Sample solution for Pierce 11.4.1(1).

We show that ascription can be formulated as a derived form by proving a theorem with the same structure as Thm. 11.3.1 (with errata corrected).

We write λ^E for the simply-typed λ-calculus with the additional rules given in Figure 11-3, and λ^I for pure simply-typed λ-calculus as given in Figure 9-1. We describe an elaboration function $e \in \lambda^E \to \lambda^I$, as follows:

$$e(\texttt{t as T}) = (\lambda\texttt{x:T.x})\ e(\texttt{t})$$
$$e(\texttt{t}) = \texttt{t} \textit{ with each subterm } \texttt{t}' \textit{ replaced by } e(\texttt{t}') \quad \textit{for any other } \texttt{t}$$

For convenience we will write $\lceil\texttt{t}\rceil$ for $e(\texttt{t})$.

Theorem 1 *For each term* $\texttt{t}$ *of* λ^E*,*

i. If $\texttt{t} \to_E \texttt{t}'$*, then* $\lceil\texttt{t}\rceil \to_I \lceil\texttt{t}'\rceil$*.*

ii. If $\lceil\texttt{t}\rceil \to_I \texttt{u}$*, then* $\exists$ *a* λ^E *term* $\texttt{t}'$ *such that* $\texttt{t} \to_E \texttt{t}'$ *and* $\lceil\texttt{t}'\rceil = \texttt{u}$*.*

iii. If $\Gamma \vdash^E \texttt{t} : \texttt{T}$*, then* $\Gamma \vdash^I \lceil\texttt{t}\rceil : \texttt{T}$*.*

iv. If $\Gamma \vdash^I \lceil\texttt{t}\rceil : \texttt{T}$*, then* $\Gamma \vdash^E \texttt{t} : \texttt{T}$*.*

Proof.

The proof of each result is by structural induction on $\texttt{t}$. In each instance, it is obvious that the only case of interest is $\texttt{t} = \texttt{t}_1\ \texttt{as T}$, so $\lceil\texttt{t}\rceil = (\lambda\texttt{x:T.x})\ \lceil\texttt{t}_1\rceil$. To shorten the proof, we consider only this case in the various sections below.

i. By hypothesis, $\texttt{t} \to_E \texttt{t}'$. There are two cases, corresponding to the one-step rules that might apply to $\texttt{t}$:

   - (E-Ascribe) In this case, $\texttt{t}_1 = \texttt{v}_1$ for some value $\texttt{v}_1$, and $\texttt{t}' = \texttt{v}_1$. So $\lceil\texttt{t}\rceil = (\lambda\texttt{x:T.x})(\lceil\texttt{v}_1\rceil) = (\lambda\texttt{x:T.x})(\texttt{v}_1)$. By (E-AppAbs), $(\lambda\texttt{x:T.x})(\texttt{v}_1) \to_I \texttt{v}_1 = \lceil\texttt{v}_1\rceil = \lceil\texttt{t}'\rceil$.
   - (E-Ascribe1) In this case, $\texttt{t}_1 \to_E \texttt{t}_1'$. We have $\lceil\texttt{t}\rceil = (\lambda\texttt{x:T.x})(\lceil\texttt{t}_1\rceil)$. By induction, $\lceil\texttt{t}_1\rceil \to_E \lceil\texttt{t}_1'\rceil$, so by (E-App2), $(\lambda\texttt{x:T.x})(\lceil\texttt{t}_1\rceil) \to_I (\lambda\texttt{x:T.x})(\lceil\texttt{t}_1'\rceil) = \lceil\texttt{t}'\rceil$.

ii. By hypothesis, $\exists\texttt{u}$ such that $\lceil\texttt{t}\rceil \to_I \texttt{u}$. There are two cases, corresponding to the one-step rules that might apply to $\lceil\texttt{t}\rceil$:

   - (E-AppAbs) In this case, $\lceil\texttt{t}_1\rceil = \texttt{v}_1$ for some value $\texttt{v}_1$, and $\texttt{u} = \texttt{v}_1$. It is easy to see that $\texttt{t}_1 = \texttt{v}_1$ as well, since only a variable can elaborate to a variable. Hence, by (E-Ascribe), we have $\texttt{t} \to_E \texttt{v}_1$ and $\lceil\texttt{v}_1\rceil = \texttt{u}$.

- (E-APP2) In this case, $\mathtt{u} = (\lambda\mathtt{x}\mathtt{:T.x})\ \mathtt{u}_1$, where $\lceil \mathtt{t}_1 \rceil \rightarrow_I \mathtt{u}_1$. By induction, $\exists \mathtt{t}_1'$ such that $\mathtt{t}_1 \rightarrow_E \mathtt{t}_1'$ and $\lceil \mathtt{t}_1' \rceil = \mathtt{u}_1$. Hence, by (E-ASCRIBE1), we have $\mathtt{t} \rightarrow_E \mathtt{t}_1'\ \mathtt{as}\ \mathtt{T}$ and $\lceil \mathtt{t}_1'\ \mathtt{as}\ \mathtt{T} \rceil = (\lambda\mathtt{x}\mathtt{:T.x})\ \lceil \mathtt{t}_1' \rceil = (\lambda\mathtt{x}\mathtt{:T.x})\ \mathtt{u}_1 = \mathtt{u}$.

iii. By hypothesis, $\Gamma \vdash^E \mathtt{t} : \mathtt{T}$. The only applicable rule is (T-ASCRIBE), so we must have $\Gamma \vdash^E \mathtt{t}_1 : \mathtt{T}$. By induction, $\Gamma \vdash^I \lceil \mathtt{t}_1 \rceil : \mathtt{T}$. So we can build the following deduction:

$$\dfrac{\dfrac{\dfrac{\mathtt{x} : \mathtt{T} \in \Gamma, \mathtt{x} : \mathtt{T}}{\Gamma, \mathtt{x} : \mathtt{T} \vdash^I \mathtt{x} : \mathtt{T}}\ \text{T-VAR}}{\Gamma \vdash^I (\lambda\mathtt{x}\mathtt{:T.x}) : \mathtt{T} \rightarrow \mathtt{T}}\ \text{T-ABS} \qquad \Gamma \vdash^I \lceil \mathtt{t}_1 \rceil : \mathtt{T}}{\Gamma \vdash^I (\lambda\mathtt{x}\mathtt{:T.x})\ \lceil \mathtt{x}_1 \rceil : \mathtt{T}}\ \text{T-APP}$$

iv. By hypothesis, $\Gamma \vdash^I \lceil \mathtt{t} \rceil : \mathtt{T}$. The only applicable rule is (T-APP), which gives $\Gamma \vdash^I (\lambda\mathtt{x}\mathtt{:T.x}) : \mathtt{T}' \rightarrow \mathtt{T}$ and $\Gamma \vdash^I \lceil \mathtt{t}_1 \rceil : \mathtt{T}'$. By the Inversion Lemma, $\mathtt{T}' = \mathtt{T}$. By induction, $\Gamma \vdash^E \mathtt{t}_1 : \mathtt{T}$. Finally, by (T-ASCRIBE), $\Gamma \vdash^E \mathtt{T}_1\ \mathtt{as}\ \mathtt{T} : \mathtt{T}$. $\square$