int x = 17
Z[1023] := 99;
double e = 2.81828
type emp = {name: string, age: int}
class Foo extends Bar { ... }
data Day = Mon | Tue | Wed | Thu | Fri | Sat | Sun
String s = "abc"
type Days = set of Day
fold :: (a -> b -> b) -> [a] -> b -> b
'a btree = LEAF of 'a | NODE of 'a * 'a btree * 'a btree

---

**Organizing the Zoo**

<table>
<thead>
<tr>
<th>Mathematical View</th>
<th>Programming Language View</th>
<th>Machine View</th>
</tr>
</thead>
</table>

Programming Language view: Types classify values and operations
Mathematical View: Types are sets of values
Machine View: Types describe memory layout of values

---

**Example: Integers**

\[ \mathbb{Z} \leftrightarrow \text{int} \leftrightarrow \mathbb{Z}^{2^1 2^1 2^1 2^1 2^1 2^1 2^1 2^1 2^1} \]
**Example: Records**

\[ \mathbb{Z} \times \mathbb{Z} \leftrightarrow \{a : \text{int}, b : \text{int}\} \leftrightarrow \begin{array}{c} a \\ b \end{array} \begin{array}{c} 42 \\ 999 \end{array} \]

**Example: Arrays**

\[ \mathbb{N} \rightarrow \mathbb{Z} \leftrightarrow \text{int[]} \leftrightarrow \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 98 \\ 99 \end{array} \begin{array}{c} 18 \\ 22 \\ 327899 \\ -14 \\ 107 \\ 2222 \end{array} \]

**Other Useful Axes for Classification**

- **Primitive Types vs. User-defined Types**
  - Is the type “built in” to the language?

- **Atomic Types vs. Type Constructors**
  - Are values constructed out of smaller parts?

- **Abstract Types vs. Concrete Types**
  - Are value representations visible to the programmer?

- **Mutable Types vs. Immutable Types**
  - Can values be modified once created?

- **Reference Types vs. Non-reference Types**
  - What are the semantics of assignment?
  - etc.
**Machine Types**

Machine language doesn't distinguish types; all values are just bit patterns until used. As such they can be loaded, stored, moved, etc. But certain type-specific operations are supported directly by hardware; the operands are in some sense implicitly typed.

Typical machine-level types:

- **Integers** of various sizes, signedness, etc. with standard arithmetic operations.
- **Bit vectors** of various sizes, with bit-level logic operations (and, or, etc.)
- **Floating point** numbers of various sizes, with standard arithmetic operations.
- **Pointers** to values stored in memory, with load and store operations.
- **Instructions**, i.e., code, which can be executed.

There is no abstraction at machine level: programs can inspect individual words and bits of any value.

**Primitive Atomic Types**

Most higher-level languages provide some built-in atomic types.

- e.g. in Java: boolean, byte, short, int, long, char, float, double

Built-in type names and operators are keywords or part of initial environment

Language may have special syntax for writing literal values (e.g. numbers, character strings)

Atomic types are usually abstract: programs cannot inspect internal details of value representation. e.g.,

- usually cannot extract mantissa or exponent from a floating point number (C/C++ is an exception)
- can't tell what convention is used to encode booleans as numbers (C/C++ is an exception)

Atomic types are often closely based on standard machine-level types, giving them an obvious representation and (usually) efficient implementation for operations

**Constructed Types**

Many languages also provide a set of mechanisms for constructing new, user-defined types from existing types.

- e.g. Java has array and class definition mechanisms

Each type constructor comes with corresponding mechanisms for:

- constructing values
- inspecting values
- (for mutable types) modifying values

For example, in Java:

- arrays are constructed using new and an optional list of initializers; their elements are inspected and modified using subscript notation (e.g. a[i]).
- class instances (objects) are constructed using new and defined constructor methods; object fields can be accessed using dot notation (e.g. p.x).

**Abstraction for Constructed Types**

Are constructed types abstract?

- User-defined types can't be completely abstract, because we must have access to their components in order to write operations over the type.
- But many languages have mechanisms for limiting component access in order to enforce some level of abstraction, e.g. Java's private fields

The line between built-in and user-defined types is not always clear-cut.

- e.g, in Java, the String type is really just an ordinary class that happens to be provided in the standard library, except that there is special syntax for string literals and concatenation.
- Built-in constructed type values usually need to be kept abstract in order to guarantee that they behave as specified.

We will examine data abstraction mechanisms in detail later.
Mathematical View of Type Constructors

It can be enlightening to view type constructors as operators on the underlying sets represented by the component types.

Early on in the history of programming languages, it became clear that a small number of type operators suffices to describe most useful data structures:

- Cartesian product ($S_1 \times S_2$)
- Disjoint union ($S_1 + S_2$)
- Mapping (by explicit enumeration or by formula) ($S_1 \rightarrow S_2$)
- Set ($P S$)
- Sequence ($S^*$)
- Recursive definition ($\mu S.T(s)$)

Records = Cartesian Products

Records, tuples, "structures", etc. Nearly every language has them.

"Take a bunch of existing types and choose one value from each."

Examples (Ada Syntax)

```ada
  type EMP is
    record
      NAME : STRING;
      AGE : INTEGER;
    end record;
  
  E: EMP := (NAME => "ANDREW", AGE => 99);
```

(ML syntax):

```ml
  type emp = string * int (unlabeled fields)
  val e : emp = ("ANDREW",99);

  type emp =
    {name: string, age: int} (labeled fields)
  val e : emp = {name="ANDREW",age=99};
```

Representation of Constructed Types

Concretely, each language defines the internal representation of values of the composite type, based on the type constructor and the types used in the construction.

Historically, most languages have provided just a few constructors, usually with the property that constructed values can be represented and accessed efficiently on conventional hardware.

For conventional languages, this is the short list:

- Records
- Unions
- Arrays

Many languages also support manipulation of pointers to values of these types, in order to allow operating on data "by reference" and to support recursive structures.

Records (continued)


Representation: Fields occupy successive memory addresses (perhaps with some padding to maintain hardware-required alignments), so total size is (roughly) sum of field sizes.

Each field lives at a known static offset from the beginning of the record, allowing very fast access using pointer arithmetic.

Because records may be large, they are often manipulated by reference, i.e., represented by a pointer. The fields within a record may also be represented this way.
**DISJOINT UNIONS**

Variant records, discriminated records, unions, etc.

"Take a bunch of existing types and choose one value from one type.”

Introduced in Pascal:

```pascal
type RESULT = record
    case found : Boolean of
        true: (value:integer);
        false: (error:STRING)
    end;

function search (...) : RESULT;...
```

Generally behave like records, with tag as an additional field.

A variant value is represented by the tag following by the representation of the particular variant. Its size is thus bounded by the size of the largest possible variant plus the tag size.

**DISJOINT UNIONS DONE PROPERLY**

ML has very clean approach to building and inspecting disjoint unions:

```ml
datatype result = FOUND of integer | NOTFOUND of string

fun search (...) : result =
    if ... then FOUND 10 else NOTFOUND "problem"

case search(...) of
    FOUND x =>
        print ("Found it : " ^ (Int.toString x))
    | NOTFOUND s =>
        print ("Couldn’t find it : " ^ s)
```

Here **FOUND** and **NOTFOUND** tags are not ordinary fields. The case expression **combines** inspection of tag and extraction of values into one operation.

**VARIANT INSECURITIES**

Pascal variant records are **insecure** because it is possible to manipulate the tag independently from the variant contents.

```pascal
tr.value := 101; { write an integer }
write tr.error; { but read a string! }
if (tr.found) then begin { check for integer }
    tr := tr1; { overwrite with arbitrary RESULT }
    x := tr.value { might now contain a string }
```

These problems were fixed in Ada by requiring tag and variant contents to be set simultaneously, and inserting a runtime check on the tag before any read of the variant contents.

**OBJECTS**

Objects in class-based OO languages, e.g. Java, can be viewed as a sort of variant record.

- The object’s class identifier is stored at the beginning of the record, and acts like a variant tag, distinguishing among different subclasses.
- The remaining record fields correspond to the object’s instance variables.

The class tag is used to control dynamic dispatch to the class’s methods, and (depending on the language) might be accessible more directly (e.g. Java’s `instanceof`).

- The class tag cannot be altered after the object is created, so there is no danger of insecurity.
### Arrays

The oldest type constructor, found in Fortran (the first real high-level language); crucial for numerical computations.

Basic implementation idea: a table laid out in adjacent memory locations permitting indexed access to any element in constant time, using the hardware’s ability to compute memory addresses.

Mathematically: A finite mapping from an index set to element set.

Index set is nearly always a set of integers 0..n − 1, or some other discrete set isomorphic to such a set.

Multidimensional arrays (matrices) can be built in several ways:

- using an index set of tuples of integers (e.g. in Fortran)
- using an element set of arrays (e.g. in C/C++/Java)

Is the size of an array part of its type? Some older languages (e.g. Fortran) took this attitude, but most modern languages are more flexible, and allow the size to be set independently for each array value when the array is first created:

- as a local variable, e.g., in Ada:
  ```
  function fred(size:integer);
  var bill: array(0..size) of real;
  ```
- or on the heap, e.g., in Java:
  ```
  int[] bill = new int[size];
  ```

Arrays are often large, and hence better manipulated by reference.

The major security issue for arrays is bounds checking of index values. In general, it’s not possible to check all bounds at compile time (though often possible in particular cases). Runtime checks are always possible; this may be costly, but it's usually worth it!

### Functions and Mappings

Mathematical mappings can also be represented by an algorithmic formula.

A function gives a “recipe” for computing a result value from an argument value.

A program function can describe an infinite mapping.

But differs from mathematical function in that:

- it must be specified by an explicit algorithm
- executing the function may have side-effects on variables.

As we have seen, it can be very handy to manipulate functions as first-class values.

What about data structures of essentially unbounded size, such as sequences (or lists)?

“Take an arbitrary number of values of some type.”

Such data structures require special treatment: they are typically represented by small segments of data linked by pointers, and dynamic storage allocation (and deallocation) is required.

The basic operations on a sequence include

- concatenation (especially concatenating a single element onto the head or tail of an existing sequence); and
- extraction of elements (especially the head).

An important example is the (unbounded) string, a sequence of chars.

Best representation depends heavily on what nature and frequency of various operations. Hard to give single, uniformly efficient implementation.
DEFINING SEQUENCES

Unless the programming language supports sequences directly, the programmer must define them using a **recursive** definition.

For example, a list of integers is either
- **empty**, or
- has a **head** which is an integer and a **tail** which is itself a list of integers.

In mathematical notation, we can write

\[
\text{list}_\mathbb{Z} = \text{empty} + (\mathbb{Z} \times \text{list}_\mathbb{Z})
\]

or, avoiding the need to name the type,

\[
\mu t.(\text{empty} + (\mathbb{Z} \times t))
\]

where the **fixpoint** operator \( \mu s.T(s) \) defines the smallest type \( s \) such that \( s = T(s) \).

ML lets us write down this type almost directly:

```
datatype intlist = EMPTY | CELL of int * intlist
```
Structural equivalence is defined inductively:

- Primitive types are equivalent if they are exactly the same type.
- Cartesian product types are equivalent if their corresponding component types are equivalent. (Record field names may or may not matter.)
- Disjoint union types are equivalent if their corresponding component types are equivalent.
- Mapping types (arrays and functions) are the same if their domain and range types are the same.
- Recursive types are a challenge. Are these two types structurally equivalent?

```plaintext
type t1 = { a:int, b: POINTER TO t1 };
type t2 = { a:int, b: POINTER TO t2 };
```

Intuitively yes, but it’s (a little) tricky for a type-checking algorithm to determine this!

The question of equivalence is more interesting if the language has type names, which arise for two main reasons:

- As a convenient shorthand to avoid giving the full type each time, e.g.

```plaintext
function f(x:int * bool * real) : int * bool * real = ...
type t = int * bool * real
function f(x:t) : t = ...
```

- As a way of improving program correctness by subdividing values into types according to their meaning within the program, e.g.

```plaintext
type polar = { r:real, a:real }
type rect = { x:real, y:real }
function polar_add(x:polar,y:polar) : polar = ...
function rect_add(x:rect,y:rect) : rect = ...
var a:polar; c:rect;
a := (150.0,30.0) (* ok *)
polar_add(a,a) (* ok *)
c := a (* type error *)
rect_add(a,c) (* type error *)
```

Whole idea here is that some structurally equivalent are treated as inequivalent.

Also: what about unnamed type expressions?

```plaintext
type t = int * int
procedure f(x: int * int) = ...
procedure g(x: t) = ...
var a:t = (3,4)
g(a); (* ok *)
f(a); (* ok or not ?? *)
```

Different types now seem too distinct; can’t even convert from one form of real to another.

Simplistic idea: Two types are equivalent iff they have the same name. Supports polar/rect distinction.

But pure name equivalence is very restrictive, e.g.:

```plaintext
type ftemp = real
type ctemp = real
var x:ftemp, y:ftemp, z: ctemp;
x := y; (* ok *)
x := 10.0; (* probably ok *)
x := z; (* type error *)
x := 1.8 * z + 32.0; (* probably type error *)
```

Different types now seem too distinct; can’t even convert from one form of real to another.
C Type Equivalence

C uses structural equivalence for array and function types, but name equivalence for struct, union, and enum types. For example:

    char a[100];
    void f(char b[]);
    f(a); /* ok */

    struct polar{float x; float y;};
    struct rect{float x; float y;};
    struct polar a;
    struct rect b;
    a = b; /* type error */

A type defined by a typedef declaration is just an abbreviation for an existing type.

Note that this policy makes it easy to check equivalence of recursive types, which can only be built using structs.

    struct fred {int x; struct fred *y;} a;
    struct bill {int x; struct fred *y;} b;
    a = b; /* type error */

Java Type Equivalence

Java uses nearly strict name equivalence, where names are either:
- One of eight built-in primitive types (int, float, boolean, etc.), or
- Declared classes or interfaces (reference types).

The only non-trivial type expressions that can appear in a source program are array types, which are compared structurally, using name equivalence for the ultimate element type. Java has no mechanism for type abbreviations.

Java types form a subtyping hierarchy:
- If class A extends class B, then A is a subtype of B.
- If class A implements interface I, then A is a subtype of I.
- If numeric type t can be coerced to numeric type u without loss of precision, then t is a subtype of u.

If T\textsubscript{1} is a subtype of T\textsubscript{2}, then a value of type T\textsubscript{1} can be used wherever a value of T\textsubscript{2} is expected.