma \vdash \texttt{f (if false then true else false)} : \texttt{Bool}}\ \textsc{T-App}$$

2. Let $\Gamma = \texttt{f}\!:\!\texttt{Bool}\!\to\!\texttt{Bool}$ and $\Gamma_1 = \Gamma, \texttt{x}\!:\!\texttt{Bool}$.

$$\dfrac{\dfrac{\dfrac{\texttt{f}\!:\!\texttt{Bool}\!\to\!\texttt{Bool} \in \Gamma_1}{\Gamma_1 \vdash \texttt{f} : \texttt{Bool}\!\to\!\texttt{Bool}}\ \textsc{T-Var} \qquad \dfrac{\dfrac{\texttt{x} \in \Gamma_1}{\Gamma_1 \vdash \texttt{x} : \texttt{Bool}}\ \textsc{T-Var} \quad \dfrac{}{\Gamma_1 \vdash \texttt{false} : \texttt{Bool}}\ \textsc{T-False} \quad \dfrac{\texttt{x} \in \Gamma_1}{\Gamma_1 \vdash \texttt{x} : \texttt{Bool}}\ \textsc{T-Var}}{\Gamma_1 \vdash \texttt{if x then false else x} : \texttt{Bool}}\ \textsc{T-If}}{\Gamma_1 \vdash \texttt{f (if x then false else x)} : \texttt{Bool}}\ \textsc{T-App}}{\Gamma \vdash \lambda\texttt{x}\!:\!\texttt{Bool.f (if x then false else x)} : \texttt{Bool}\!\to\!\texttt{Bool}}\ \textsc{T-Abs}$$