Sample solution for Pierce 22.2.3.

Here are three possible solutions. *Any* solution must be a substitution instance of the first one shown.

```
([X ↦ Z → U → V, Y ↦ Z → U], V)
([X ↦ Nat → Nat → Nat, Y ↦ Nat → Nat, Z ↦ Nat], Nat)
([X ↦ Nat → W → Nat → Nat, Y ↦ Nat → W, Z ↦ Nat], Nat → Nat)
```

Sample solutions for Pierce 23.4.1.

Derivation for `id`:

$$\dfrac{\dfrac{\dfrac{x : X \in X, x : X}{X, x : X \vdash x : X}\;\text{T-Var}}{X \vdash \lambda x{:}X.x : X \to X}\;\text{T-Abs}}{\vdash \lambda X.\lambda x{:}X.x : \forall X.X \to X}\;\text{T-TAbs}$$

Derivation for `double`:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{f : X \to X \in X, f : X \to X, a : X}{X, f : X \to X, a : X \vdash f : X \to X}\;\text{T-Var} \quad \dfrac{\dfrac{f : X \to X \in X, f : X \to X, a : X}{X, f : X \to X, a : X \vdash f : X \to X}\;\text{T-Var} \quad \dfrac{a : X \in X, f : X \to X, a : X}{X, f : X \to X, a : X \vdash a : X}\;\text{T-Var}}{X, f : X \to X, a : X \vdash fa : X}\;\text{T-App}}{X, f : X \to X, a : X \vdash f(fa) : X}\;\text{T-App}}{X, f : X \to X \vdash \lambda a{:}X.f(fa) : X \to X}\;\text{T-Abs}}{X \vdash \lambda f{:}X \to X.\lambda a{:}X.f(fa) : (X \to X) \to X \to X}\;\text{T-Abs}}{\vdash \lambda X.\lambda f{:}X \to X.\lambda a{:}X.f(fa) : \forall X.(X \to X) \to X \to X}\;\text{T-TAbs}$$

Derivation for `selfApp`:

$$\dfrac{\dfrac{\dfrac{\dfrac{x : \forall X.X \to X \in x : \forall X.X \to X}{x : \forall X.X \to X \vdash x : \forall X.X \to X}\;\text{T-Var}}{x : \forall X.X \to X \vdash x[\forall X.X \to X] : (\forall X.X \to X) \to (\forall X.X \to X)}\;\text{T-TApp} \quad \dfrac{x : \forall X.X \to X \in x : \forall X.X \to X}{x : \forall X.X \to X \vdash x : \forall X.X \to X}\;\text{T-Var}}{x : \forall X.X \to X \vdash x[\forall X.X \to X]x : \forall X.X \to X}\;\text{T-App}}{\vdash \lambda x{:}\forall X : X \to X.x[\forall X.X \to X]x : (\forall X.X \to X) \to (\forall X.X \to X)}\;\text{T-Abs}$$

Derivation for `quadruple`, taking $\Gamma = \texttt{double} : \forall X.(X \to X) \to X \to X$

$$\dfrac{\dfrac{\dfrac{\dfrac{\texttt{double} : \forall X.(X \to X) \to X \to X \in \Gamma, X}{\Gamma, X \vdash \texttt{double} : \forall X.(X \to X) \to X \to X}\;\text{T-Var}}{\Gamma, X \vdash \texttt{double}[X \to X] : ((X \to X) \to X \to X) \to (X \to X) \to X \to X}\;\text{T-TApp} \quad \dfrac{\dfrac{\texttt{double} : \forall X.(X \to X) \to X \to X \in \Gamma, X}{\Gamma, X \vdash \texttt{double} : \forall X.(X \to X) \to X \to X}\;\text{T-Var}}{\Gamma, X \vdash \texttt{double}[X] : (X \to X) \to X \to X}\;\text{T-TApp}}{\Gamma, X \vdash \texttt{double}[X \to X](\texttt{double}[X]) : (X \to X) \to X \to X}\;\text{T-App}}{\Gamma \vdash \lambda X.\texttt{double}[X \to X](\texttt{double}[X]) : \forall X.(X \to X) \to X \to X}\;\text{T-TAbs}$$

Sample solution for Pierce 23.5.1.

The goal is to show Preservation for System F as shown in Figure 23-1.

Be sure to note the erratum for p. 342, which explains that the context $\Gamma, \mathtt{x} : \mathsf{T}$ is considered well-formed only if every type variable free in T is bound in Γ.

Following the hint in the answers, we begin by proving the following Lemma about preservation under type substitutions.

Lemma 1 *If* $\Gamma, \mathsf{X}, \Delta \vdash \mathtt{t} : \mathsf{T}$, *then* $\Gamma, [\mathsf{X} \mapsto \mathsf{S}]\Delta \vdash [\mathsf{X} \mapsto \mathsf{S}]\mathtt{t} : [\mathsf{X} \mapsto \mathsf{S}]\mathsf{T}$.

Proof. By induction on the depth of the derivation of $\Gamma, \mathsf{X}, \Delta \vdash \mathtt{t} : \mathsf{T}$, then by cases on the final typing rule.

- (T-VAR) Here $\mathtt{t} = \mathtt{z}$ and $\mathtt{z} : \mathsf{T} \in \Gamma, \mathsf{X}, \Delta$. Clearly $[\mathsf{X} \mapsto \mathsf{S}]\mathtt{z} = \mathtt{z}$. There are two sub-cases.
  i. If $\mathtt{z} : \mathsf{T} \in \Gamma$, then $\mathtt{z} : \mathsf{T} \in \Gamma, [\mathsf{X} \mapsto \mathsf{S}]\Delta$, so by T-VAR, $\Gamma, [\mathsf{X} \mapsto \mathsf{S}]\Delta \vdash \mathtt{z} : \mathsf{T}$. By the well-formedness condition on contexts, $\mathsf{X} \notin FV(\mathsf{T})$, so $\mathsf{T} = [\mathsf{X} \mapsto \mathsf{S}]\mathsf{T}$, giving the desired result.
  ii. If $\mathtt{z} : \mathsf{T} \in \Delta$, then $([\mathsf{X} \mapsto \mathsf{S}]\Delta)(\mathtt{z}) = [\mathsf{X} \mapsto \mathsf{S}]\mathsf{T}$, so by T-VAR, $\Gamma, [\mathsf{X} \mapsto \mathsf{S}]\Delta \vdash \mathtt{z} : [\mathsf{X} \mapsto \mathsf{S}]\mathsf{T}$, as desired.

- (T-ABS) Here $\mathtt{t} = \lambda \mathtt{x}{:}\mathsf{T}_1.\mathtt{t}_2$ where $\Gamma, \mathsf{X}, \Delta, \mathtt{x} : \mathsf{T}_1 \vdash \mathtt{t}_2 : \mathsf{T}_2$ and $\mathsf{T} = \mathsf{T}_1 \to \mathsf{T}_2$. By induction, $\Gamma, [\mathsf{X} \mapsto \mathsf{S}]\Delta, \mathtt{x} : [\mathsf{X} \mapsto \mathsf{S}]\mathsf{T}_1 \vdash [\mathsf{X} \mapsto \mathsf{S}]\mathtt{t}_2 : [\mathsf{X} \mapsto \mathsf{S}]\mathsf{T}_2$. So by T-ABS, $\Gamma, [\mathsf{X} \mapsto \mathsf{S}]\Delta \vdash \lambda \mathtt{x}{:}[\mathsf{X} \mapsto \mathsf{S}]\mathsf{T}_1.[\mathsf{X} \mapsto \mathsf{S}]\mathtt{t}_2 : [\mathsf{X} \mapsto \mathsf{S}]\mathsf{T}_1 \to [\mathsf{X} \mapsto \mathsf{S}]\mathsf{T}_2$, i.e. $\Gamma, [\mathsf{X} \mapsto \mathsf{S}]\Delta \vdash [\mathsf{X} \mapsto \mathsf{S}](\lambda \mathtt{x}{:}\mathsf{T}_1.\mathtt{t}_2) : [\mathsf{X} \mapsto \mathsf{S}](\mathsf{T}_1 \to \mathsf{T}_2)$, as desired.

- (T-APP) Here $\mathtt{t} = \mathtt{t}_1\mathtt{t}_2$ where $\Gamma, \mathsf{X}, \Delta \vdash \mathtt{t}_1 : \mathsf{T}_{11} \to \mathsf{T}_{12}$, $\Gamma, \mathsf{X}, \Delta \vdash \mathtt{t}_2 : \mathsf{T}_{11}$, and $\mathsf{T} = \mathsf{T}_{12}$. By induction $\Gamma, [\mathsf{X} \mapsto \mathsf{S}]\Delta \vdash [\mathsf{X} \mapsto \mathsf{S}]\mathtt{t}_1 : [\mathsf{X} \mapsto \mathsf{S}](\mathsf{T}_{11} \to \mathsf{T}_{12})$, and $\Gamma, [\mathsf{X} \mapsto \mathsf{S}]\Delta \vdash [\mathsf{X} \mapsto \mathsf{S}]\mathtt{t}_2 : [\mathsf{X} \mapsto \mathsf{S}]\mathsf{T}_{11}$. Distributing the substitution in the first of these derivations and applying T-APP, we get $\Gamma, [\mathsf{X} \mapsto \mathsf{S}]\Delta \vdash ([\mathsf{X} \mapsto \mathsf{S}]\mathtt{t}_1)([\mathsf{X} \mapsto \mathsf{S}]\mathtt{t}_2) : [\mathsf{X} \mapsto \mathsf{S}]\mathsf{T}_{12}$; undistributing the substitution on the term gives us the desired result.

- (T-TABS) Here $\mathtt{t} = \lambda \mathsf{Y}.\mathtt{t}_2$ where $\Gamma, \mathsf{X}, \Delta, \mathsf{Y} \vdash \mathtt{t}_2 : \mathsf{T}_2$ and $\mathsf{T} = \forall \mathsf{Y}.\mathsf{T}_2$. By induction, $\Gamma, [\mathsf{X} \mapsto \mathsf{S}]\Delta, \mathsf{Y} \vdash [\mathsf{X} \mapsto \mathsf{S}]\mathtt{t}_2 : [\mathsf{X} \mapsto \mathsf{S}]\mathsf{T}_2$. So by T-TABS, $\Gamma, [\mathsf{X} \mapsto \mathsf{S}]\Delta \vdash \lambda \mathsf{Y}.[\mathsf{X} \mapsto \mathsf{S}]\mathtt{t}_2 : \forall \mathsf{Y}.[\mathsf{X} \mapsto \mathsf{S}]\mathsf{T}_2$. Pulling the substitutions out of the quantifiers gives us the desired result.

- (T-TAPP) Here $\mathtt{t} = \mathtt{t}_1[\mathsf{T}_2]$, where $\Gamma, \mathsf{X}, \Delta \vdash \mathtt{t}_1 : \forall \mathsf{Y}.\mathsf{T}_{12}$ and $\mathsf{T} = [\mathsf{Y} \mapsto \mathsf{T}_2]\mathsf{T}_{12}$. By induction, $\Gamma, [\mathsf{X} \mapsto \mathsf{S}]\Delta \vdash [\mathsf{X} \mapsto \mathsf{S}]\mathtt{t}_1 : [\mathsf{X} \mapsto \mathsf{S}](\forall \mathsf{Y}.\mathsf{T}_{12})$. Pushing the substitution inside the quantifier and applying T-TAPP gives $\Gamma, [\mathsf{X} \mapsto \mathsf{S}]\Delta \vdash ([\mathsf{X} \mapsto \mathsf{S}]\mathtt{t}_1)[[\mathsf{X} \mapsto \mathsf{S}]\mathsf{T}_2] : [\mathsf{Y} \mapsto ([\mathsf{X} \mapsto \mathsf{S}]\mathsf{T}_2)]([\mathsf{X} \mapsto \mathsf{S}]\mathsf{T}_{12})$. Pulling the substitutions to the left gives $\Gamma, [\mathsf{X} \mapsto \mathsf{S}]\Delta \vdash [\mathsf{X} \mapsto \mathsf{S}](\mathtt{t}_1[\mathsf{T}_2]) : [\mathsf{X} \mapsto \mathsf{S}]([\mathsf{Y} \mapsto \mathsf{T}_2]\mathsf{T}_{12})$, as desired.

Next, we need a revised version of Lemma 9.3.8.

Lemma 2 *If* $\Gamma, \mathtt{x} : \mathtt{S} \vdash \mathtt{t} : \mathtt{T}$ *and* $\Gamma \vdash \mathtt{s} : \mathtt{S}$*, then* $\Gamma \vdash [\mathtt{x} \mapsto \mathtt{s}]\mathtt{t} : \mathtt{T}$.

Proof. By induction on the depth of the derivation of $\Gamma, \mathtt{x} : \mathtt{S} \vdash \mathtt{t} : \mathtt{T}$, and then by cases on the final typing rule. Cases T-VAR, T-ABS, and T-APP are exactly as in the proof of Lemma 9.3.8 on pp. 106-107. The remaining cases are:

- (T-TABS) Here $\mathtt{t} = \lambda\mathtt{X}.\mathtt{t}_2$ where $\Gamma, \mathtt{x} : \mathtt{S}, \mathtt{X} \vdash \mathtt{t}_2 : \mathtt{T}_2$ and $\mathtt{T} = \forall\mathtt{X}.\mathtt{T}_2$. By induction, $\Gamma, \mathtt{X} \vdash [\mathtt{x} \mapsto \mathtt{s}]\mathtt{t}_2 : \mathtt{T}_2$. So by T-TABS, $\Gamma \vdash \lambda\mathtt{X}.[\mathtt{x} \mapsto \mathtt{s}]\mathtt{t}_2 : \forall\mathtt{X}.[\mathtt{x} \mapsto \mathtt{s}]\mathtt{T}_2$, i.e., $\Gamma \vdash [\mathtt{x} \mapsto \mathtt{s}](\lambda\mathtt{X}.\mathtt{t}_2) : [\mathtt{x} \mapsto \mathtt{s}](\forall\mathtt{X}.\mathtt{T}_2)$, as desired.

- (T-TAPP) Here $\mathtt{t} = \mathtt{t}_1[\mathtt{T}_2]$, where $\Gamma, \mathtt{x} : \mathtt{S} \vdash \mathtt{t}_1 : \forall\mathtt{X}.\mathtt{T}_{12}$ and $\mathtt{T} = [\mathtt{X} \mapsto \mathtt{T}_2]\mathtt{T}_{12}$. By induction $\Gamma \vdash [\mathtt{x} \mapsto \mathtt{s}]\mathtt{t}_1 : \forall\mathtt{X}.\mathtt{t}_{12}$. So, by T-TAPP, $\Gamma \vdash ([\mathtt{x} \mapsto \mathtt{s}]\mathtt{t}_1)[\mathtt{T}_2] : [\mathtt{X} \mapsto \mathtt{T}_2]\mathtt{T}_{12}$, i.e., $\Gamma \vdash [\mathtt{x} \mapsto \mathtt{s}](\mathtt{t}_1[\mathtt{T}_2]) : [\mathtt{X} \mapsto \mathtt{T}_2]\mathtt{T}_{12}$, as desired.

Finally, we can prove the preservation theorem itself.

Theorem 1 *If* $\Gamma \vdash \mathtt{t} : \mathtt{T}$ *and* $\mathtt{t} \to \mathtt{t}'$*, then* $\Gamma \vdash \mathtt{t}' : \mathtt{T}$.

Proof. By induction on the derivation of $\Gamma \vdash \mathtt{t} : \mathtt{T}$, then case analysis on the final step. The cases for T-VAR, T-ABS, and T-TABS can't occur, as no E-rule applies to them. The T-APP case is exactly as in the proof of Thm 9.39 on p. 507 (where the Substitution Lemma now refers to our Lemma 2).

The remaining case is T-TAPP. Here $\mathtt{t} = \mathtt{t}_1[\mathtt{T}_2]$, $\Gamma \vdash \mathtt{t}_1 : \forall\mathtt{X}.\mathtt{T}_{12}$ and $\mathtt{T} = [\mathtt{X} \mapsto \mathtt{T}_2]\mathtt{T}_{12}$. There are two possible evaluation rules

- (E-TAPP) Here $\mathtt{t}_1 \to \mathtt{t}_1'$ and $\mathtt{t}' = \mathtt{t}_1'[\mathtt{T}_2]$. By induction, $\Gamma \vdash \mathtt{t}_1' : \forall\mathtt{X}.\mathtt{T}_{12}$. Hence, by T-TAPP, $\Gamma \vdash \mathtt{t}_1'[\mathtt{T}_2] : [\mathtt{X} \mapsto \mathtt{T}_2]\mathtt{T}_{12}$, as desired.

- (E-TAPPTABS) Here $\mathtt{t}_1 = \lambda\mathtt{X}.\mathtt{t}_{12}$ and $\mathtt{t}' = [\mathtt{X} \mapsto \mathtt{T}_2]\mathtt{t}_{12}$. By inversion, $\Gamma, \mathtt{X} \vdash \mathtt{t}_{12} : \mathtt{T}_{12}$. So by Lemma 1, $\Gamma \vdash [\mathtt{X} \mapsto \mathtt{T}_2]\mathtt{t}_{12} : [\mathtt{X} \mapsto \mathtt{T}_2]\mathtt{T}_{12}$, as desired.