ECE 510 OCE
BDDs and Their Applications

Lecture 4. BDD-based Representations

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Overview

- BDD-based representation of functions, functions with dcs, relations, minterms, cubes, sets, sets of sets, state machines, partitions, set systems, graphs, covering tables, matrices - what else?:)
- Common features of all successful BDD-based representations
- Detailed discussion of project possibilities
- Selecting representation for your problem

Incompletely Specified Functions

- Completely specified function (CSF) is a mapping $B^n \rightarrow B$, where B = {0,1}, n > 0
- Incompletely specified function (ISF) is a set of two CSFs, one represents ON-set (F1), while the other represents DC-set (FDC)
- Alternatively, an ISF can be represented by the interval (F1, F2), where F1 is ON-set and F2 is the sum of ON-set (F1) and DC-set (FDC)

$$F1 \leq F \leq F2$$

Relations

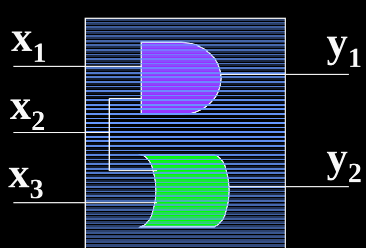
- Relation is a mapping $B^n \rightarrow B^m$, where B = {0,1}, n > 0, m > 0.
- If m = 1, a relation is a function
- For example, $F(x_1, x_2, y_1, y_2, y_3)$, n = 2, m = 3

x_1	x_2	y_1	y_2	y_3
0	0	1	-	0
0	1	0	1	-
1	0	1	1	0
1	1	0	1	-

Relations Are Reducible to Functions

- Relation over variables $(x_1,x_2,\ldots,x_n)$ and $(y_1,y_2,\ldots,y_m)$ can be represented as a boolean function, which is 1 for a given minterm iff this minterm represents related assignments of variables $(x_1,x_2,\ldots,x_n)$ and $(y_1,y_2,\ldots,y_m)$.

Example

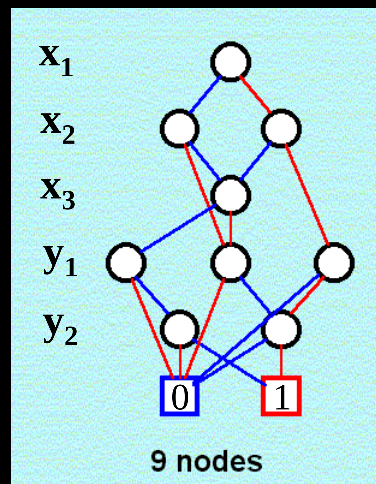


x_1	x_2	x_3	y_1	y_2
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	0
1	0	1	0	1
1	1	0	1	1
1	1	1	1	1

This is a relation with $n = 3$, $m = 2$

Example (continued)

x_1	x_2	x_3	y_1	y_2	F
0	0	0	0	0	1
0	0	1	0	1	1
0	1	0	0	1	1
0	1	1	0	1	1
1	0	0	0	0	1
1	0	1	0	1	1
1	1	0	1	1	1
1	1	1	1	1	1
other					0



Minterms and Cubes

- Given a function $F(x_1,x_2,\ldots,x_n)$, a product of *all* its variables in arbitrary polarities is a minterm
- A product of its variables, which does not necessarily include all variables, is a cube
- Each minterm is a cube; the reverse is not true

Implicit Cube Representation

- To represent cubes of n-variable function $F(x_1,x_2,\ldots,x_n)$, two sets of n vars are used:

  signature variables $S = (s_1,s_2,\ldots,s_n)$

  and polarity variables $P = (p_1,p_2,\ldots,p_n)$

  s_i variable is true iff i-th variable is present in the cube; p_i variable is true iff i-th variable enters this cube in positive polarity
- For example, cube $x_2x_3'x_4$ is represented by the pair [(0111), (1101)]

BDD-based Representation of Relations Between Cubes

- Characteristic function of the cube is

  $\chi(s,p)(x) = \Pi_k\ (s_k \Rightarrow (x_k = p_k))$
- Cubes $C_1(s_1,p_1)$ and $C_2(s_2,p_2)$ are identical iff

  $\Pi_k\ [\ (s_{1k} = s_{1k})\ \&\ (s_{1k} \Rightarrow (p_{1k} = p_{2k}))\ ]$
- Cube $C_1(s_1,p_1)$ contains cube $C_2(s_2,p_2)$ iff

  $\Pi_k\ [\ s_{1k} \Rightarrow (s_{2k}\ \&\ (p_{1k} = p_{2k}))\ ]$

Characteristic Function of a Set

- Function $F: B^n \rightarrow B$, $B = \{0,1\}$, defines a subset of minterms of B^n, on which it is 1.
- Given a binary encoding of a set of elements, characteristic function of a *subset* of this set is a boolean function, which is 1 for minterms encoding the subset and 0 for other minterms.

Example

- Problem: Given the set $\{p_0, p_1, p_2, p_3, p_4, p_5\}$ and its encoding:

  $p_0 - \bar{x}_2\bar{x}_1\bar{x}_0$ $p_2 - \bar{x}_2 x_1\bar{x}_0$ $p_4 - x_2\bar{x}_1\bar{x}_0$

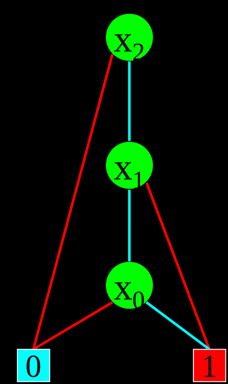
  $p_1 - \bar{x}_2\bar{x}_1 x_0$ $p_3 - \bar{x}_2 x_1 x_0$ $p_5 - x_2\bar{x}_1 x_0$

  find characteristic function of subset $\{p_0, p_2, p_3\}$ and represent the subset using BDD
- Solution: Define a function over the encoding variables (x_0,x_1,x_2) such that it is equal to 1 for minterms representing subset $\{p_0, p_2, p_3\}$.

$$\varphi_{\{p0,\ p2,\ p3\}}(x_0,x_1,x_2) = \bar{x}_2\bar{x}_1\bar{x}_0 + \bar{x}_2 x_1\bar{x}_0 + \bar{x}_2 x_1 x_0$$

BDD Representation of the Characteristic Function

$$\mathbf{F} = \bar{x}_2\bar{x}_1\bar{x}_0$$
$$+\ \bar{x}_2 x_1 \bar{x}_0$$
$$+\ \bar{x}_2 x_1 x_0$$



Set Manipulation

Operations on sets can be reduced to boolean operations on their characteristic functions

- Empty set: $\chi_{\varnothing} = 0$
- Union of sets: $\chi_{S \cup T} = \chi_S + \chi_T$
- Intersection of sets: $\chi_{S \cap T} = \chi_S \,\&\, \chi_T$
- Difference of sets: $\chi_{S - T} = \chi_S \,\&\, (\chi_T)'$
- Subset relation ($S \subset T$): $\chi_{S - T} = \chi_S \,\&\, (\chi_T)' = 0$

Sets of Sets

- If BDD representation of sets is known, let us apply it two times:
  - first, to represent elements of a set
  - next, to represent sets themselves
- To do this, we first encode the elements of a set; next encode sets themselves
- The simplest way to do this is to use a unique binary code for each set and represent it using labeling variables

FSMs: Transition Relation

- FSM is $\{ I, O, S, \delta, \lambda)$
  (r inputs, m states, n outputs)
- Transition relation is a boolean function
  $T: B^r \times B^m \times B^m \rightarrow B, B=\{0,1\}$
  such that $T(i, x, y) = 1$ iff state y can be reached in one transition from state x when input i is applied

FSMs: Output Relation

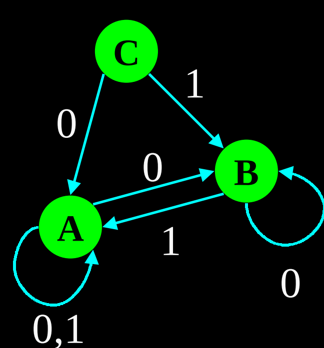
- Output relation is a boolean function

  $O: B^r \times B^m \times B^n \rightarrow B$

  such that $O(i, x, o) = 1$ iff output o can be produced when the FSM is in state x and input i is applied

Example

Ins	CS	code	NS	code
0	A	00	B	10
0,1	A	00	A	00
0	B	10	B	10
1	B	10	A	00
0	C	01	B	10
1	C	01	A	00



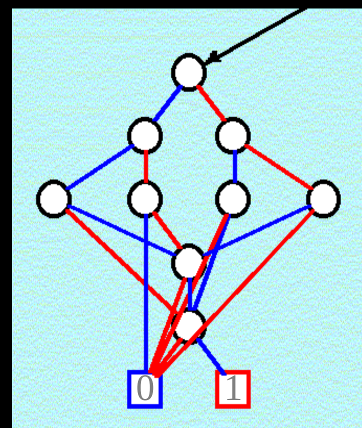
Example (continued)

Transition relation

$T(I,a_1,a_2,b_1,b_2) =$

$i'a_1'a_2'b_1b_2'+$
$a_1'a_2'b_1'b_2' +$
$i'a_1a_2'b_1b_2' +$
$ia_1a_2'b_1'b_2' +$
$i'a_1a_2'b_1b_2' +$
$ia_1a_2'b_1'b_2'$

i
a_1
b_1
a_2
b_2



Equivalence Relation

- Equivalence relation is a boolean function $E(x_1, x_2)$ such that $E(x_1, x_2) = 1$, iff x_1 and x_2 are equivalent.
- Equivalence relation is
  - reflexive: $\forall x\ E(x,x) = 1$
  - symmetric: $\forall x,y\ E(x,y) = E(y,x)$
  - transitive: $\forall x,y,z\ [E(x,y)\ \&\ E(y,z)] \Rightarrow E(x,z)$

Partitions

- Partition π on a set of elements S is a collection of disjoint subsets of S, whose set union is S
- It is written: $\pi = \{B_a\}$ such that
  - $B_a \cap B_b = \emptyset, \forall\ a \neq b$
  - $\cup\{B_a\} = S$
- Often used partitions are partitions of states of FSM $M=\{S,I,O,\delta,\lambda\}$ satisfying certain properties, for example, substitution property:

$$\forall s,t \in S:\ s \equiv t(\pi) \Rightarrow \forall a \in I\ \ \delta(s,a) \equiv \delta(t,a)(\pi)$$

Equivalence Relations and Partitions

- There is a one-to-one correspondence between equivalence relations and partitions
- Each equivalence relation corresponds to a partition of objects, such that each block of the partition consists of objects equivalent with respect to the given equivalence relation, and vice versa

Example

- Problem: Given the set $\{p_0, p_1, p_2, p_3, p_4, p_5\}$ and its encoding:

  $p_0 - \bar{x}_2\bar{x}_1\bar{x}_0$ $p_2 - \bar{x}_2 x_1\bar{x}_0$ $p_4 - x_2\bar{x}_1\bar{x}_0$

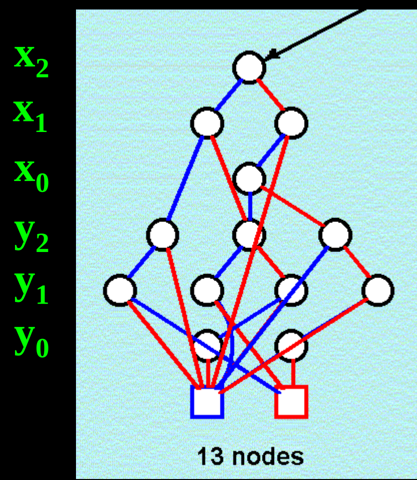
  $p_1 - \bar{x}_2\bar{x}_1 x_0$ $p_3 - \bar{x}_2 x_1 x_0$ $p_5 - x_2\bar{x}_1 x_0$

  find characteristic function of partition $\pi = \{ \{p_0,p_1\}, \{p_2,p_3,p_4\}, \{p_5\} \}$ and represent it using BDDs
- Solution: Define an equivalence relation $E(x_0,x_1,x_2, y_0,y_1,y_2)$ such that it is equal to 1 only for those pairs of minterms that correspond to codes of objects belonging to some block.

Example (continued)

X \ Y	000	001	010	011	100	101	110	111
000	1	1	0	0	0	0	0	0
001	1	1	0	0	0	0	0	0
010	0	0	1	1	1	0	0	0
011	0	0	1	1	1	0	0	0
100	0	0	1	1	1	0	0	0
101	0	0	0	0	0	1	0	0
110	0	0	0	0	0	0	0	0
111	0	0	0	0	0	0	0	0

Implicit Representation of Partition π



Implicit Representation of Matrices

- Suppose each cell of the matrix contains one of a finite number of entries
- Encode columns and rows of the matrix using minterms composed of variables $X=(x_0,x_1,\ldots,x_n)$ and $Y=(y_0,y_1,\ldots,y_m)$
- For each type of entry, create a relation $M_i(X,Y)$ such that $M(X_1,Y_1)=1$, iff the matrix has an entry of this type on the intersection of column X_1 and row Y_1

Properties of BDDs

- Canonicity
- Compactness (with some exceptions)
- Fast computation (with some exceptions)
- Represent a variety of discrete objects
  - Boolean functions and relations
  - Compositional sets and sets of sets
  - Partitions of states
  - Graphs and matrices
- Facilitate symbolic methods
  - Two-level minimization
  - State traversal of FSMs, etc.