HW2 Additional sample solution information

Question 1. Contextual Semantics for Pure λ-calculus.

Following the outline on p. 2 of the Contextual Semantics handout, we need to define a basic computational stepping relation $\rightarrow_{cmp}$, a grammar of contexts $\mathtt{C}$, and a contextual stepping relation $\rightarrow_{ctx}$.

The computational stepping relation is just given by rule E-AbsApp from the small-step semantics:

$$(\lambda \mathtt{x}.\mathtt{t})\ \mathtt{v} \rightarrow_{cmp} \mathtt{t}[\mathtt{v}/\mathtt{x}] \qquad \text{(E-AbsApp)}$$

The grammar is $\mathtt{C} ::= [\ ] \mid \mathtt{C}\ \mathtt{t} \mid \mathtt{v}\ \mathtt{C}$.

The contextual stepping relation is exactly as for the language of the handout, namely:

$$\frac{\mathtt{t} \rightarrow_{cmp} \mathtt{t}'}{\mathtt{C}[\mathtt{t}] \rightarrow_{ctx} \mathtt{C}[\mathtt{t}']} \qquad \text{(E-Step)}$$

Question 5. Pierce 5.3.6.

In addition to what's given in the book, the solution for full beta-reduction should include a rule allowing unrestricted reduction under lambdas:

$$\frac{\mathtt{t}_1 \rightarrow \mathtt{t}_1'}{\lambda \mathtt{x}.\mathtt{t}_1 \rightarrow \lambda \mathtt{x}.\mathtt{t}_1'} \qquad \text{(E-Abs)}$$

Question 6. Pierce 5.3.8

A correct solution for the big-step formulation is:

$$\lambda \mathtt{x}.\mathtt{t} \Downarrow \lambda \mathtt{x}.\mathtt{t} \qquad \text{(B-Value)}$$

$$\frac{\mathtt{t}_1 \Downarrow \lambda \mathtt{x}.\mathtt{t}_{12} \qquad \mathtt{t}_2 \Downarrow \mathtt{v}_2 \qquad [\mathtt{x} \mapsto \mathtt{v}_2]\mathtt{t}_{12} \Downarrow \mathtt{v}}{\mathtt{t}_1\ \mathtt{t}_2 \Downarrow \mathtt{v}} \qquad \text{(B-App)}$$

Theorem: $\mathtt{t} \rightarrow^* \mathtt{v} \iff \mathtt{t} \Downarrow \mathtt{v}$.

Proof

($\Longrightarrow$). [Thanks to Nicholas for this improved proof structure.]

First, we prove a lemma: if $\mathtt{t} \rightarrow \mathtt{t}'$ and $\mathtt{t}' \Downarrow \mathtt{v}$ then $\mathtt{t} \Downarrow \mathtt{t}$. Proof is by structural induction on the derivation of $\mathtt{t} \rightarrow \mathtt{t}'$, casing over the rule used at the root of this derivation.

- (E-AppAbs) $\mathtt{t} = (\lambda \mathtt{x}.\mathtt{t}_{12})\ \mathtt{v}_2$ and $\mathtt{t}' = [\mathtt{x} \mapsto \mathtt{v}_2]\mathtt{t}_{12}$. Since $\lambda \mathtt{x}.\mathtt{t}_{12}$ and $\mathtt{v}_2$ are already values that evaluate to themselves under B-Value, we can immediately apply B-App.
- (E-App1) $\mathtt{t} = \mathtt{t}_1\ \mathtt{t}_2$ and $\mathtt{t}' = \mathtt{t}_1'\ \mathtt{t}_2$, where $\mathtt{t}_1 \rightarrow \mathtt{t}_1'$. Since $\mathtt{t}'$ is not a value, the derivation of $\mathtt{t}' \Downarrow \mathtt{v}$ must be rooted by a use of B-App, so we have (i) $\mathtt{t}_1' \Downarrow \lambda \mathtt{x}.\mathtt{t}_{12}$, (ii) $\mathtt{t}_2 \Downarrow \mathtt{v}_2$ and (iii) $[\mathtt{x} \mapsto \mathtt{v}_2]\mathtt{t}_{12} \Downarrow \mathtt{v}$. Since $\lambda \mathtt{x}.\mathtt{t}_{12}$ is a value, we can apply the inductive hypothesis to (i) to obtain (i') $\mathtt{t}_1 \Downarrow \lambda \mathtt{x}.\mathtt{t}_{12}$. Then we can re-apply B-App to (i'),(ii), and (iii) to obtain $\mathtt{t} \Downarrow \mathtt{v}$.
- (E-App2) $\mathtt{t} = \mathtt{v}_1\ \mathtt{t}_2$ and $\mathtt{t}' = \mathtt{v}_1\ \mathtt{t}_2'$, where $\mathtt{t}_2 \rightarrow \mathtt{t}_2'$. Similar to previous case, where we apply induction to $\mathtt{t}_2' \Downarrow \mathtt{v}_2$ to obtain $\mathtt{t}_2 \Downarrow \mathtt{v}_2$.

Now given the lemma, the main proof is by induction on the length of $vt \rightarrow^* \mathtt{v}$. If there are no steps, $\mathtt{t} = \mathtt{v}$ and $\mathtt{t} \Downarrow \mathtt{v}$ follows immediately by B-VALUE. If there is more than one step, we have $\mathtt{t} \rightarrow^* \mathtt{t}' \rightarrow \mathtt{v}$, where by induction $\mathtt{t}' \Downarrow \mathtt{v}$. Then we can immediately apply the lemma to get $\mathtt{t} \Downarrow \mathtt{v}$.

($\Longleftarrow$) We proceed by induction on the big-step derivation, casing on the rule used at the root.

- (B-ABS) Immediate, since $\lambda\mathtt{x}.\mathtt{t}$ is already a value.
- (B-APP) We have $\mathtt{t} = \mathtt{t}_1\ \mathtt{t}_2$, $\mathtt{t}_1 \Downarrow \lambda\mathtt{x}.\mathtt{t}_{12}$, $\mathtt{t}_2 \Downarrow \mathtt{v}_2$, and $[\mathtt{x} \mapsto \mathtt{v}_2]\mathtt{t}_{12} \Downarrow \mathtt{v}$. By induction, we have (i) $\mathtt{t}_1 \rightarrow^* \lambda\mathtt{x}.\mathtt{t}_{12}$, (ii) $\mathtt{t}_2 \rightarrow^* \mathtt{v}_2$, and (iii) $[\mathtt{x} \mapsto \mathtt{v}_2]\mathtt{t}_{12} \rightarrow^* \mathtt{v}$. By (i) and repeated use of (E-APP1) [there is an inductive argument hiding here, but it is hardly worth spelling out], we can conclude $\mathtt{t}_1\ \mathtt{t}_2 \rightarrow^* (\lambda\mathtt{x}.\mathtt{t}_{12})\ \mathtt{t}_2$. Then by (ii) and repeated use of (E-APP2) [ditto], we can conclude $(\lambda\mathtt{x}.\mathtt{t}_{12})\ \mathtt{t}_2 \rightarrow^* (\lambda\mathtt{x}.\mathtt{t}_{12})\ \mathtt{v}_2$. By (E-APPABS) and (iii), $(\lambda\mathtt{x}.\mathtt{t}_{12})\ \mathtt{v}_2 \rightarrow [\mathtt{x} \mapsto \mathtt{v}_2]\mathtt{t}_{12} \rightarrow^* \mathtt{v}$. Combining these sequences gives the desired result.