Sample solutions for Pierce 23.4.1.

Derivation for \( \text{id} \):

\[
\begin{align*}
\text{T-VAR} & \quad x : X \in X, x : X \quad \frac{}{} \\
\text{T-VAR} & \quad x, x : X \vdash x : X \\
\text{T-ABS} & \quad \frac{}{\lambda x : X. x : X \to X} \\
\text{T-TABS} & \quad \frac{}{\vdash \lambda x. \lambda x : X. x : \forall X.X \to X}
\end{align*}
\]

Derivation for \( \text{double} \):

\[
\begin{align*}
\text{T-VAR} & \quad f : X \to X \in X, f : X \to X, a : X \\
\text{T-VAR} & \quad x, f : X \to X, a \vdash f : X \to X \\
\text{T-APP} & \quad \frac{}{x, f : X \to X, a : X \vdash f(a) : X} \\
\text{T-ABS} & \quad \frac{}{x, f : X \to X \vdash \lambda a : X. f(a) : X \to X} \\
\text{T-ABS} & \quad \frac{}{\vdash \lambda f : X \to X. \lambda a : X. f(a) : (X \to X) \to X \to X} \\
\text{T-TABS} & \quad \frac{}{\vdash \lambda x. \lambda f : X \to X. \lambda a : X. f(a) : \forall X.(X \to X) \to X \to X}
\end{align*}
\]

Derivation for \( \text{selfApp} \):

\[
\begin{align*}
\text{T-VAR} & \quad x : \forall X.X \to X \in x : \forall X.X \to X \\
\text{T-APP} & \quad x : \forall X.X \to X \vdash x : \forall X.X \to X \\
\text{T-APP} & \quad \frac{}{x : \forall X.X \to X \vdash x[\forall X.X \to X] : (\forall X.X \to X) \to (\forall X.X \to X)} \\
\text{T-VAR} & \quad x : \forall X.X \to X \vdash x : \forall X.X \to X \\
\text{T-APP} & \quad \frac{}{x : \forall X.X \to X \vdash x[\forall X.X \to X] : (\forall X.X \to X) \to (\forall X.X \to X)} \\
\text{T-ABS} & \quad \frac{}{\vdash \lambda x : \forall X.X \to X. x[\forall X.X \to X] : (\forall X.X \to X) \to (\forall X.X \to X)}
\end{align*}
\]

Derivation for \( \text{quadruple} \), taking \( \Gamma = \text{double} : \forall X.(X \to X) \to X \to X \)

\[
\begin{align*}
\text{T-VAR} & \quad \text{double} : \forall X.(X \to X) \to X \to X \in \Gamma, X \\
\text{T-APP} & \quad \frac{}{\Gamma, X \vdash \text{double} : \forall X.(X \to X) \to X \to X} \\
\text{T-VAR} & \quad \text{double} : \forall X.(X \to X) \to X \to X \in \Gamma, X \\
\text{T-APP} & \quad \frac{}{\Gamma, X \vdash \text{double} : \forall X.(X \to X) \to X \to X} \\
\text{T-VAR} & \quad \text{double} : \forall X.(X \to X) \to X \to X \in \Gamma, X \\
\text{T-APP} & \quad \frac{}{\Gamma, X \vdash \text{double} : \forall X.(X \to X) \to X \to X} \\
\text{T-APP} & \quad \frac{}{\Gamma \vdash \lambda X. \text{double} : X \to X.(\text{double}[X]) : (X \to X) \to X \to X} \\
\text{T-TABS} & \quad \frac{}{\Gamma \vdash \lambda X. \text{double} : X \to X.(\text{double}[X]) : \forall X.(X \to X) \to X \to X}
\end{align*}
\]
Sample solution for Pierce 23.5.1.

The goal is to show Preservation for System F as shown in Figure 23-1.

Be sure to note the erratum for p. 342, which explains that the context \( \Gamma, x : T \) is considered well-formed only if every type variable free in \( T \) is bound in \( \Gamma \).

Following the hint in the answers, we begin by proving the following Lemma about preservation under type substitutions.

**Lemma 1** If \( \Gamma, x, \Delta \vdash t : T \), then \( \Gamma, [x \mapsto S] \Delta \vdash [x \mapsto S] t : [x \mapsto S] T \).

**Proof.** By induction on the depth of the derivation of \( \Gamma, x, \Delta \vdash t : T \), then by cases on the final typing rule.

- **(T-VAR)** Here \( t = z \) and \( z : T \in \Gamma, x, \Delta \). Clearly \( [x \mapsto S] z = z \). There are two sub-cases.
  
  i. If \( z : T \in \Gamma \), then \( z : T \in \Gamma, [x \mapsto S] \Delta \), so by T-VAR, \( \Gamma, [x \mapsto S] \Delta \vdash z : T \). By the well-formedness condition on contexts, \( x \notin FV(T) \), so \( T = [x \mapsto S] T \), giving the desired result.

  ii. If \( z : T \in \Delta \), then \( ([x \mapsto S] \Delta)(z) = [x \mapsto S] T \), so by T-VAR, \( \Gamma, [x \mapsto S] \Delta \vdash z : [x \mapsto S] T \), as desired.

- **(T-ABS)** Here \( t = \lambda x : T_1. t_2 \) where \( \Gamma, x, \Delta, x : T_1 \vdash t_2 : T_2 \) and \( T = T_1 \rightarrow T_2 \). By induction, \( \Gamma, [x \mapsto S] \Delta, x : [x \mapsto S] T_1 \vdash [x \mapsto S] t_2 : [x \mapsto S] T_2 \). So by T-ABS, \( \Gamma, [x \mapsto S] \Delta \vdash \lambda x : [x \mapsto S] T_1. [x \mapsto S] t_2 : [x \mapsto S] T_1 \rightarrow [x \mapsto S] T_2 \), i.e. \( \Gamma, [x \mapsto S] \Delta \vdash [x \mapsto S] (\lambda x : T_1. t_2) : [x \mapsto S] (T_1 \rightarrow T_2) \), as desired.

- **(T-APP)** Here \( t = t_1 t_2 \) where \( \Gamma, x, \Delta \vdash t_1 : T_{11} \rightarrow T_{12} \), \( \Gamma, x, \Delta \vdash t_2 : T_{11} \), and \( T = T_{12} \). By induction, \( \Gamma, [x \mapsto S] \Delta \vdash [x \mapsto S] t_1 : [x \mapsto S] (T_{11} \rightarrow T_{12}) \), and \( \Gamma, [x \mapsto S] \Delta \vdash [x \mapsto S] t_2 : [x \mapsto S] T_{11} \). Distributing the substitution in the first of these derivations and applying T-APP, we get \( \Gamma, [x \mapsto S] \Delta \vdash ([x \mapsto S] t_1) ([x \mapsto S] t_2) : [x \mapsto S] T_{12} \); undistributing the substitution on the term gives us the desired result.

- **(T-TABS)** Here \( t = \forall Y. t_2 \) where \( \Gamma, x, \Delta, y \vdash t_2 : T_2 \) and \( T = \forall Y. T_2 \). By induction, \( \Gamma, [x \mapsto S] \Delta, y \vdash [x \mapsto S] t_2 : [x \mapsto S] T_2 \). So by T-TABS, \( \Gamma, [x \mapsto S] \Delta \vdash \forall Y. [x \mapsto S] t_2 : \forall Y. [x \mapsto S] T_2 \). Pulling the substitutions out of the quantifiers gives us the desired result.

- **(T-TAPP)** Here \( t = t_1 [T_2] \), where \( \Gamma, x, \Delta \vdash t_1 : \forall Y. T_{12} \) and \( T = [Y \mapsto T_2] T_{12} \). By induction, \( \Gamma, [x \mapsto S] \Delta \vdash [x \mapsto S] t_1 : [x \mapsto S] (\forall Y. T_{12}) \). Pushing the substitution inside the quantifier and applying T-TAPP gives \( \Gamma, [x \mapsto S] \Delta \vdash ([x \mapsto S] t_1) ([x \mapsto S] T_2) : [Y \mapsto ([x \mapsto S] T_2)] ([x \mapsto S] T_{12}) \). Pulling the substitutions to the left gives \( \Gamma, [x \mapsto S] \Delta \vdash [x \mapsto S] (t_1 [T_2]) : [x \mapsto S] ([Y \mapsto T_2] T_{12}) \), as desired.
Next, we need a revised version of Lemma 9.3.8.

**Lemma 2** If $\Gamma, x : S \vdash t : T$ and $\Gamma \vdash s : S$, then $\Gamma \vdash [x \mapsto s]t : T$.

**Proof.** By induction on the depth of the derivation of $\Gamma, x : S \vdash t : T$, and then by cases on the final typing rule. Cases T-VAR, T-ABS, and T-APP are exactly as in the proof of Lemma 9.3.8 on pp. 106-107. The remaining cases are:

- (T-TABS) Here $t = \lambda X. t_2$ where $\Gamma, x : S, X \vdash t_2 : T_2$ and $T = \forall X. T_2$. By induction, $\Gamma, X \vdash [x \mapsto s]t_2 : T_2$. So by T-TABS, $\Gamma \vdash \lambda X. [x \mapsto s]t_2 : \forall X. [x \mapsto s]T_2$, i.e., $\Gamma \vdash [x \mapsto s](\lambda X. t_2) : [x \mapsto s](\forall X. T_2)$, as desired.

- (T-TAPP) Here $t = t_1[T_2]$, where $\Gamma, x : S \vdash t_1 : \forall X. T_{12}$ and $T = [X \mapsto T_2]T_{12}$. By induction $\Gamma \vdash [x \mapsto s]t_1 : \forall X. T_{12}$. So, by T-TAPP, $\Gamma \vdash ([x \mapsto s]t_1)[T_2] : [X \mapsto T_2]T_{12}$, i.e., $\Gamma \vdash [x \mapsto s](t_1[T_2]) : [X \mapsto T_2]T_{12}$, as desired.

Finally, we can prove the preservation theorem itself.

**Theorem 1** If $\Gamma \vdash t : T$ and $t \rightarrow t'$, then $\Gamma \vdash t' : T$.

**Proof.** By induction on the derivation of $\Gamma \vdash t : T$, then case analysis on the final step. The cases for T-VAR, T-ABS, and T-TABS can’t occur, as no E-rule applies to them. The T-APP case is exactly as in the proof of Thm 9.39 on p. 507 (where the Substitution Lemma now refers to our Lemma 2).

The remaining case is T-TAPP. Here $t = t_1[T_2]$, $\Gamma \vdash t_1 : \forall X. T_{12}$ and $T = [X \mapsto T_2]T_{12}$. There are two possible evaluation rules

- (E-TAPP) Here $t_1 \rightarrow t_1'$ and $t' = t_1'[T_2]$. By induction, $\Gamma \vdash t_1' : \forall X. T_{12}$. Hence, by T-TAPP, $\Gamma \vdash [t_1'[T_2]] : [X \mapsto T_2]T_{12}$, as desired.

- (E-TAPP-TABS) Here $t_1 = \lambda X. t_{12}$ and $t' = [X \mapsto T_2]t_{12}$. By inversion, $\Gamma, X \vdash t_{12} : T_{12}$. So by Lemma 1, $\Gamma \vdash [X \mapsto T_2]t_{12} : [X \mapsto T_2]T_{12}$, as desired.