Mystery Problem

Recall from the midterm the simply typed λ-calculus extended with Booleans and Pairs, which will be denoted \( \lambda_{\rightarrow, B, X} \). This problem concerns the big-step evaluation rules for \( \lambda_{\rightarrow, B, X} \). Consider the following theorem and accompanying “proof.”

**Theorem.** If \( \vdash t : T \), then there exists a \( v \) such that \( t \Downarrow v \) and \( \vdash v : T \).

**Proof.** By induction on the typing derivation \( \vdash t : T \). We proceed by case analysis on the root (“final”) rule in the derivation.

- **Case T-VAR**: This case is impossible, since the typing environment is empty.
- **Case T-ABS**: In this case \( t \) has the form \( \lambda x : T_1 . t_1 \), so it is already a value, and the conclusion is immediate.
- **Case T-APP**: \( t = t_1 t_2 \)
  \[ \vdash t_1 : T_1 \rightarrow T_2 \]
  \[ \vdash t_2 : T_1 \]
  \[ T = T_2 \]

  By induction on the derivation for \( t_1 \), there exists a value \( v_1 \) such that \( t_1 \Downarrow v_1 \) and \( \vdash v_1 : T_1 \rightarrow T_2 \). By the Canonical Forms Lemma, \( v \) can be written as \( \lambda x : T_1 . t_11 \). Hence we must have \( x : T_1 \vdash t_11 : T_2 \). By induction on the derivation for \( t_2 \), there exists a \( v_2 \) such that \( t_2 \Downarrow v_2 \) and \( \vdash v_2 : T_1 \). By the Substitution Lemma, \( \vdash [x \mapsto v_2]t_11 : T_2 \). Finally, by induction on this derivation, there exists a \( v \) such that \( [x \mapsto v_2]t_11 \Downarrow v \) and \( \vdash v : T_2 \). Combining our results and applying B-APP, we get \( t_1 t_2 \Downarrow v \) and \( \vdash v : T \), as required.

- **Case T-TRUE**: Similar to T-ABS.
- **Case T-FALSE**: Similar to T-ABS.
- **Case T-IF**: \( t = \text{if } t_1 \text{ then } t_2 \text{ else } t_3 \)
  \[ \vdash t_1 : \text{Bool} \]
  \[ \vdash t_2 : T \]
  \[ \vdash t_3 : T \]

  By induction on the derivation for \( t_1 \), there exists a value \( v_1 \) such that \( t_1 \Downarrow v_1 \) and \( \vdash v_1 : \text{Bool} \). By the Canonical Forms Lemma, \( v_1 \) must be either true or false. Suppose \( v_1 = \text{true} \) (the other case is similar). By induction on the derivation for \( t_2 \), there exists a \( v_2 \) such that \( t_2 \Downarrow v_2 \) and \( \vdash v_2 : T \). By B-IF-TURE, \( t \Downarrow v_2 \), as required.

- **Case T-PAIR**: \( t = \{t_1, t_2\} \)
  \[ \vdash t_1 : T_1 \]
  \[ \vdash t_2 : T_2 \]
  \[ T = T_1 \times T_2 \]

  By induction on the deriviation for \( t_1 \), there exists a \( v_1 \) such that \( t_1 \Downarrow v_1 \) and \( \vdash v_1 : T_1 \). Similarly, there exists a \( v_2 \) such that \( t_2 \Downarrow v_2 \) and \( \vdash v_2 : T_2 \). Then \( v = \{v_1, v_2\} \) is a value. By B-PAIR, \( t \Downarrow v \) and by T-PAIR, \( \vdash v : T_1 \times T_2 \), as required.
• Case T-PROJ1: \( t = t_0 . 1 \)

\[
\vdash t_0 : T_1 \times T_2 \\
T = T_1
\]

By induction on the derivation for \( t_0 \), there exists a \( v_0 \) such that \( t_0 \Downarrow v_0 \) and \( \vdash v_0 : T_1 \times T_2 \).

By the Canonical Forms Lemma, \( v_0 \) must be of the form \( \{ v_1, v_2 \} \) for some values \( v_1 \) and \( v_2 \). By B-PROJ1, \( t \Downarrow v_1 \), and, by inversion, \( \vdash v_1 : T_1 \), as required.

• Case T-PROJ2: Similar to T-PROJ1.

This theorem is actually true, but the “proof” has a fatal flaw. Identify precisely where and how the proof goes wrong. (Note: There might also be minor flaws, typos, etc. But what you’re looking for is not just an easily fixed error.)

**Rules for Simply Typed \( \lambda \)-calculus with Booleans and Pairs (\( \lambda \to, B, \times \))**

**Typing rules**

\[
\begin{align*}
x : T &\in \Gamma \\
\Gamma \vdash x : T &\quad (T-VAR) \\
\Gamma, x : T_1 \vdash t_2 : T_2 &\quad (T-ABS) \\
\Gamma \vdash \lambda x : T_1 . t_2 : T_1 \to T_2 \\
\Gamma \vdash t_1 : T_11 \to T_12 &\quad (T-APP) \\
\Gamma, t_1 : T_11 \vdash t_12 &\quad (T-TRUE) \\
\Gamma \vdash \text{true} : \text{Bool} &\quad (T-FALSE) \\
\Gamma \vdash \text{false} : \text{Bool} &\quad (T-IF) \\
\Gamma \vdash t_1 : \text{Bool} &\quad (T-PAIR) \\
\Gamma \vdash t_2 : T &\quad (T-PROJ1) \\
\Gamma \vdash t_3 : T &\quad (T-PROJ2)
\end{align*}
\]

**Big-step evaluation rules**

\[
\begin{align*}
v \Downarrow v &\quad (B-VALUE) \\
t_1 \Downarrow (\lambda x : t_11 . t_12) &\quad t_2 \Downarrow v_2 &\quad [x \mapsto v_2] t_12 \Downarrow v \\
&\quad t_1 \Downarrow v_1 \quad t_2 \Downarrow v_2 &\quad (B-APP) \\
\text{if } t_1 \text{ then } t_2 \text{ else } t_3 &\Downarrow v_2 &\quad (B-IFTRUE) \\
\text{if } t_1 \text{ then } t_2 \text{ else } t_3 &\Downarrow v_3 &\quad (B-IFFALSE) \\
\{ t_1, t_2 \} &\Downarrow \{ v_1, v_2 \} &\quad (B-PAIR) \\
&\quad t \Downarrow \{ v_1, v_2 \} &\quad (B-PROJ1) \\
&\quad t.1 \Downarrow v_1 &\quad (B-PROJ2) \\
&\quad t.2 \Downarrow v_2
\end{align*}
\]