SySc 545/645 Information Theory Final Prep

Monday 12/6/04 10:15 – 12:05, regular classroom

1. Given probabilities, know how to find Huffman codes for arbitrary alphabet sizes.

2. On the midterm exam there was a question about the transition probabilities of a Markov process. In particular, I asked for a simple polynomial equation that the transition probabilities must satisfy to maximize the entropy rate. You should understand the principles involved in the answer to that question.

3. Problem 8.1 in the book concerns preprocessing Y the output of a channel. If $g : Y \to \tilde{Y}$ is invertible, then capacity is unchanged. But in some cases, a *singular* g may not decrease capacity. For example, $g(y) = y - (y \mod 2)$ does not reduce the capacity of the noisy typewriter. What are the most general conditions on g that assure that capacity is not changed?

4. Given the transition probabilities $P_{Y|X}$ that characterize a discrete memoryless channel without feedback:

   (a) What is the the definition of *information* channel capacity?

   (b) What is the the definition of *operational* channel capacity?

   It may be helpful to recall that the channel coding theorem can be paraphrased as saying that the *information* channel capacity equals the *operational* channel capacity.

5. (Problem 9 Chapter 8) The Z channel has binary input and output alphabets and transition probabilities $P(y|x)$ given by the following matrix:
$$Q = \begin{bmatrix} 1 & 0 \\ q & (1-q) \end{bmatrix} \qquad x, y \in \{0, 1\}$$
   Find the capacity of the Z channel and the maximizing input probability distribution.

6. (Problem 10 Chapter 8) For the Z channel of the previous problem, assume that we choose a $(2^{nR}, n)$ code at random, where each codeword is a sequence of fair coin tosses. Find the maximum rate R such that the probability of error $P_\epsilon^{(n)}$, averaged over the randomly generated codes, tends to zero as the block length n tends to infinity. **Be sure that you can explain why the error probability goes to zero.**

7. In the proof of the channel coding theorem (Theorem 8.7.1) an average over all possible $(2^{nR}, n)$ codebooks is considered. If the channel is binary, the block size is $n = 1024$, and $R = 0.75$, what is the number of codebooks over which the average is calculated. Compare this number to Avogadro's number (6.02×10^{23}).

8. A Hamming code with block length 15 can be characterized by the matrix

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(compare with Eqn. 8.124 on page 210 of the book). Each legal input codeword x satisfies

$$Hx = 0$$

(all arithmetic is understood to be mod 2). We suppose that the codewords are used with equal probability and are transmitted across a binary symmetric channel characterized by a crossover probability $\epsilon = 10^{-4}$. The 15 bits received can be represented by $y = x + e$ where e is the error pattern. Note that

$$H(x + e) = He.$$

   (a) How many codewords (legal input x's) are there?

   (b) If $y = 001011100100101$ what is $\hat{x}(y)$, the best guess for x?

   (c) If the best decoding scheme is used, what is the probability of error? (Give bounds like $\frac{1}{41} < P(\text{error}) < \frac{1}{37}$.)

   (d) Find a formula for the block length of Hamming codes. For each block length, give the number of data bits and the number of check bits.

(e) How do the results for parts 8a through 8c change for other block lengths?

9. A channel has binary input and output alphabets and transition probabilities $P(y|x)$ given by the following matrix:

$$Q = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \qquad x, y \in \{0, 1\}$$

Find the capacity of the channel and the maximizing input probability distribution.

10. The noisy Greek typewriter has 24 possible inputs and outputs, i.e., $\mathcal{X} = \mathcal{Y} = \{1, 2, \ldots, 24\}$, and the channel can be characterized by $y(t) = x(t) + z(t) \mod 24$, where $z(t)$ is zero, one, or two with probability $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

    (a) What is the channel capacity?

    (b) Characterize the entire set of distributions that achieve C.

    (c) What is the dimension of the set?

    Rework the problem assuming that the Z distribution is $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$.

11. In the proof of the channel coding theorem there is a calculation of the probability of error. In that calculation, one argues[1], "By the symmetry of the code construction, the average probability of error averaged over all codes does not depend on the particular index that was sent." Explain the symmetry argument.

[1] See the top of page 201 in the text, or page 13 of Fraser's notes.