Sample solution for Pierce 11.4.1(1).

We show that ascription can be formulated as a derived form by proving a theorem with the same structure as Thm. 11.3.1 (with errata corrected).

We write λ^E for the simply-typed λ-calculus with the additional rules given in Figure 11-3, and λ^I for pure simply-typed λ-calculus as given in Figure 9-1. We describe an elaboration function $e \in \lambda^E \to \lambda^I$, as follows:

$$\begin{array}{rcl} e(\mathtt{t\ as\ T}) & = & (\lambda\mathtt{x}\mathtt{:T.x})\ e(\mathtt{t}) \\ e(\mathtt{x}) & = & \mathtt{x} \\ e(\lambda\mathtt{x}\mathtt{:T.t}) & = & \lambda\mathtt{x}\mathtt{:T.}e(\mathtt{t}) \\ e(\mathtt{t\ u}) & = & e(\mathtt{t})\ e(\mathtt{u}) \end{array}$$

For convenience we will write $\lceil \mathtt{t} \rceil$ for $e(\mathtt{t})$.

Theorem 1 *For each term* $\mathtt{t}$ *of* λ^E*,*

i. If $\mathtt{t} \to_E \mathtt{t}'$*, then* $\lceil \mathtt{t} \rceil \to_I \lceil \mathtt{t}' \rceil$*.*

ii. If $\lceil \mathtt{t} \rceil \to_I \mathtt{u}$*, then* $\exists$ *a* λ^E *term* $\mathtt{t}'$ *such that* $\mathtt{t} \to_E \mathtt{t}'$ *and* $\lceil \mathtt{t}' \rceil = \mathtt{u}$*.*

iii. If $\Gamma \vdash^E \mathtt{t} : \mathtt{T}$*, then* $\Gamma \vdash^I \lceil \mathtt{t} \rceil : \mathtt{T}$*.*

iv. If $\Gamma \vdash^I \lceil \mathtt{t} \rceil : \mathtt{T}$*, then* $\Gamma \vdash^E \mathtt{t} : \mathtt{T}$*.*

Proof.

The proof of each result is by structural induction on $\mathtt{t}$. In each instance, we consider only the interesting case where $\mathtt{t} = \mathtt{t}_1$ $\mathtt{as\ T}$, so $\lceil \mathtt{t} \rceil = (\lambda\mathtt{x}\mathtt{:T.x})\ \lceil \mathtt{t}_1 \rceil$. A really careful proof would consider all the other cases too.

i. By hypothesis, $\mathtt{t} \to_E \mathtt{t}'$. There are two cases, corresponding to the one-step rules that might apply to $\mathtt{t}$:

- (E-Ascribe) In this case, $\mathtt{t}_1 = \mathtt{v}_1$ for some value $\mathtt{v}_1$, and $\mathtt{t}' = \mathtt{v}_1$. So $\lceil \mathtt{t} \rceil = (\lambda\mathtt{x}\mathtt{:T.x})(\lceil \mathtt{v}_1 \rceil) = (\lambda\mathtt{x}\mathtt{:T.x})(\mathtt{v}_1)$. By (E-AppAbs), $(\lambda\mathtt{x}\mathtt{:T.x})(\mathtt{v}_1) \to_I \mathtt{v}_1 = \lceil \mathtt{v}_1 \rceil = \lceil \mathtt{t}' \rceil$.
- (E-Ascribe1) In this case, $\mathtt{t}_1 \to_E \mathtt{t}_1'$. We have $\lceil \mathtt{t} \rceil = (\lambda\mathtt{x}\mathtt{:T.x})(\lceil \mathtt{t}_1 \rceil)$. By induction, $\lceil \mathtt{t}_1 \rceil \to_E \lceil \mathtt{t}_1' \rceil$, so by (E-App2), $(\lambda\mathtt{x}\mathtt{:T.x})(\lceil \mathtt{t}_1 \rceil) \to_I (\lambda\mathtt{x}\mathtt{:T.x})(\lceil \mathtt{t}_1' \rceil) = \lceil \mathtt{t}' \rceil$.

ii. By hypothesis, $\exists \mathtt{u}$ such that $\lceil \mathtt{t} \rceil \to_I \mathtt{u}$. There are two cases, corresponding to the one-step rules that might apply to $\lceil \mathtt{t} \rceil$:

- (E-AppAbs) In this case, $\lceil \mathtt{t}_1 \rceil = \mathtt{v}_1$ for some value $\mathtt{v}_1$, and $\mathtt{u} = \mathtt{v}_1$. It is easy to see that $\mathtt{t}_1 = \mathtt{v}_1$ as well, since only a variable can elaborate to a variable. Hence, by (E-Ascribe), we have $\mathtt{t} \to_E \mathtt{v}_1$ and $\lceil \mathtt{v}_1 \rceil = \mathtt{u}$.

- (E-App2) In this case, $\mathtt{u} = (\lambda\mathtt{x}{:}\mathtt{T}.\mathtt{x})\ \mathtt{u}_1$, where $\lceil \mathtt{t}_1 \rceil \rightarrow_I \mathtt{u}_1$. By induction, $\exists \mathtt{t}_1'$ such that $\mathtt{t}_1 \rightarrow_E \mathtt{t}_1'$ and $\lceil \mathtt{t}_1' \rceil = \mathtt{u}_1$. Hence, by (E-Ascribe1), we have $\mathtt{t} \rightarrow_E \mathtt{t}_1'\ \mathtt{as}\ \mathtt{T}$ and $\lceil \mathtt{t}_1'\ \mathtt{as}\ \mathtt{T} \rceil = (\lambda\mathtt{x}{:}\mathtt{T}.\mathtt{x})\ \lceil \mathtt{t}_1' \rceil = (\lambda\mathtt{x}{:}\mathtt{T}.\mathtt{x})\ \mathtt{u}_1 = \mathtt{u}$.

iii. By hypothesis, $\Gamma \vdash^E \mathtt{t} : \mathtt{T}$. The only applicable rule is (T-Ascribe), so we must have $\Gamma \vdash^E \mathtt{t}_1 : \mathtt{T}$. By induction, $\Gamma \vdash^I \lceil \mathtt{t}_1 \rceil : \mathtt{T}$. So we can build the following deduction:

$$\dfrac{\dfrac{\dfrac{\mathtt{x} : \mathtt{T} \in \Gamma, \mathtt{x} : \mathtt{T}}{\Gamma, \mathtt{x} : \mathtt{T} \vdash^I \mathtt{x} : \mathtt{T}}\ \text{T-Var}}{\Gamma \vdash^I (\lambda\mathtt{x}{:}\mathtt{T}.\mathtt{x}) : \mathtt{T} \rightarrow \mathtt{T}}\ \text{T-Abs} \qquad \Gamma \vdash^I \lceil \mathtt{t}_1 \rceil : \mathtt{T}}{\Gamma \vdash^I (\lambda\mathtt{x}{:}\mathtt{T}.\mathtt{x})\ \lceil \mathtt{x}_1 \rceil : \mathtt{T}}\ \text{T-App}$$

iv. By hypothesis, $\Gamma \vdash^I \lceil \mathtt{t} \rceil : \mathtt{T}$. The only applicable rule is (T-App), which gives $\Gamma \vdash^I (\lambda\mathtt{x}{:}\mathtt{T}.\mathtt{x}) : \mathtt{T}' \rightarrow \mathtt{T}$ and $\Gamma \vdash^I \lceil \mathtt{t}_1 \rceil : \mathtt{T}'$. By the Inversion Lemma, $\mathtt{T}' = \mathtt{T}$. By induction, $\Gamma \vdash^E \mathtt{t}_1 : \mathtt{T}$. Finally, by (T-Ascribe), $\Gamma \vdash^E \mathtt{T}_1\ \mathtt{as}\ \mathtt{T} : \mathtt{T}$. $\square$