Chapter 3

Impulse Sampling

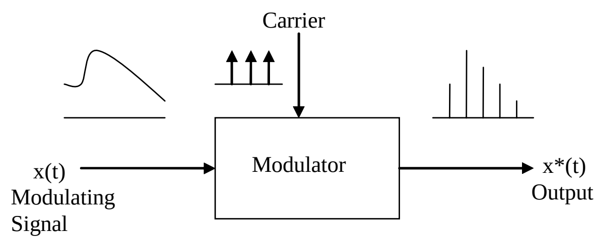
If a continuous-time signal x(t) is sampled in a periodic manner, mathematically the sampled signal may be represented by

$$x^*(t) = \sum_{k=-\infty}^{\infty} x(t)\ \delta(t-kT) = x(t)\ \sum_{k=-\infty}^{\infty} \delta(t-kT) \tag{1}$$

or

$$x^*(t) = \sum_{k=-\infty}^{\infty} x(kT)\ \delta(t-kT) \tag{2}$$

x*(t) is the sampled version of the continuous-time signal x(t).
Low limits may be changed to $k = 0,\ if\ x(t) = 0, \quad t < 0$



Relationship between z transform and Laplace transform

Taking the Laplace transform of equation (2)

$$\begin{aligned} X^*(s) &= \mathcal{L}[x^*(t)] = x(0)\ \mathcal{L}[\delta(t)] + x(T)\ \mathcal{L}[\delta(t-T)] + x(2T)\ \mathcal{L}[\delta(t-2T)] + \cdots \\ &= x(0) + x(T)e^{-Ts} + x(2T)e^{-2Ts} + \cdots \\ &= \sum_{k=0}^{\infty} x(kT)e^{-kTs} \end{aligned}$$

Now, define

$$e^{Ts} = z$$

so,

$$s = \frac{1}{T}\ ln\ z$$

$$X^*(s)\ |_{s=\frac{1}{T}lnz} = \sum_{k=0}^{\infty} x(kT)\ z^{-k}$$

or

$$X^*(s)\ |_{s=\frac{1}{T}lnz} = X(z)$$

This shows how the z transform is related to the Laplace transform.
Note: the notation *X(z)* does not signify *X(s)* with s replaced by z, but rather $X^*(s = \frac{1}{T}lnz)$

Data Hold

Data hold is a process of generating a continuous-time signal *h(t)* from a discrete-time sequence *x(kT)*. The signal *h(t)* during the time interval $kT \le t \le (k+1)T$ may be approximated by a polynomial in τ as follows:

$$h(kT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \cdots + a_1\tau + a_0$$

where $\quad 0 \le \tau \le T$
note: $\quad h(kT) = x(kT)$

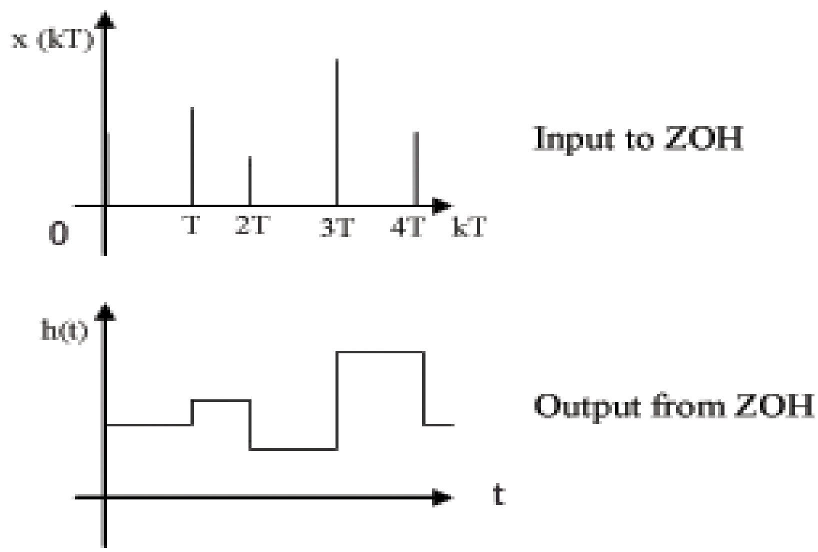
$$\Rightarrow \qquad h(kT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \cdots + a_1\tau + x(kT)$$

If the data hold circuit is an n^{th}-order polynomial extrapolator, it is called an ***n^th^*-order hold**. It uses the past $n+1$ discrete data $x((k-n)T)$, $x((k-n+1)T),\ \cdots,\ x(kT)$ to generate $h(kT+\tau)$.

ZERO-ORDER HOLD

If $n = 0$ in the above equation, we have a zero order hold so that

$$h(kT+\tau) = x(kT) \qquad 0 \le \tau < T, \qquad k = 0,\ 1,\ 2,\ \cdots$$



Transfer Function of ZOH

$$\begin{aligned} h(t) &= x(0)\ [u(t) - u(t-T)] + x(T)\ [u(t-T)] - u(t-2T)] + \\ &\quad x(2T)[u(t-2T) - u(t-3T)] + \cdots \\ &= \sum_{k=0}^{\infty} x(kT)\ [u(t-kT)] - u(t-(k+1)T)] \end{aligned}$$

Now, $\mathcal{L}[u(t-kT)] = \frac{e^{-kTs}}{s}$

$$\begin{aligned} thus, \qquad \mathcal{L}[h(t)] &= H(s) = \sum_{k=0}^{\infty} x(kT)\ \frac{e^{-kTs} - e^{-(k+1)Ts}}{s} \\ &= \underbrace{\frac{1-e^{-Ts}}{s}}_{G_{h0(s)}} \underbrace{\sum_{k=0}^{\infty} x(kT) e^{-kTs}}_{X^*(s)} \end{aligned}$$

Thus, transfer function of ZOH

$$= \frac{H(s)}{X^*(s)} = \frac{1-e^{-Ts}}{s}$$

FIRST-ORDER HOLD

$$h(kT+\tau) = a_1\tau + x(kT), \qquad 0 \le \tau < T, \qquad k = 0,\ 1,\ 2, \cdots$$

now

$$h((k-1)T) = x((k-1)T)$$

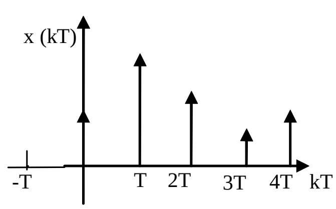
so that

$$h((k-1)T) = -a_1 T + x(kT) = x((k-1)T)$$

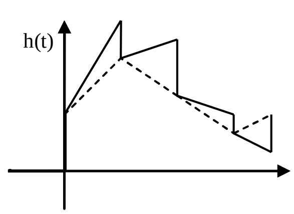
or

$$a_1 = \frac{x(kT) - x((k-1)T)}{T}$$

$$\Rightarrow \qquad h(kT+\tau) = x(kT) + \frac{x(kT) - x((k-1)T)}{T}\ \tau, \qquad 0 \le \tau < T$$



Input to FOH

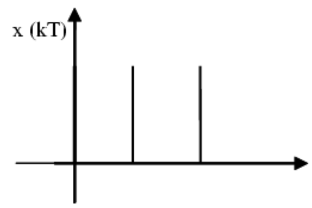


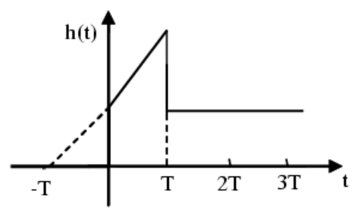
Output from FOH

TRANSFER FUNCTION

Consider a unit step function input

$$x^*(t) = \sum_{k=0}^{\infty} u(kT)\ \delta(t-kT) = \sum_{k=0}^{\infty} \delta(t-kT)$$





$$h(t) = (1+\frac{t}{T})\ u(t) - \frac{(t-T)}{T}\ u(t-T) - u(t-T)$$

$$\begin{aligned} \Rightarrow \qquad H(s) &= (\frac{1}{s}+\frac{1}{Ts^2}) - \frac{1}{Ts^2}\ e^{-Ts} - \frac{1}{s}\ e^{-Ts} \\ &= \frac{1-e^{-Ts}}{s} + \frac{1-e^{-Ts}}{Ts^2} \\ &= (1-e^{-Ts})\ \frac{Ts+1}{Ts^2} \end{aligned}$$

The Laplace transform of unit step

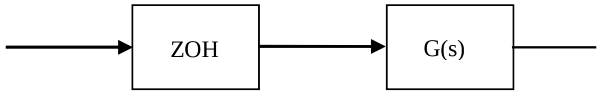
$$X^*(s) = \mathcal{L}[u^*(t)] = \frac{1}{1-e^{-Ts}}$$

thus

$$\begin{aligned} G_{h1}(s) &= \frac{H(s)}{X^*(s)} = (1-e^{-Ts})^2\ \frac{Ts+1}{Ts^2} \\ &= \left(\frac{1-e^{-Ts}}{s}\right)^2\ \frac{Ts+1}{T} \end{aligned}$$

Obtaining z transform of functions involving the term $\frac{1-e^{-Ts}}{s}$

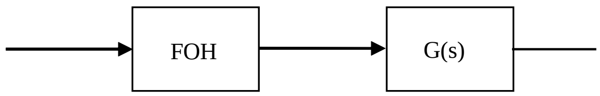
Suppose the transfer function G(s) follows a zero-order hold



$$X(s) = \frac{1-e^{-Ts}}{s}\, G(s)$$

$$X(z) = \mathcal{Z}[X(s)] = (1-z^{-1})\, \mathcal{Z}\left[\frac{G(s)}{s}\right]$$

Suppose a transform function G(s) follows a first-order hold



$$X(s) = (\frac{1-e^{-Ts}}{s})^2\, \frac{Ts+1}{T}\, G(s)$$

$$X(z) = (1-z^{-1})^2\, \mathcal{Z}[\frac{Ts+1}{Ts^2} G(s)]$$

Reconstructing Original Signals from Sampled Signals

Sampling Theorem:

Define $\omega_s = \frac{2\pi}{T}$ where T is the sampling period, then if

$$\omega_s > 2\omega_1$$

where ω_1 is the highest-frequency component present in the continuous-time signal x(t), then the signal x(t) can be reconstructed completely from the sampled signal $x^*(t)$.

Intuitive proof

The frequency spectrum of a sampled signal $x^*(t)$ is given by

$$X^*(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j\omega + j\omega_s k)$$

* See next page

Spectrum of Sampled Signal

Fourier series representation of a train of impulses

$$\sum_{k=-\infty}^{\infty} \delta(t-kT) = \sum_{n=-\infty}^{\infty} C_n \, e^{j(\frac{2\pi n}{T})t} \tag{3}$$

where

$$\begin{aligned} C_n &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \sum_{k=-\infty}^{\infty} \delta(t-kT)e^{-jn(\frac{2\pi t}{T})} \, dt \\ &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)e^{-jn(\frac{2\pi t}{T})} \, dt \end{aligned}$$

$\delta(t)$ is the only value in the integral limit range. So

$$C_n = \frac{1}{T} \tag{4}$$

note: by sifting property, $\int_{-\infty}^{\infty} \delta(t) \; g(t) \; dt = \; g(0)$

Thus, (3) & (4) $\Rightarrow$

$\sum_{k=-\infty}^{\infty} \delta(t-kT) = \; \frac{1}{T}\sum_{n=-\infty}^{\infty} \; e^{j(\frac{2\pi n}{T})t}$

$\nwarrow$ Fourier series representation of sum of impulses

$$x^*(t) = \sum_{k=-\infty}^{\infty} x(t) \; \delta(t-kT)$$

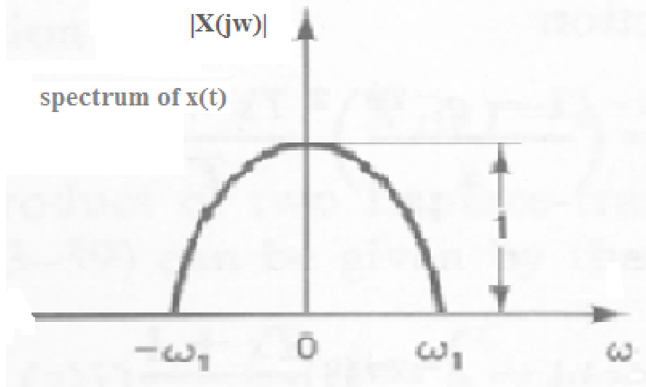
$$\Rightarrow \qquad \mathcal{L}\left\{x^*(t)\right\} = \int_{-\infty}^{\infty} x(t) \left\{\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}\right\} \; e^{-sT} \; dt$$

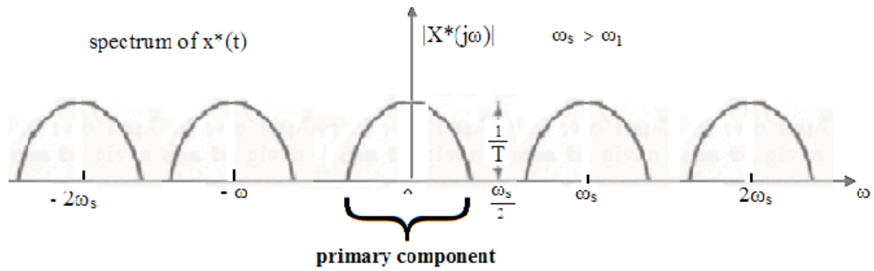
where $\quad \omega_s = \frac{2\pi}{T}$

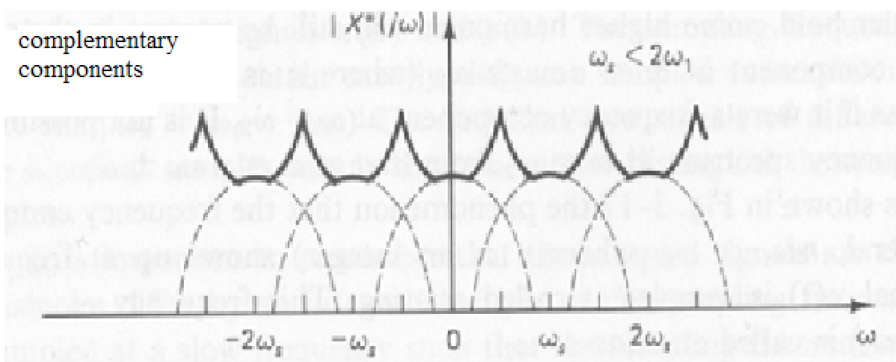
$$\begin{aligned} X^*(s) &= \frac{1}{T}\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) \; e^{jn\omega_s t} \; e^{-st} \; dt \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) \; e^{-(s-jn\omega_s)t} \; dt \end{aligned}$$

$\nearrow$ Laplace transform with a change of variable

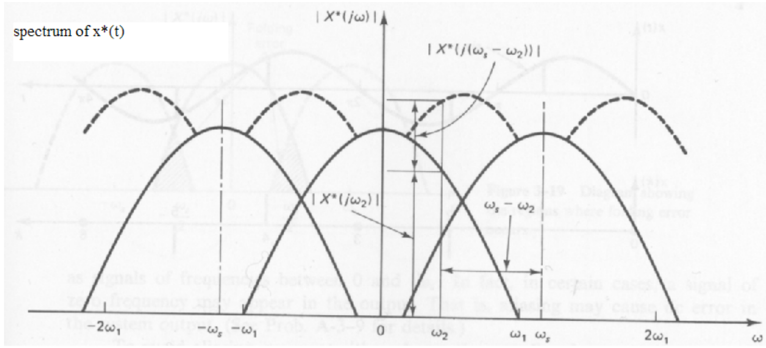
$$X^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(s-jn\omega_s)$$


|X(jw)|
spectrum of x(t)
1
$-\omega_1$
0
ω_1
ω


spectrum of x*(t)
$|X^*(j\omega)|$
$\omega_s > \omega_1$
$\frac{1}{T}$
$-2\omega_s$
$-\omega$
$\frac{\omega_s}{2}$
ω_s
$2\omega_s$
ω
primary component


complementary components
$|X^*(j\omega)|$
$\omega_s < 2\omega_1$
$-2\omega_s$
$-\omega_s$
0
ω_s
$2\omega_s$
ω

Aliasing



If $\omega_s < 2\omega_1$, aliasing occurs

Consider frequency component ω_2 which falls in the overlap. It has two components

$$|X^*(j\omega_2)| \qquad and \qquad \underbrace{|X^*(j(\omega_s - \omega_2))|}_{\text{from spectrum centered at } \omega = \omega_s}$$

Therefore the frequency spectrum of the sampled signal at $\omega = \omega_2$ includes components not only at frequency ω_s but also at frequency $\omega_s - \omega_2$ (in general, at $n\omega_s \pm \omega_2$, where n is an integer).

The frequency $\underbrace{\omega_s - \omega_2}$ is known as an <u>alias</u> of ω_2.

In general $n\omega_s \pm \omega_2$

Aliasing

Two frequencies x(t) and y(t) differ from each other by an integral multiple of ω_s, sampling frequency $T = \frac{2\pi}{\omega_s}$.

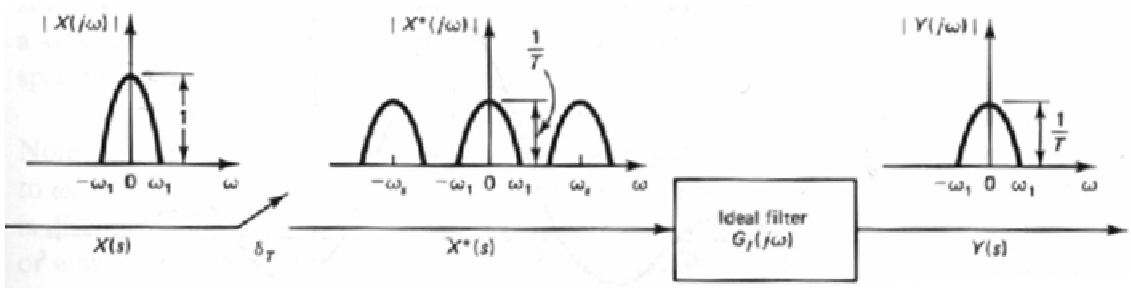
$$x(kT) = sin(\omega_2 kT + \theta)$$

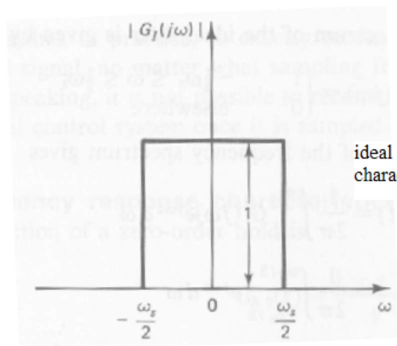
$$\begin{aligned} y(kT) &= sin((\omega_2 + n\omega_s)kT + \theta) \\ &= \sin(\omega_2 kT + 2\pi kn + \theta) \\ &= \sin(w_2 kT + \theta) \end{aligned}$$

$\Rightarrow$ Two signals of different frequencies can have identical samples, means that we cannot distinguish between them from their samples.

If $\omega_s < 2\omega_1$, the original spectrum is contaminated, so x(t) cannot be reconstructed.

If $\omega_s > 2\omega_1$, the original signal can be recovered by using an ideal low pass filter.



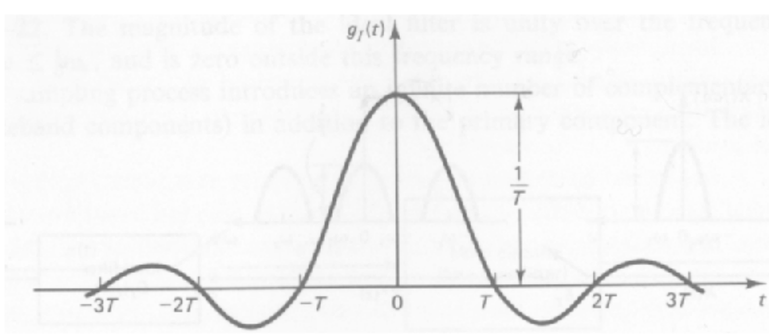


ideal low pass filter characteristics

$$G_I(j\omega) = \begin{cases} 1 & -\frac{1}{2}\omega_s \leq \omega \leq \frac{1}{2}\omega_s \\ 0 & elsewhere \end{cases}$$

the inverse Fourier transform gives

$$g_I(t) = \frac{1}{T} \frac{sin(\omega_s \frac{t}{2})}{\omega_s \frac{t}{2}} \qquad \leftarrow \textit{unit impulse response}$$



Non-causal signal $\Rightarrow$ Filter is not realizable

Use a ZOH to approximate ideal low pass filter.

Frequency response of ZOH

$$G_{h_0} = \frac{1 - e^{-Ts}}{s}$$

Since

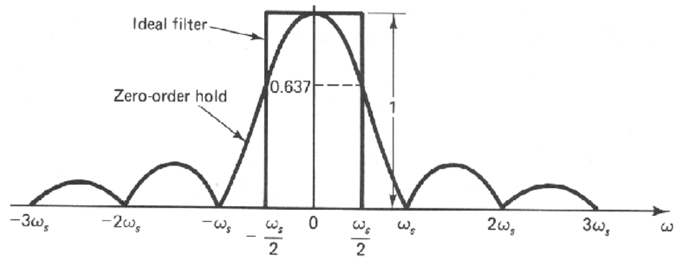
$$\frac{1 - e^{-Ts}}{s} = \frac{e^{-T\frac{s}{2}}(e^{T\frac{s}{2}} - e^{-T\frac{s}{2}})}{s}$$

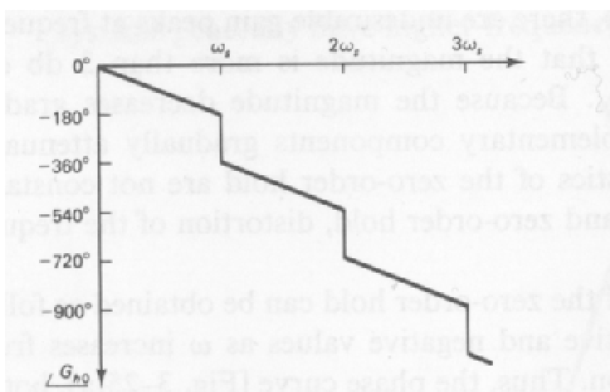
$$s = j\omega \qquad \Rightarrow \qquad \frac{e^{\frac{-j\omega}{2}T}(e^{T\frac{j\omega}{2}} - e^{-T\frac{j\omega}{2}})}{j\omega}$$

$$\frac{\sin\frac{\omega T}{2} e^{\frac{j\omega}{2}T}}{\frac{\omega}{2}}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\Rightarrow \qquad G_{h_0(j\omega)} = T\,\frac{sin(\omega\ \frac{T}{2})}{\omega\frac{T}{2}}\ e^{-j\omega\frac{T}{2}}$$





The ZOH does not approximate an ideal low pass filter very well. Higher order holds do a better job but are more complex and have more time delay, which reduces stability margin. So ZOH's get used a lot.

The Pulse Transfer Function

Convolution Summation

$$
\begin{aligned}
y(kT) &= \sum_{k=0}^{\infty} g(kT - hT)\; x(kT) \\
&= \sum_{k=0}^{\infty} x(kT - hT)\; g(kT) \\
&\equiv x(kT) \; * g(kT)
\end{aligned}
$$

where g(kT) is the system's weighting sequence. i.e. $g(k) = \mathcal{Z}^{-1}\{G(z)\}$

Starred Laplace Transfrom

$$Y(s) = G(s)\; X^*(s)$$

Note: $X^*(s)$ is periodic with period $\frac{2\pi}{\omega_s}$ *since* $X^*(s) = X^*(s \pm j\omega_s k)$, *for* $k = 0,\; 1,\; 2, \cdots$. G(s) is not periodic.

Taking the starred transform { See notes on next page }

$$Y^*(s) = [G(s)X^*(s)]^* = [G(s)]^* X^*(s) = G^*(s)X^*(s)$$

Note: $X^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(s + j\omega_s k) + \frac{1}{2}x(0^+)$

$$Y(z) = G(z)\; X(z)$$

Since the z transform can be seen to be the starred Laplace transform with e^{Ts} replaced by z; i.e. $X^*(s) = X(z)$

$$Y^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G(s + j\omega_s k)X^*(s + j\omega_s k)$$

and

$$X^*(s) = \frac{1}{T}\sum_{h=-\infty}^{\infty} X(s + j\omega_s h) + \frac{1}{2}x(0+)$$

we have

$$X^*(s + j\omega_s k) = \frac{1}{T}\sum_{h=-\infty}^{\infty} X(s + j\omega_s h + j\omega_s k) + \frac{1}{2}x(0+)$$

By letting h + k = m, we obtain

$$X^*(s + j\omega_s k) = \frac{1}{T}\sum_{m=-\infty}^{\infty} X(s + j\omega_s m) + \frac{1}{2}x(0+) = X^*(s)$$

Substitution of this last equation into the expression for Y*(s) gives

$$Y^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G(s + j\omega_s k)X^*(s)$$

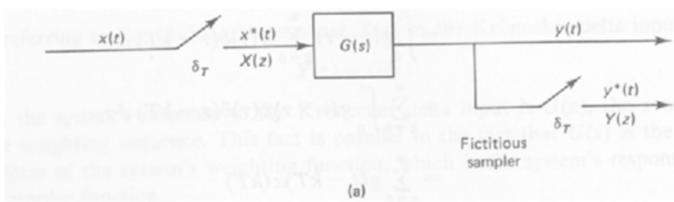
Since G*(s) can be given by

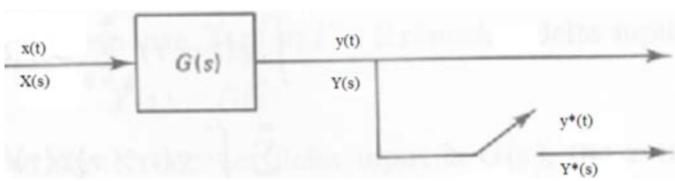
$$G^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G(s + j\omega_s k)$$

we obtain

$$Y^*(s) = G^*(s)X^*(s)$$

Obtaining The Pulse Transfer Function





The presence or absence of an *input* sampler is crucial in determining the pulse transfer function of a system.
For figure a,

$$Y(s) = G(s)\ X^*(s)$$

$$\Rightarrow \quad Y^*(s) = G^*(s)\ X^*(s)$$

$$\Rightarrow \quad Y(z) = G(z)\ X(z)$$

For figure b,

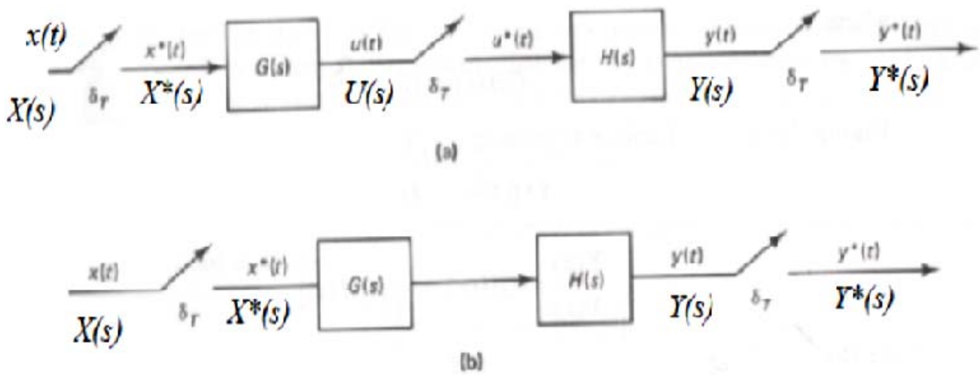
$$Y(s) = G(s)\ X(s)$$

$$\Rightarrow \quad Y^*(s) = [G(s)\ X(s)]^* = [GX(s)]^* = GX(z) \neq G(z)X(z)$$

Methods for obtaining the z transform

1. $X(z) = \mathcal{Z}\left[X(s)\ expanded\ into\ partial\ fractions\ X_i(s)\right] = \sum_i \left\{\mathcal{Z}[X_i(s)]\right\}$
2. $X(z) = \sum_{k=0}^{\infty} x(kT) z^{-k}$
3. $X(z) = \sum \left[residue\ of\ \frac{X(s)z}{z-e^{Ts}}\ at\ pole\ of\ X(s)\right]$

Pulse transfer function of cascaded elements



For figure (a)

$$U(s) = G(s)\ X^*(s) \qquad \Rightarrow U^*(s) = G^*(s)\ X^*(s)$$

$$Y(s) = H(s)\ U^*(s) \qquad \Rightarrow Y^*(s) = H^*(s)\ U^*(s)$$

$$\begin{aligned} \Rightarrow \qquad Y^*(s) &= H^*(s)\ G^*(s)\ X^*(s) \\ &= G^*(s)\ H^*(s)\ X^*(s) \end{aligned}$$

$$\Rightarrow \qquad Y(z) = G(z)\ H(z)\ X(z) \qquad \Rightarrow \qquad \frac{Y(z)}{X(z)} = G(z)\ H(z)$$

For figure (b)

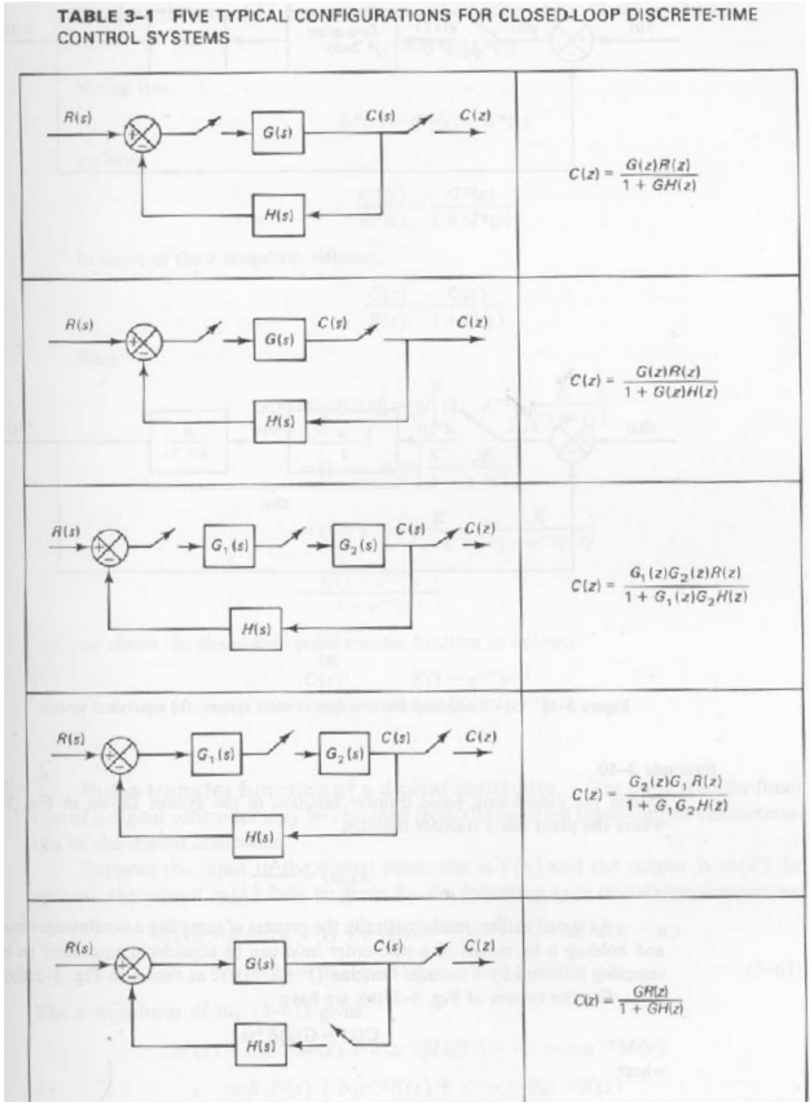
$$Y(s) = G(s)\ H(s)\ X^*(s) = GH(s)\ X^*(s)$$

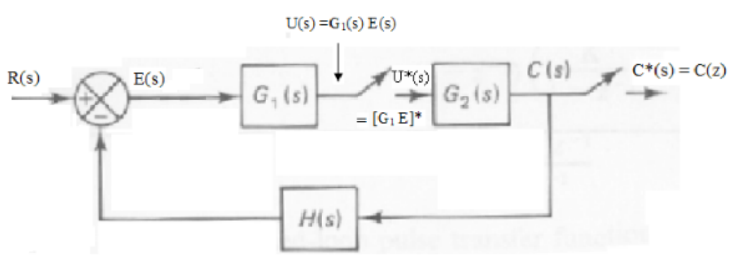
$$\Rightarrow \qquad Y^*(s) = [GH(s)]^*\ X^*(s)$$

$$\Rightarrow \qquad Y(z) = GH(z)\ X(z) \qquad \Rightarrow \qquad \frac{Y(z)}{X(z)} = GH(z) = \mathcal{Z}[GH(s)]$$

$$Note\ that \qquad G(z)\ H(z) \neq GH(z)\ = \mathcal{Z}[GH(s)]$$

Pulse transfer function of closed-loop systems





$$C(s) = G_2(s)\; U^*(s) \tag{5}$$

$$\Rightarrow \qquad C^*(s) = G_2^*(s)\; U^*(s) \tag{6}$$

$$E(s) = R(s) - H(s)\; C(s) \tag{7}$$

$$\begin{aligned} U(s) = G_1(s)\; E(s) &= G_1(s)\; R(s) - G_1(s)\; H(s)\; C(s) \\ &= G_1(s)\; R(s) - G_1(s)\; H(s)\; G_2(s)\; U^*(s) \end{aligned}$$

$$
\begin{aligned}
U^*(s) &= [G_1(s)\ R(s)]^* - [G_1(s)\ G_s(s)\ H(s)\ U^*(s)]^* \\
&= [G_1R(s)]^* - [G_1G_2H(s)]^*\ U^*(s)
\end{aligned}
$$

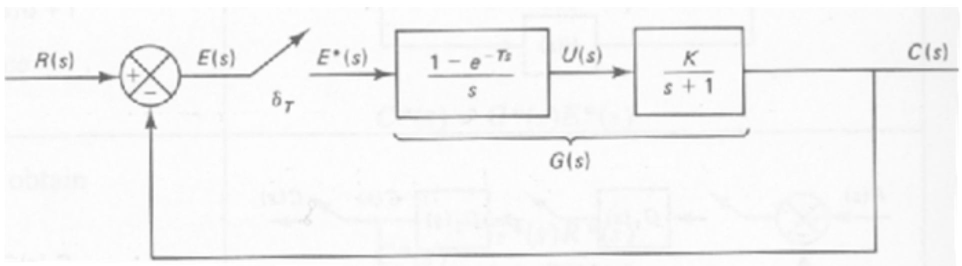
$$
\Rightarrow \qquad \{1 + [G_1G_2H(s)]^*\}\ U^*(s) = [G_1R(s)]^*
$$

$$
\Rightarrow \qquad U^*(s) = \frac{[G_1R(s)]^*}{1 + [G_1G_2H(s)]^*} \tag{8}
$$

(6)&(8) $\quad \Rightarrow$

$$
C^*(s) = \frac{G_2^*(s)\ [G_1R(s)]^*}{1 + [G_1G_2H(s)]^*} \qquad \Rightarrow \qquad C(z) = \frac{G_2(z)\ G_1R(z)}{1 + G_1G_2H(z)}
$$

Example



Find $\frac{C(z)}{R(z)}$

$$
\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)}, \qquad \textbf{for} \qquad H(s) = 1
$$

now

$$
\begin{aligned}
G(z) = \mathcal{Z}[G(s)] &= \mathcal{Z}\left[(1 - e^{-Ts})\frac{K}{s(s+1)}\right] \\
&= (1 - z^{-1})\ \mathcal{Z}\left[\frac{K}{s} - \frac{K}{s+1}\right] \\
&= (1 - z^{-1})\ \left(\frac{K}{1 - z^{-1}} - \frac{K}{1 - e^{-T}z^{-1}}\right) \\
&= \frac{K(1 - e^{-T})z^{-1}}{1 - e^{-T}z^{-1}}
\end{aligned}
$$

closed-loop

$$
\Rightarrow \qquad \frac{C(z)}{R(z)} = \frac{K(1 - e^{-T})z^{-1}}{1 + [K - (K+1)e^{-T}]z^{-1}}
$$

Obtaining response between consecutive sampling instants

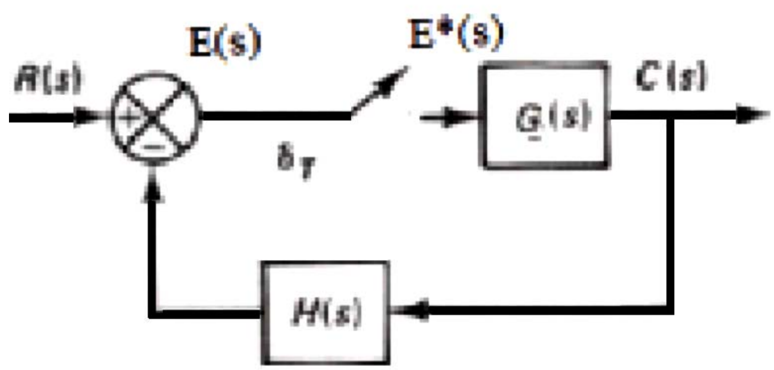
The z transform does not give the response between sampling instants. Can use the following methods to do so.

1. The Laplace transform method
2. The modified z transform method
3. The state space method

Will look only at first method for now.

1. The Laplace transform method

Example: Consider the following system



We know that

$$C(s) = G(s)\ E^*(s) = G(s)\frac{R^*(s)}{1+GH^*(s)}$$

Thus

$$c(t) = \mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}\left\{G(s)\ \frac{R^*(s)}{1+GH^*(s)}\right\}$$

Mapping between the s plane and the z plane

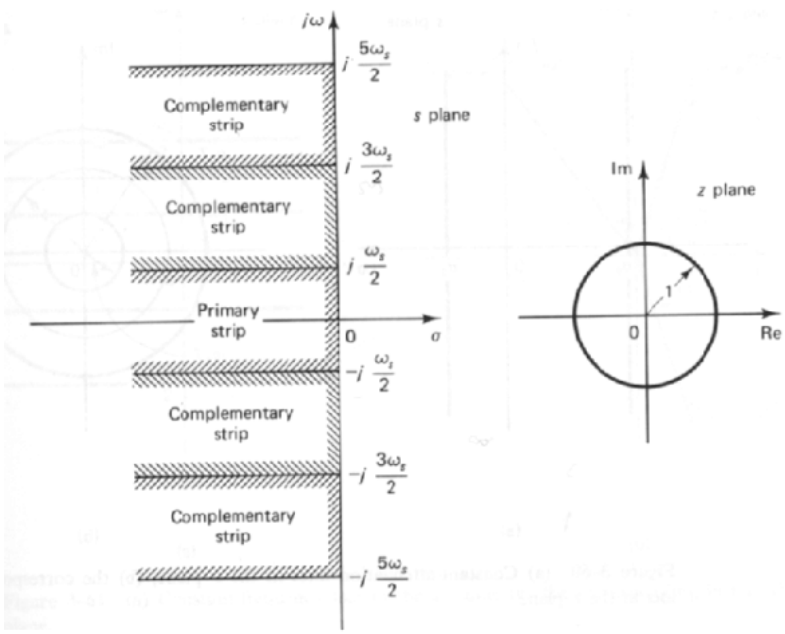
$z = e^{Ts}$
let $s = \sigma + \ j\omega$

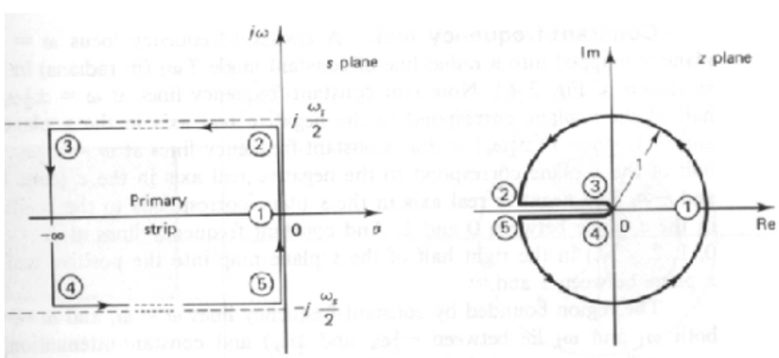
$$z = e^{T(\sigma + j\omega)} \ = e^{T\sigma} \ e^{jT\omega} = e^{T\sigma} e^{j(T\omega + 2\pi k)}$$

now

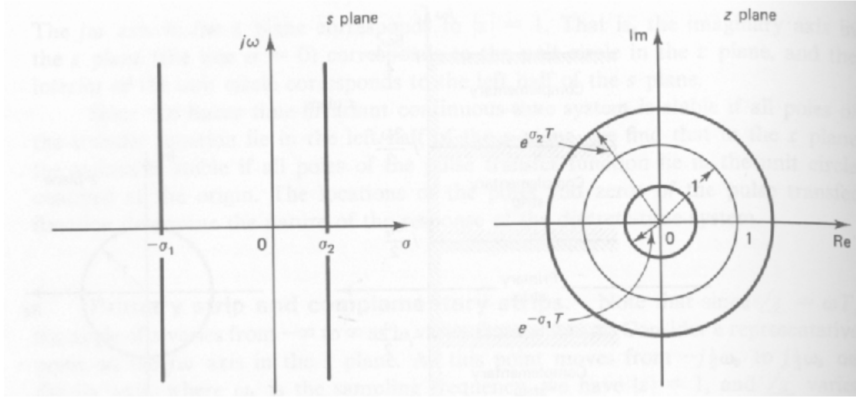
$$|z| = e^{T\sigma} < 1 \qquad for\ \sigma < 0, i.e.\ LHP$$

$\Rightarrow$ $\quad j\omega$ axis in s plane corresponds to $|z| = 1$, the unit circle in z plane

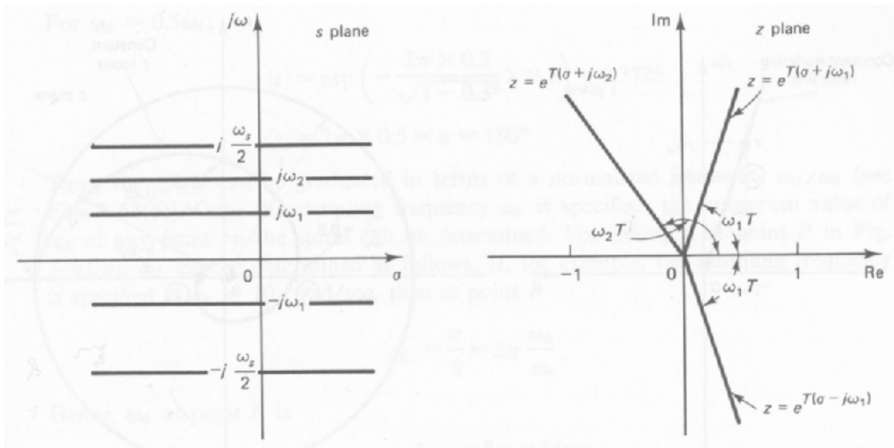




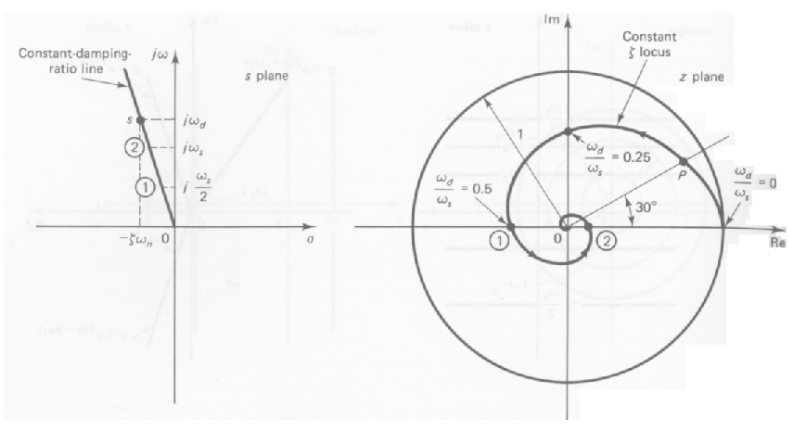
Constant-attenuation loci

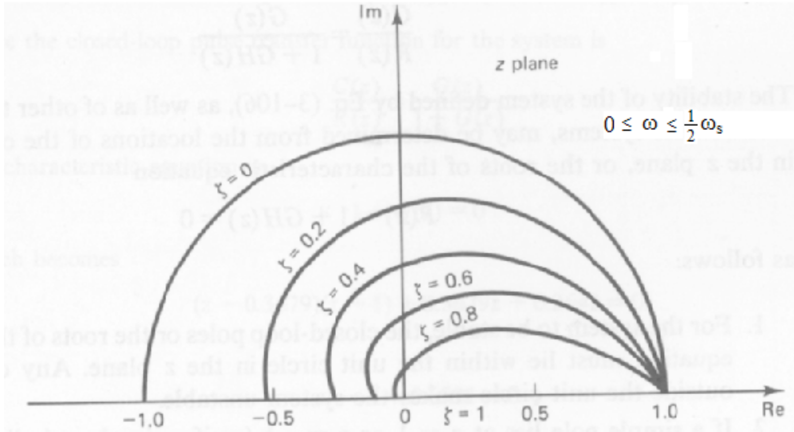


Constant-frequency loci



Constant-damping ratio line





Stability analysis of closed-loop systems in the z domain

Consider the following pulse transfer function

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)}$$

Stability may be accessed by looking at the roots of the the characteristic equation.

$$P(z) = 1 + GH(z) = 0$$

1. For stability, require that the roots $|z_i| < 1$
2. If simple pole $z_i = 1$ *or* $z_i = -1$ *or* $(z_i = 1$ *and* $z_i = -1)$
   or z_i *where* z_i *complex such that* $|z| = 1$
   $\Rightarrow$ critical stability

3. Zeros do not affect absolute stability

Stability tests without finding roots

1. Jury test
2. Bilinear transformation and Routh criteria

The Jury test

$$F(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0 = 0$$

$$a_n > 0$$

General Form of the Jury stability table

Row	z^0	z^1	z^2	…	z^{n-k}	…	z^{n-1}	z^n
1	a^0	a^1	a^2	…	a^{n-k}	…	a^{n-1}	a^n
2	a^n	a^{n-1}	a^{n-2}	…	a^k	…	a^1	a^0
3	b^0	b^1	b^2	…	b^{n-k}	…	b^{n-1}	
4	b^{n-1}	b^{n-2}	b^{n-3}	…	b^k	…	b^0	
5	c^0	c^1	c^2	…		c^{n-2}		
6	c^{n-2}	c^{n-3}	c^{n-4}			c^0		
: :	: :	: :	: :	: :				
2n-5	p^0	p^1	p^2	p^3				
2n-4	p^3	p^2	p^1	p^0				
2n-3	q^0	q^1	q^2					

where

$$
\begin{aligned}
b_k &= \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix} \\
c_k &= \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix} \\
d_k &= \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix} \\
&\vdots \\
q_0 &= \begin{vmatrix} p_0 & p_3 \\ p_3 & p_0 \end{vmatrix} \\
q_2 &= \begin{vmatrix} p_0 & p_1 \\ p_3 & p_2 \end{vmatrix}
\end{aligned}
$$

The necessary and sufficient condition for the F(z) to have no roots on and outside the unit circle are

$$F(1) > 0$$

$$F(-1) \begin{cases} > 0 & n\ even \\ < 0 & n\ odd \end{cases}$$

$$\left.\begin{aligned} |a_0| <& \quad a_n \\ |b_0| >& \quad |b_{n-1}| \\ |c_0| >& \quad |c_{n-2}| \\ |d_0| >& \quad |d_{n-3}| \\ |q_0| >& \quad |q_2| \end{aligned}\right\} (n-1)\ constraints$$

For a second order system, n=2, Jury's table contains only one row ⇒

$$F(1) > 0$$

$$F(-1) > 0$$

$$|a_0| < a_n$$

Bilinear transformation and Routh stability criterion

Transform the z plane to the w plane by

$$z = \frac{\mathrm{w}+1}{\mathrm{w}-1}$$

$$\Rightarrow \qquad \mathrm{w} = \frac{z+1}{z-1}$$

which maps the inside of the unit circle in the z plane into the left half of the w plane. The unit circle in z plane maps into the imaginary axis in the w plane and the outside of the unit circle in z plane maps into RHP of ω plane.

The w plane is similar to s plane (but not quantitatively) $\Rightarrow$ can use Routh test.

Let

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$$

$$\Rightarrow \qquad a_0 \left(\frac{\mathrm{w}+1}{\mathrm{w}-1}\right)^n + a_1 \left(\frac{\mathrm{w}+1}{\mathrm{w}-1}\right)^{n-1} + \cdots + a_{n-1} \left(\frac{\mathrm{w}+1}{\mathrm{w}-1}\right) + a_0 = 0$$

$$Q(\mathrm{w}) = b_0 \mathrm{w}^n + b_1 \mathrm{w}^{n-1} + \cdots + b_{n-1}\mathrm{w} + b_n = 0$$

$\nearrow$ Requires a lot of computation but can now use Routh test.