SySc 645 Information Theory:
Final Exam. Solutions

1. (30 points) A Hamming code with block length 7 can be characterized by the matrix

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Each legal input codeword x satisfies

$$Hx = 0$$

(all arithmetic is understood to be mod 2). The 7 bits received can be represented by $y = x + e$ where e is the error pattern. Assume that the bits are transmitted over a memoryless binary symmetric channel with the probability of error in each bit of ϵ which is much less than $\frac{1}{2}$.

   (a) How many codewords (legal input x's) are there?
   Answer: 16

   (b) If $y = 1110010$ what is $\hat{x}(y)$, the best guess for x?
   Answer: 1110000

   (c) Assuming that you attempt to correct errors in received words, find P_E, the probability that you will make an error in guessing the transmitted codeword:

      i. First write out an exact algebraic expression.
      Answer: $P_E = 1 - (1-\epsilon)^7 - 7\epsilon(1-\epsilon)^6$

      ii. Give an approximate numerical value for the expression assuming that $\epsilon = 10^{-9}$.
      Answer: $P_E \approx \binom{7}{2}\epsilon^2 = 21 \times 10^{-18}$

2. (20 points) Given the transition probabilities $P_{Y|X}$ that characterize a discrete memoryless channel without feedback:

   (a) What is the definition of *information* channel capacity?
   Answer: $C_{\text{info}} = \max_{P_X} I(X;Y)$

   (b) What is the definition of *operational* channel capacity?
   Answer: C_{op} = Supremum of achievable rates, where a rate R is achievable if there is a sequence of $(2^{nR}, n)$ codes in which the limit of the maximal probability of error goes to zero as $n \to \infty$.

3. (10 points) In the proof of the channel coding theorem an average over all possible $(2^{nR}, n)$ codebooks is considered. If the channel is binary, and $R = \frac{1}{2}$, define $N(n)$ as the number of codebooks over which the average is calculated. For what integer value of n is $N(n)$ closest to Avogadro's number? In other words what integer n minimizes

$$\left|N(n) - 6.02 \times 10^{23}\right|?$$

Answer: 7: The formula is $N(n) = 2^{n2^{nR}}$, and for $R = .5$, $N(7) = 6.92 \times 10^{23}$

4. (10 points) A channel has input $x \in 0,1,2,3$, output $y \in 0,1,2,3$, and transition probabilities $P(y|x)$ given by the following matrix:

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{2} \end{bmatrix}$$

Find the capacity of the channel and the maximizing input probability distribution.
Answer: $P_X = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ $C = \frac{1}{4}$bits

5. (15 points) Suppose that X is distributed with the exponential density

$$f(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

   (a) Find the differential entropy $h(X)$.
   Answer: 1 nat

   (b) Find a function ϕ_1 so that if $y = \phi_1(x)$ then $h(Y) = 0$.
   Answer: $\phi(x) = \frac{x}{e}$

   (c) Find a function ϕ_2 with $\phi_2(1) = 0$ so that if $w = \phi_2(x)$ then $h(W) = 0$.
   Answer: $\phi(x) = \frac{x-1}{e}$

6. (15 points) Consider the ordinary Shannon Gaussian channel with two correlated looks at X, ie, $Y = (Y_1, Y_2)$, where

$$Y_1 = X + Z_1$$
$$Y_2 = X + Z_2$$

with a power constraint P on X, and $(Z_1, Z_2) \sim \mathcal{N}_2(0, K)$ where

$$K = \begin{bmatrix} N & N\rho \\ N\rho & N \end{bmatrix}$$

Find the capacity for

   (a) $\rho = 1$
   Answer: $\frac{1}{2}\log\left(1 + \frac{P}{N}\right)$

   (b) $\rho = 0$
   Answer: $\frac{1}{2}\log\left(1 + \frac{2P}{N}\right)$

   (c) $\rho = -1$
   Answer: ∞