To determine how long to keep a given variable (or temporary) in a register, need to know the range of instructions for which the variable is live.

A variable or temporary is live immediately following an instruction if its current value will be needed in the future (i.e., it will be used again, and it won’t be changed before that use).

It’s easy to calculate liveness for a consecutive series of instructions without branches, just by working backwards.

But if a value can stay in a register over a jump, things get harder.
Linear Scan Allocation

Using live ranges turns out to be computationally expensive (more later). A simple alternative is to approximate each live range by a live interval. This is the consecutive interval of instructions between the first and last use of each variable.

Live after instruction:

\[
\begin{align*}
1 & \text{ mov } 0, \%t1 \\
2 & \text{ L1: add } \%t1, 1, \%t2 \\
3 & \text{ add } \%t3, \%t2, \%t3 \\
4 & \text{ mul } \%t2, 2, \%t1 \\
5 & \text{ cmp } \%t1, 1, 1000 \\
6 & \text{ bl L1} \\
7 & \text{ return } \%t3
\end{align*}
\]

Live ranges: \%t1: 1, 4, 5, 6 \%t2: 2, 3 \%t3: 1, 2, 3, 4, 5, 6

Live intervals: \%t1: [1, 6] \%t2: [2, 3] \%t3: [1, 6]

(Revised) Basic idea: if two temporaries have non-overlapping live intervals, they can occupy the same physical register.

Linear Scan Allocation Algorithm Details

1. Compute \text{startpoint}[i] and \text{endpoint}[i] of live interval \(i\) for each variable. Store the intervals in a list in order of increasing start point.

2. Initialize set \text{active} := \emptyset and pool of free registers = all usable registers.

3. For each live interval \(i\) in order of increasing start point:

   a) For each interval \(j\) in \text{active}, in order of increasing end point

      i. If endpoint\[j\] \(\geq\) startpoint\[i\] break to step (b).
      ii. Remove \(j\) from \text{active}.
      iii. Add register\[j\] to pool of free registers.

   b) Set register\[i\] := next register from pool of free registers, and remove it from pool. (If pool is already empty, need to spill.)

   c) Add \(i\) to \text{active}, sorted by increasing end point.

A more complicated form of linear scan does take advantage of knowing precise live ranges (expressed as "holes" in live intervals).

Computing Liveness

Liveness is another property that is computed using dataflow analysis. Classic algorithm is defined on standard CFG (not necessarily in SSA).

\[
\begin{align*}
\text{gen}[n] &= \text{set of variables used by node } n \\
\text{kill}[n] &= \text{set of variables defined by node } n
\end{align*}
\]

\[
\begin{align*}
s & \quad \text{gen}[s] \quad \text{kill}[s] \\
t < b \quad \text{bop c} & \quad \{b, c\} \quad \{t\} \\
t < M[b] & \quad \{b\} \quad \{t\} \\
M[a] < b & \quad \{a, b\} \quad \{t\} \\
\text{if a relop b then L} & \quad \{a, b\} \quad \{t\} \\
t < f(a1, \ldots an) & \quad \{a1, \ldots an\} \quad \{t\}
\end{align*}
\]

Note that flow equations for this problem are backwards, i.e., data flows in reverse direction from control flow.

\[
\begin{align*}
\text{in}[n] &= \text{gen}[n] \cup (\text{out}[n] - \text{kill}[n]) \\
\text{out}[n] &= \bigcup_{s \in \text{succ}[n]} \text{in}[s]
\end{align*}
\]

In this case we want the fixed point solution having the smallest sets.

Liveness Example

\[
\begin{align*}
1 & \quad a \leftarrow 0 \\
2 & \quad L: \quad b \leftarrow a + 1 \\
3 & \quad c \leftarrow c + b \\
4 & \quad a \leftarrow b * 2 \\
5 & \quad \text{if a < N goto L} \\
6 & \quad f(c)
\end{align*}
\]

Assume that \(a, b, c\) are local variables not used after the termination of this code fragment.

We can extract the problem parameters:

\[
\begin{align*}
n & \quad \text{succ}[n] \quad \text{gen}[n] \quad \text{kill}[n] \\
6 & \quad - \quad c \quad - \\
5 & \quad 2, 6 \quad a \quad - \\
4 & \quad 5 \quad b \quad a \\
3 & \quad 4 \quad b, c \quad c \\
2 & \quad 3 \quad a \quad b \\
1 & \quad 2 \quad - \quad a
\end{align*}
\]
**SOLVING EXAMPLE**

Here's a solution. The control flow equations are solved as usual, but it is more efficient (takes fewer iterations) to fill them in in (roughly) reverse execution order, computing $\text{out}[i]$ before $\text{in}[i]$. That's because liveness is a "backwards" flow problem.

<table>
<thead>
<tr>
<th>iteration 0</th>
<th>iteration 1</th>
<th>iteration 2</th>
<th>iteration 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>-</td>
<td>-</td>
<td>c</td>
</tr>
<tr>
<td>5</td>
<td>-</td>
<td>-</td>
<td>c</td>
</tr>
<tr>
<td>4</td>
<td>-</td>
<td>-</td>
<td>a,c</td>
</tr>
<tr>
<td>3</td>
<td>-</td>
<td>-</td>
<td>b,c</td>
</tr>
<tr>
<td>2</td>
<td>-</td>
<td>-</td>
<td>b,c</td>
</tr>
<tr>
<td>1</td>
<td>-</td>
<td>-</td>
<td>a,c</td>
</tr>
</tbody>
</table>

Note that this in example, no more than two of \{a,b,c\} are ever simultaneously live, so two registers will suffice to hold these variables at all times.

**VISUALIZING RESULTS**

<table>
<thead>
<tr>
<th>Live ranges</th>
<th>Live Intervals</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>c</td>
</tr>
</tbody>
</table>

| 1 | a <- 0 |
| 2 | b <- a + 1 |
| 3 | c <- c + b |
| 4 | a <- b * 2 |
| 5 | if a < N goto L |
| 6 | f(c) |

**STATIC VS. DYNAMIC LIVENESS EXAMPLE**
Static Liveness (2)

Is a live-out at node 2? It depends on whether control flow ever reaches node 4.

A smart compiler could answer no.

A smarter compiler could answer similar questions about more complicated programs.

But no compiler can ever always answer such questions correctly. This is a consequence of the uncomputability of the Halting Problem.

So we must be content with static liveness, which talks about paths of control-flow edges, and is just a conservative approximation of dynamic liveness, which talks about actual execution paths.

Coloring Register Interference Graphs

More general mechanism for doing register allocation is by translation to graph coloring.

- Build a register interference graph, which has
  - a node for each logical register.
  - an edge between two nodes if the corresponding registers are simultaneously live.
- Attempt to color the nodes of the graph so that no two nodes connected by an edge have the same color.
  (Like coloring a map, where nodes = countries and edges connect countries with a common border.)
- If we have \( k \) physical registers, we try to color with \( k \) colors.
- If this fails, we must spill and try again. (Nasty in practice.)

Example

Live after instr.

```
ld a,t0 ; a:t0  t0
ld b,t1 ; b:t1  t0  t1
sub t0,t1,t2 ; t:t2  t0  t2
ld c,t3 ; c:t3  t0  t2  t3
sub t0,t3,t4 ; u:t4  t2  t4
add t2,t4,t5 ; v:t5  t4  t5
add t5,t4,t6 ; d:t6  t6
st t6,d
```

```
___________
| \         |
|  t0 ----- t1    t2  |
|  | __________//  |
|  |    ______/    |
|  \    /     |
|  t3  t4------ t5  t6
```

Coloring Heuristics

In general case, determining whether a graph can be \( k \)-colored is hard (N.P. Complete, and hence probably exponential).

But a simple heuristic will usually find a \( k \)-coloring if there is one.

1. Choose a node with fewer than \( k \) neighbors.

2. Remove that node. Note that if we can color the resulting graph with \( k \) colors, we can also color the original graph, by giving the deleted node a color different from all its neighbors.

3. Repeat until either
   - there are no nodes with fewer than \( k \) neighbors, in which case we must spill; or
   - the graph is gone, in which case we can color the original graph by adding the deleted nodes back in one at a time and coloring them.

For our example, this heuristic finds a 3-coloring, which is the best we can do.
LIVENESS AND COLORING FOR SSA GRAPH

Computing liveness is easier for SSA graphs:

- Don’t need kill sets!
- Must be careful about $\phi$-nodes:
  - Assignments must be done in parallel.
  - Arguments are not really live simultaneously.

Recent work indicates that interference graphs derived from SSA code can be colored in polynomial time

- Should allow considerably simpler coloring-based allocation algorithms.