Name:____________________________________

CS 578 Programming Language Semantics – Mid-term Exam May 3, 2022

This exam has 4 questions; most have several sub-parts. The worth of each question and sub-part is indicated in square brackets. There are 75 points in total, and you have 75 minutes for the exam. Please write your answers on the exam paper in the spaces provided. The exam is closed book.

All the questions concern the simply typed λ-calculus extended with Booleans and Pairs, which will be denoted $\lambda_{\to,\mathbb{B},\times}$. For your reference, syntax and semantic rules for $\lambda_{\to,\mathbb{B},\times}$ are provided at the end of the exam.

1. [15 pts.]

Consider the $\lambda_{\rightarrow,\mathbb{B},\times}$ term

$$\mathtt{t} = (\lambda \mathtt{y}\!:\!(\mathtt{Bool} \to \mathtt{Bool}) \times \mathtt{Bool}.(\mathtt{y.1})(\mathtt{y.2}))\,\{\lambda \mathtt{x}\!:\!\mathtt{Bool.x}, \mathtt{false}\}$$

When answering the following questions, you may abbreviate `Bool` by `B` and `false` by `F` to save writing time.

(a) [5 pts.] Show the sequence of one-step evaluation transitions (in the small-step semantics) that lead from `t` to the normal form `false`. It is *not* necessary to give the full derivation for each transition. (Hint: Four steps are needed.)

(b) [10 pts.] Draw a derivation tree using the typing rules to show that $\vdash$ `t` : `Bool`. (Hint: Your tree should have 11 nodes.)

2. [20 pts.]

Consider the following meta-properties that may apply to a language with a small-step semantics.

- **Determinacy** (of one-step evaluation): If $\mathtt{t} \to \mathtt{t}'$ and $\mathtt{t} \to \mathtt{t}''$ then $\mathtt{t}' = \mathtt{t}''$.
- **Uniqueness** (of normal forms): If $\mathtt{t} \to^* \mathtt{u}$ and $\mathtt{t} \to^* \mathtt{u}'$, where $\mathtt{u}$ and $\mathtt{u}'$ are both normal forms, then $\mathtt{u} = \mathtt{u}'$.
- **Termination** (of evaluation): For every term $\mathtt{t}$ there is some normal form $\mathtt{t}'$ such that $\mathtt{t} \to^* \mathtt{t}'$.
- **Progress**: If $\vdash \mathtt{t} : \mathtt{T}$ then either $\mathtt{t}$ is a value or else $\exists \mathtt{t}'$ such that $\mathtt{t} \to \mathtt{t}'$.
- **Preservation**: If $\vdash \mathtt{t} : \mathtt{T}$ and $\mathtt{t} \to \mathtt{t}'$ then $\vdash \mathtt{t}' : \mathtt{T}$.

For each of the following languages, state which, if any, of the properties are **false**, and, for each such property, give a brief counter-example demonstrating that the property does not hold.

(a) [5 pts.] Language $\lambda_{\to,\mathbb{B},\times}$.

(b) [5 pts.] Language $\lambda_{\to,\mathbb{B},\times}$ with the addition of a small-step rule

$$\{\mathtt{v}_1, \mathtt{v}_2\}.1 \to \mathtt{v}_2 \qquad \text{(E-FUNNY1)}$$

(c) [5 pts.] Language $\lambda_{\to,\mathbb{B},\times}$ with the addition of a small-step rule

$$\frac{\mathtt{t}_2 \to \mathtt{t}_2'}{\{\mathtt{t}_1,\mathtt{t}_2\} \to \{\mathtt{t}_1,\mathtt{t}_2'\}} \qquad \text{(E-FUNNY2)}$$

(d) [5 pts.] Language $\lambda_{\to,\mathbb{B},\times}$ with the addition of the typing rule

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{Bool} \qquad \Gamma \vdash \mathtt{t}_2 : \mathtt{Bool}}{\Gamma \vdash \{\mathtt{t}_1,\mathtt{t}_2\} : \mathtt{Bool}} \qquad \text{(T-FUNNY3)}$$

3. [25 pts.]

The following Preservation theorem holds for the small-step semantics of $\lambda_{\rightarrow,\mathbb{B},\times}$:

Theorem. If $\vdash \texttt{t} : \texttt{T}$ and $\texttt{t} \rightarrow \texttt{t}'$, then $\vdash \texttt{t}' : \texttt{T}$.

An incomplete proof of this theorem is given below. Complete the proof by filling in the three missing cases (marked by a ?). You may assume the following lemmas without proof:

- **Substitution Lemma**: If $\Gamma, \texttt{x} : \texttt{S} \vdash \texttt{t} : \texttt{T}$ and $\Gamma \vdash \texttt{s} : \texttt{S}$, then $\Gamma \vdash [\texttt{x} \mapsto \texttt{s}]\texttt{t} : \texttt{T}$.
- **Inversion Lemma** of the usual form.

Proof. By induction on the typing derivation $\vdash \texttt{t} : \texttt{T}$. We proceed by case analysis on the final rule in the derivation.

- Case T-VAR: ?

- Case T-ABS: No one-step rule applies, so this case cannot occur.
- Case T-APP: ?

- Case T-TRUE: No one-step rule applies, so this case cannot occur.
- Case T-FALSE: Similar to T-TRUE.

- Case T-IF: $\texttt{t} = \texttt{if } \texttt{t}_1 \texttt{ then } \texttt{t}_2 \texttt{ else } \texttt{t}_3$
  $\vdash \texttt{t}_1 : \texttt{Bool}$
  $\vdash \texttt{t}_2 : \texttt{T}$
  $\vdash \texttt{t}_3 : \texttt{T}$

  There are three cases, based on the possible one-step rules.

  - E-IFTRUE: Here $\texttt{t}_1 = \texttt{true}$ and $\texttt{t}' = \texttt{t}_2$, so result is immediate.
  - E-IFFALSE: Similar.
  - E-IF: Here $\texttt{t}_1 \to \texttt{t}_1'$ and $\texttt{t}' = \texttt{if } \texttt{t}_1' \texttt{ then } \texttt{t}_2 \texttt{ else } \texttt{t}_3$. By induction, $\vdash \texttt{t}_1' : \texttt{Bool}$. By T-IF, $\vdash \texttt{t}' : \texttt{T}$.

- Case T-PAIR: $\texttt{t} = \{\texttt{t}_1, \texttt{t}_2\}$
  $\vdash \texttt{t}_1 : \texttt{T}_1$
  $\vdash \texttt{t}_2 : \texttt{T}_2$
  $\texttt{T} = \texttt{T}_1 \times \texttt{T}_2$

  There are two cases, based on the possible one-step rules

  - E-PAIR1: Here $\texttt{t}_1 \to \texttt{t}_1'$ and $\texttt{t}' = \{\texttt{t}_1', \texttt{t}_2\}$. By induction, $\vdash \texttt{t}_1' : \texttt{T}_1$. By T-PAIR, $\vdash \{\texttt{t}_1', \texttt{t}_2\} : \texttt{T}_1 \times \texttt{T}_2$.
  - E-PAIR2: Similar.

- Case T-PROJ1: ?

- Case T-PROJ2: Similar to T-PROJ1.

4. [15 pts.]

This question asks about other semantic presentations of $\lambda_{\rightarrow,\mathbb{B},\times}$.

(a) [5 pts.] Here is a partial set of **big-step** evaluation rules for $\lambda_{\rightarrow,\mathbb{B},\times}$. Add the three missing rules.

$$\mathtt{v} \Downarrow \mathtt{v} \qquad \text{(B-VALUE)}$$

$$\frac{\mathtt{t}_1 \Downarrow (\lambda\mathtt{x}\mathtt{:}\mathtt{T}_{11}\mathtt{.}\mathtt{t}_{12}) \qquad \mathtt{t}_2 \Downarrow \mathtt{v}_2 \qquad [\mathtt{x} \mapsto \mathtt{v}_2]\mathtt{t}_{12} \Downarrow \mathtt{v}}{\mathtt{t}_1\;\mathtt{t}_2 \Downarrow \mathtt{v}} \qquad \text{(B-APP)}$$

$$\frac{\mathtt{t}_1 \Downarrow \mathtt{true} \qquad \mathtt{t}_2 \Downarrow \mathtt{v}_2}{\mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3 \Downarrow \mathtt{v}_2} \qquad \text{(B-IFTRUE)}$$

$$\frac{\mathtt{t}_1 \Downarrow \mathtt{false} \qquad \mathtt{t}_3 \Downarrow \mathtt{v}_3}{\mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3 \Downarrow \mathtt{v}_3} \qquad \text{(B-IFFALSE)}$$

(b) [5 pts.] Now consider a **contextual** semantics for $\lambda_{\to,\mathbb{B},\times}$. As usual, the top-level evaluation relation is given by a single rule

$$\frac{\texttt{t} \to_{cmp} \texttt{t}'}{\texttt{C[t]} \to_{ctx} \texttt{C[t}'\texttt{]}} \qquad \text{(E-STEP)}$$

The computation rules $\to_{cmp}$ are just a subset of the small-step evaluation rules. Which ones? (Just list their names.)

(c) [5 pts.] Still considering the **contextual** semantics for $\lambda_{\to,\mathbb{B},\times}$, complete this definition for the grammar of contexts:

$\texttt{C} ::= [\ ] \mid \texttt{C}\,\texttt{t} \mid \ldots$

Syntax and Rules for Simply Typed λ-calculus with Booleans and Pairs ($\lambda_{\to,\mathbb{B},\times}$)

Syntactic forms:

`t` ::=		*terms:*
	`x`	*variable*
	λ`x:T.t`	*abstraction*
	`t t`	*application*
	`true`	*constant true*
	`false`	*constant false*
	`if t then t else t`	*conditional*
	`{t,t}`	*pair*
	`t.1`	*first projection*
	`t.2`	*second projection*
`v` ::=		*values:*
	λ`x:T.t`	*abstraction value*
	`true`	*true value*
	`false`	*false value*
	`{v,v}`	*pair value*
`T` ::=		*types:*
	$\mathtt{T} \to \mathtt{T}$	*type of functions*
	`Bool`	*type of booleans*
	$\mathtt{T}_1 \times \mathtt{T}_2$	*type of pairs*
Γ ::=		*contexts:*
	$\emptyset$	*empty context*
	$\Gamma, \mathtt{x} : \mathtt{T}$	*term variable binding*

Typing rules:

$$\frac{\mathtt{x} : \mathtt{T} \in \Gamma}{\Gamma \vdash \mathtt{x} : \mathtt{T}} \tag{T-Var}$$

$$\frac{\Gamma, \mathtt{x} : \mathtt{T}_1 \vdash \mathtt{t}_2 : \mathtt{T}_2}{\Gamma \vdash \lambda \mathtt{x}{:}\mathtt{T}_1 . \mathtt{t}_2 : \mathtt{T}_1 \to \mathtt{T}_2} \tag{T-Abs}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_{11} \to \mathtt{T}_{12} \qquad \Gamma \vdash \mathtt{t}_2 : \mathtt{T}_{11}}{\Gamma \vdash \mathtt{t}_1\ \mathtt{t}_2 : \mathtt{T}_{12}} \tag{T-App}$$

$$\Gamma \vdash \mathtt{true} : \mathtt{Bool} \tag{T-True}$$

$$\Gamma \vdash \mathtt{false} : \mathtt{Bool} \tag{T-False}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{Bool} \qquad \Gamma \vdash \mathtt{t}_2 : \mathtt{T} \qquad \Gamma \vdash \mathtt{t}_3 : \mathtt{T}}{\Gamma \vdash \mathtt{if}\ \mathtt{t}_1\ \mathtt{then}\ \mathtt{t}_2\ \mathtt{else}\ \mathtt{t}_3 : \mathtt{T}} \tag{T-If}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_1 \qquad \Gamma \vdash \mathtt{t}_2 : \mathtt{T}_2}{\Gamma \vdash \{\mathtt{t}_1, \mathtt{t}_2\} : \mathtt{T}_1 \times \mathtt{T}_2} \tag{T-Pair}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_{11} \times \mathtt{T}_{12}}{\Gamma \vdash \mathtt{t}_1 . 1 : \mathtt{T}_{11}} \tag{T-Proj1}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \mathtt{T}_{11} \times \mathtt{T}_{12}}{\Gamma \vdash \mathtt{t}_1 . 2 : \mathtt{T}_{12}} \tag{T-Proj2}$$

Small-step evaluation rules:

$$\frac{\mathtt{t}_1 \to \mathtt{t}_1'}{\mathtt{t}_1\mathtt{t}_2 \to \mathtt{t}_1'\mathtt{t}_2} \tag{E-App1}$$

$$\frac{\mathtt{t}_2 \to \mathtt{t}_2'}{\mathtt{v}_1\mathtt{t}_2 \to \mathtt{v}_1\mathtt{t}_2'} \tag{E-App2}$$

$$(\lambda\mathtt{x}\mathtt{:}\mathtt{T}_{11}\mathtt{.}\mathtt{t}_{12})\ \mathtt{v}_2 \to [\mathtt{x} \mapsto \mathtt{v}_2]\mathtt{t}_{12} \tag{E-AppAbs}$$

$$\mathtt{if\ true\ then\ t}_2\ \mathtt{else\ t}_3 \to \mathtt{t}_2 \tag{E-IfTrue}$$

$$\mathtt{if\ false\ then\ t}_2\ \mathtt{else\ t}_3 \to \mathtt{t}_3 \tag{E-IfFalse}$$

$$\frac{\mathtt{t}_1 \to \mathtt{t}_1'}{\mathtt{if\ t}_1\ \mathtt{then\ t}_2\ \mathtt{else\ t}_3 \to \mathtt{if\ t}_1'\ \mathtt{then\ t}_2\ \mathtt{else\ t}_3} \tag{E-If}$$

$$\{\mathtt{v}_1\mathtt{,}\mathtt{v}_2\}\mathtt{.1} \to \mathtt{v}_1 \tag{E-PairBeta1}$$

$$\{\mathtt{v}_1\mathtt{,}\mathtt{v}_2\}\mathtt{.2} \to \mathtt{v}_2 \tag{E-PairBeta2}$$

$$\frac{\mathtt{t}_1 \to \mathtt{t}_1'}{\mathtt{t}_1\mathtt{.1} \to \mathtt{t}_1'\mathtt{.1}} \tag{E-Proj1}$$

$$\frac{\mathtt{t}_1 \to \mathtt{t}_1'}{\mathtt{t}_1\mathtt{.2} \to \mathtt{t}_1'\mathtt{.2}} \tag{E-Proj2}$$

$$\frac{\mathtt{t}_1 \to \mathtt{t}_1'}{\{\mathtt{t}_1\mathtt{,}\mathtt{t}_2\} \to \{\mathtt{t}_1'\mathtt{,}\mathtt{t}_2\}} \tag{E-Pair1}$$

$$\frac{\mathtt{t}_2 \to \mathtt{t}_2'}{\{\mathtt{v}_1\mathtt{,}\mathtt{t}_2\} \to \{\mathtt{v}_1\mathtt{,}\mathtt{t}_2'\}} \tag{E-Pair2}$$