INFORMAL SEMANTICS

- Grammars can be used to define the legal programs of a language, but not what they do! (Actually, most languages place further, non-grammatical restrictions on legal programs, e.g., type-correctness.)
- Language behavior is usually described, documented, and implemented on the basis of natural-language (e.g., English) descriptions.
- Descriptions are usually structured around the language’s grammar, e.g., they describe what each nonterminal does.
- Natural-language descriptions tend to be imprecise, incomplete, and inconsistent.

EXAMPLE: FORTRAN DO-LOOPS

"DO n i = m1,m2,m3
Repeat execution through statement n, beginning with i = m1, incrementing by m3, while i is less than or equal to m2. If m3 is omitted, it is assumed to be 1. m’s and i’s cannot be subscripted. m’s can be either integer numbers or integer variables; i is an integer variable."

Consider:

    DO 100 I = 10,9,1
    ...
    100 CONTINUE

How many times is the body executed?

EXPERIMENTAL SEMANTICS

Try it and see!

Implementation becomes standard of correctness.
This is certainly precise: compiler source code becomes specification.
But it is:
- difficult to understand;
- awkward to use;
- subject to accidental change;
- wholly non-portable.
Aims:
• **Rigorous** and **unambiguous** definition in terms of a well-understood formalism, e.g., logic, naive set theory, etc.
• Independence from **implementation**. Definition should describe how the language behaves as abstractly as possible.

Uses:
• Provably-correct implementations.
• Provably-correct programs.
• Basis for language comparison.
• Basis for language design.
  (But usually not basis for learning a language.)

Main varieties: **Operational, Denotational, Axiomatic.**
Each has different purposes and strengths. In this course, we’ll mostly focus on operational semantics, with brief looks at the others.

---

**Semantics from Interpreters**

In the homework, we’re building **definitional interpreters** for small languages that display key programming language constructs.

Our goal is to study the interpreter code in order to understand **implementation** issues associated with each language.

In addition, the interpreter serves as a form of **semantic** definition for each language construct. In effect, it defines the meaning of the language in terms of the semantics of Scala.

(Of course, you’re also learning more about the semantics of Scala as we go!)

---

**Semantics and Erroneous Programs**

An important part of a language specification is distinguishing valid from invalid programs.

It is useful to define three classes of errors that make programs invalid. (Of course, even valid programs may behave differently than the programmer intended!)

**Static errors** are violations of the language specification that can be detected at compilation time (or, in an interpreter, before interpretation begins)
• Includes: **lexical** errors, **syntactic** errors (caught during parsing), **type** errors, etc.
• Compiler or interpreter issues an error pinpointing erroneous location in source program.
• Language **semantics** are usually defined only for programs that have no static errors.

---

**Runtime Errors**

**Checked runtime errors** are violations that the language implementation is required to detect and report at runtime, in a clean way.
• Examples in Scala or Java: division by zero, array bounds violations, dereferencing a null pointer.
• Depending on language, implementation may issue an error message and die, or raise an exception (which can be caught by the program).
• Language semantics must specify behavior precisely.

**Unchecked runtime errors** are violations that the implementation need not detect.
• Subsequent behavior of the computation is **arbitrary**. (Error is often not manifested until much later in execution.)
• Examples in C: division by zero, dereferencing a null pointer, array bounds violations.
• Language semantics probably don’t specify behavior.
• **Safe** languages like Scala, Java, or Python have **no** such errors!
Axiomatic Semantics

Interpreters give an operational semantics for imperative statements. (We’ll see other, more formal, operational approaches to semantics shortly.)

An important alternative approach is to give a logical interpretation to statements.

• The state of an imperative program is defined by the values of all its variables.
• We can characterize a state by giving a logical predicate (or assertion), mentioning the variables, which is satisfied by the values of the variables in that state.
• We can define the semantics of statements by saying how they affect (arbitrary) predicates.

Triples involving Assertions

We write a Hoare triple
\[
\{ P \} S \{ Q \}
\]
to mean that if the precondition \( P \) is true before the execution of \( S \) then the postcondition \( Q \) will be true after the execution of \( S \).

Note that the triple says nothing about what happens if \( S \) doesn’t terminate. So we are only characterizing statements that terminate.

Examples of triples (not all stating true things!)

\[
\{ y \geq 3 \} x := y + 1 \{ x \geq 4 \}
\]

\[
\{ x + y = c \} \text{ while } x > 0 \text{ do } \begin{array}{l}
y := y + 1; \\
x := x - 1
\end{array} \{ x + y = c \}
\]

\[
\{ y = 2 \} x := y + 1 \{ x = 4 \}
\]

\[
\{ y = 2 \} x := y + z \{ x = 4 \}
\]

Axioms and Rules of Inference

How do we distinguish true triples from false?

Who’s to say which ones are true?

It all depends on the semantics of statements!

If we work in a suitably structured language, we can give a fixed set of axioms and rules of inference, one for each kind of statement. We then take as true the set of triples that can be logically deduced from these axioms and rules.

Of course, we want to choose axioms and rules that capture our intuitive understanding of what the statements do, and they need to be as strong as possible.

Assignment Axiom

\[
\{ P[E/x] \} x := E \{ P \}
\]

where \( P[E/x] \) means \( P \) with all instances of \( x \) replaced by \( E \).

This axiom may seem backwards at first, but it makes sense if we start from the postcondition. For example, if we want to show \( x \geq 4 \) after the execution of

\[
x := y + 1
\]

then the necessary precondition is \( y + 1 \geq 4 \), i.e., \( y \geq 3 \).
MORE RULES FOR STATEMENTS

Conditional Rule
\[ \{ P \land E \} S_1 \{ Q \}, \{ P \land \neg E \} S_2 \{ Q \} \]
\[
\{ P \} \text{ if } E \text{ then } S_1 \text{ else } S_2 \text{ endif } \{ Q \} \]

Composition Rule
\[ \{ P \} S_1 \{ Q \}, \{ Q \} S_2 \{ R \} \]
\[
\{ P \} S_1 ; S_2 \{ R \} \]

While Rule
\[ \{ P \land E \} S \{ P \} \]
\[
\{ P \} \text{ while } E \text{ do } S \{ P \land \neg E \} \]

Consequence Rule
\[ P \Rightarrow P', \{ P' \} S \{ Q' \}, Q' \Rightarrow Q \]
\[
\{ P \} S \{ Q \} \]

Here \( P \Rightarrow Q \) means that “\( P \) implies \( Q \),” i.e., “\( Q \) is true whenever \( P \) is true,” i.e. “\( P \) is false or \( Q \) is true.” Hence we always have \( \text{False} \Rightarrow Q \) for any \( Q \).

PROOF TREE EXAMPLE

\begin{align*}
\{ x + y + 1 = c + 1 \} & \quad & \{ x - 1 + y = c \} \\
\begin{aligned}
& y := y + 1 \\
& \{ x + y = c + 1 \}
\end{aligned} & \quad & \begin{aligned}
& x := x - 1 \\
& \{ x + y = c \}
\end{aligned}
\end{align*}

\begin{align*}
\{ x + y = c \land x ! = 0 \} & \quad & \{ x + y = c + 1 \} \\
\begin{aligned}
& y := y + 1; x := x - 1 \\
& \{ x + y = c + 1 \}
\end{aligned} & \quad & \begin{aligned}
& \{ x + y = c \}
\end{aligned}
\end{align*}

\begin{align*}
\{ x + y = c \land x ! = 0 \} & \quad & \{ x + y = c \}
\end{align*}

\begin{align*}
\{ x + y = c \land x ! = 0 \} & \quad & \{ x + y = c \}
\end{align*}

\begin{align*}
\{ x = c \land y = 0 \} & \quad & \{ x + y = c \}
\end{align*}

\begin{align*}
\text{while } x != 0 \text{ do } & \quad & \text{while } x != 0 \text{ do } \\
\{ x + y = c \land x ! = 0 \} & \quad & \{ x + y = c \land x ! = 0 \}
\end{align*}

\begin{align*}
\{ x + y + 1 = c + 1 \} & \quad & \{ x + y + 1 = c + 1 \}
\end{align*}

\begin{align*}
\begin{aligned}
& y := y + 1; \{ x + y = c + 1 \}
\end{aligned} & \quad & \begin{aligned}
& x := x - 1; \{ x + y = c \}
\end{aligned}
\end{align*}

\begin{align*}
\{ x = c \land \neg x ! = 0 \} & \quad & \{ x = c \land \neg x ! = 0 \}
\end{align*}

\begin{align*}
\{ x = c \land y = 0 \} & \quad & \{ y = c \}
\end{align*}

BOOKKEEPING RULES

ANNOTATED PROGRAM EXAMPLE

Proof trees can be unwieldy. Because the structure of the tree corresponds directly to the structure of the program code, it is common to use an alternative representation of proofs in which we annotate programs with assertions/assumptions.

\begin{align*}
\{ x = c \land y = 0 \} & \quad & \{ x + y = c \}
\end{align*}

\begin{align*}
\text{while } x != 0 \text{ do } & \quad & \text{while } x != 0 \text{ do }
\end{align*}

\begin{align*}
\{ x + y = c \land x ! = 0 \} & \quad & \{ x + y = c \land x ! = 0 \}
\end{align*}

\begin{align*}
\{ x + y + 1 = c + 1 \} & \quad & \{ x + y + 1 = c + 1 \}
\end{align*}

\begin{align*}
\begin{aligned}
& y := y + 1; \{ x + y = c + 1 \}
\end{aligned} & \quad & \begin{aligned}
& x := x - 1; \{ x + y = c \}
\end{aligned}
\end{align*}

\begin{align*}
\{ x = c \land \neg x ! = 0 \} & \quad & \{ x = c \land \neg x ! = 0 \}
\end{align*}

\begin{align*}
\{ y = c \} & \quad & \{ y = c \}
\end{align*}

To verify that this is a valid proof, we have to check that the annotations are consistent with each other and with the rules and axioms.
**Merits and Problems of Axiomatic Semantics**

Gives a very clean semantics for structured statements. But things get more complicated if we add features like:

- expressions with side-effects
- statements that break out of loops
- procedures
- non-trivial data structures and aliases

Useful for formal proofs of program properties (though these are seldom done by hand).

Thinking in terms of assertions is good for informal reasoning about programs.

There are beginning to be useful automated theorem proving support tools too.

Other forms of semantic definition also use similar logical structures.

**Operational Semantics**

Define behavior of language by a set of mechanical state transition rules.

Typical mechanisms:

- Define a simple Von Neumann-style abstract stack machine and describe how each syntactic construct can be compiled into stack machine instructions.
- Characterize the state of the abstract machine (environment, store, stack, etc.) and give a set of evaluation rules describing how each syntactic construct affects the state.

Some useful things to do with an operational semantics:

- Build an implementation for a real machine by interpreting or compiling the abstract machine code.
- Explicate the meaning of a language feature by proving that it has the same behavior as a combination of simpler features.
- Prove that correctly typed programs have no unchecked runtime errors.

**Formal Operational Semantics**

So far, we’ve presented operational semantics using interpreters. These have the advantage of being precise and executable. But they are not ideally compact or abstract.

Another way to present operational semantics is using state transition judgments, for appropriately defined machine states.

For example, consider a simple language of imperative expressions, in which variables must be defined before use, using a let or fun construct.

\[
\text{exp} ::= \text{var} \mid \text{int} \\
\quad \mid \left(\left(\text{'+'} \text{exp} \text{exp} \right)\right) \\
\quad \mid \left(\left(\text{'let'} \text{var} \text{exp} \text{exp} \right)\right) \\
\quad \mid \left(\left(\text{':='} \text{var} \text{exp} \right)\right) \\
\quad \mid \left(\left(\text{'ifnz'} \text{exp} \text{exp} \text{exp} \right)\right) \\
\quad \mid \left(\left(\text{'while'} \text{exp} \text{exp} \right)\right) \\
\quad \text{etc.}
\]

For simplicity, we assume that the only values are integers; ifnz expects an integer condition and tests whether it is non-zero.

**State Machine**

To evaluate this language, we choose a machine state consisting of:

- the current environment \(E\), which maps each in-scope variable to a location \(l\).
- the current store \(S\), which maps each location \(l\) to an integer value \(v\).
- the current expression \(e\), to be evaluated.

We give the state transitions in the form of judgments:

\[
\langle e, E, S \rangle \Downarrow \langle v, S' \rangle
\]

Intuitively, this says that evaluating expression \(e\) in environment \(E\) and store \(S\) yields the value \(v\) and the (possibly) changed store \(S'\).
**OPERATIONAL SEMANTICS BY INFERENCE**

To describe the machine’s operation, we give rules of inference that state when a judgment can be derived from judgments about sub-expressions.

The form of a rule is

\[
\begin{align*}
\text{premises} & \quad \text{conclusion} \\
\text{(Name of rule)} & \\
\end{align*}
\]

We can view evaluation of the program as the process of building an inference tree.

This particular formulation, where each expression results in a value based on the values of the subexpressions, is called big-step semantics.

This notation has similarities to axiomatic semantics: the notion of derivation is essentially the same, but the content of judgments is different.

---

**Environments and Stores, Formally**

- We write \( E(x) \) means the result of looking up \( x \) in environment \( E \). (This notation is because an environment is like a function taking a name as argument and returning a meaning as result.)
- We write \( E + \{ x \mapsto l \} \) for the environment obtained from existing environment \( E \) by extending it with a new binding from \( x \) to location \( l \). If \( E \) already has a binding for \( x \), this new binding replaces it.

The domain of an environment, \( \text{dom}(E) \), is the set of names bound in \( E \). Analogously with environments, we’ll write

- \( S(l) \) to mean the value at location \( l \) of store \( S \)
- \( S + \{ l \mapsto v \} \) to mean the store obtained from store \( S \) by extending (or updating) it so that location \( l \) maps to value \( v \).
- \( \text{dom}(S) \) for the set of locations bound in store \( S \).

---

**Evaluation Rules (1)**

\[
\begin{align*}
\langle x, E, S \rangle & \Downarrow \langle v, S \rangle & \langle x, E, S \rangle & \Downarrow \langle v, S \rangle & \langle x, E, S \rangle & \Downarrow \langle v, S \rangle \\
\text{(Var)} & & \text{(Int)} & & \text{(Add)} \\
\end{align*}
\]

\[
\begin{align*}
\langle e_1, E, S \rangle & \Downarrow \langle v_1, S' \rangle & \langle e_2, E, S' \rangle & \Downarrow \langle v_2, S'' \rangle \\
\langle e_1 + e_2, E, S \rangle & \Downarrow \langle v_1 + v_2, S'' \rangle \\
\text{(Add)} & & \\
\langle e_1, E, S \rangle & \Downarrow \langle v_1, S' \rangle & \langle e_2, E, S + \{ l \mapsto v_1 \} \rangle & \Downarrow \langle v_2, S'' \rangle \\
\langle \text{let } x \ e \ e_2, E, S \rangle & \Downarrow \langle v_2, S'' \rangle \\
\text{(Let)} & & \\
\langle e, E, S \rangle & \Downarrow \langle v, S' \rangle & \langle := x \ e, E, S \rangle & \Downarrow \langle v, S' + \{ l \mapsto v \} \} \rangle \\
\text{(Assign)} & & \\
\end{align*}
\]

**Evaluation Rules (2)**

\[
\begin{align*}
\langle e_1, E, S \rangle & \Downarrow \langle v_1, S' \rangle & v_1 \neq 0 & \langle e_2, E, S' \rangle & \Downarrow \langle v_2, S'' \rangle \\
\langle \text{ifnz } e_1 e_2 e_3, E, S \rangle & \Downarrow \langle v_2, S'' \rangle \\
\text{(If-nzero)} & & \\
\langle e_1, E, S \rangle & \Downarrow \langle 0, S' \rangle & \langle e_3, E, S' \rangle & \Downarrow \langle v_3, S'' \rangle \\
\langle \text{ifnz } e_1 e_2 e_3, E, S \rangle & \Downarrow \langle v_3, S'' \rangle \\
\text{(If-zero)} & & \\
\langle e_1, E, S \rangle & \Downarrow \langle v_1, S' \rangle & v_1 \neq 0 & \langle e_2, E, S' \rangle & \Downarrow \langle v_2, S'' \rangle \\
\langle \text{while } e_1 e_2, E, S'' \rangle & \Downarrow \langle v_3, S''' \rangle \\
\langle \text{while } e_1 e_2, E, S \rangle & \Downarrow \langle v_3, S''' \rangle \\
\text{(While-nzero)} & & \\
\langle e_1, E, S \rangle & \Downarrow \langle 0, S' \rangle \\
\langle \text{while } e_1 e_2, E, S \rangle & \Downarrow \langle 0, S' \rangle \\
\text{(While-zero)} & & \\
\end{align*}
\]
Notes on the Rules

- The structure of the rules guarantees that at most one rule is applicable at any point.
- The store relationships constrain the order of evaluation.
- If no rules are applicable, the evaluation gets stuck; this corresponds to a runtime error in an interpreter.

We can view the interpreter for the language as implementing a bottom-up exploration of the inference tree. A function like

```python
def eval(e: Exp, env: Env, st: Store) : (Value, Store) = ...
```

returns a value \( v \) and store \( st' \) such that

\[
\langle e, env, st \rangle \Downarrow \langle v, st' \rangle
\]

The implementation of `eval` dispatches on the syntactic form of \( e \), chooses the appropriate rule, and makes recursive calls on `eval` corresponding to the premises of that rule.

Question: how deep can the derivation tree get?