CS311 Computational Structures

Context-free Languages: Grammars and Automata

Lecture 8

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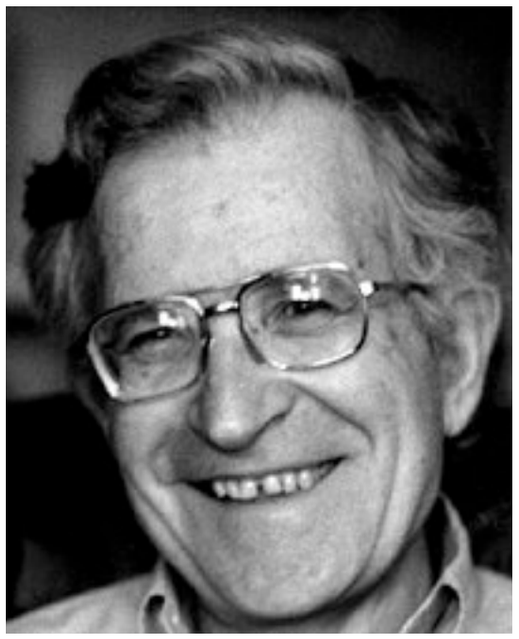
Chomsky hierarchy



In 1957, Noam Chomsky published *Syntactic Structures*, an landmark book that defined the so-called Chomsky hierarchy of languages

original name	language generated	productions:
Type-3 Grammars	Regular	$A \to \alpha$ and $A \to \alpha B$
Type-2 Grammars $\cap$	Contex-free	$A \to \gamma$
Type-1 Grammars $\cap$	Context-sensitive	$\alpha A \beta \to \alpha \gamma \beta$
Type-0 Grammars $\cap$	Recursively-enumerable	no restriction

A, B: variables, a, b terminals, α, β sequences of terminals and variables

Regular languages

- Closed under $\cup \cap * \cdot$ and $\overline{}$
- Recognizable by finite-state automata
- Denoted by Regular Expressions
- Generated by Regular Grammars

Context-free Grammars

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- More general productions than regular grammars

  S → w where w is any string of terminals and non-terminals

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  S →(A)
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Context-free languages more general than regular languages

- $\{a^n b^n \mid n \geq 0\}$ is not regular
  - but it *is* context-free
- Why are they called “context-free”?
  - Context-sensitive grammars allow more than one symbol on the lhs of productions
    - xAy → x(S)y can only be applied to the non-terminal A when it is in the *context* of x and y

Context-free grammars are widely used for programming languages

- From the definition of Algol-60:

procedure_identifier::= **identifier**.
actual_parameter::= **string_literal** | **expression** | **array_identifier** | **switch_identifier** | **procedure_identifier**.
letter_string::= **letter** | **letter_string letter**.
parameter_delimiter::= "," | ")" **letter_string** ":" "(".
actual_parameter_list::= **actual_parameter** | **actual_parameter_list parameter_delimiter actual_parameter**.
actual_parameter_part::= **empty** | "(" **actual_parameter_list**} ")".
function_designator::= **procedure_identifier actual_parameter_part**.

- We say: “most programming languages are context-free”
  - This isn’t strictly true
  - … but we pretend that it is!

Example

adding_operator::= "+" | "−" .
multiplying_operator::= "×" | "/" | "÷" .
primary::= **unsigned_number** | **variable** | **function_designator** | "(" **arithmetic_expression** ")".
factor::= **primary** | **factor** | **factor power primary**.
term::= **factor** | **term multiplying_operator factor**.
simple_arithmetic_expression::= **term** | **adding_operator term** |
    simple_arithmetic_expression adding_operator term.
if_clause::= **if Boolean_expression then**.
arithmetic_expression::= **simple_arithmetic_expression** |
    if_clause simple_arithmetic_expression else arithmetic_expression.

if a < 0 then U+V else if a * b < 17 then U/V else if k <> y then V/U else 0

Example derivation in a Grammar

- Grammar: start symbol is *A*

  $A \rightarrow aAa$
  $A \rightarrow B$
  $B \rightarrow bB$
  $B \rightarrow \varepsilon$

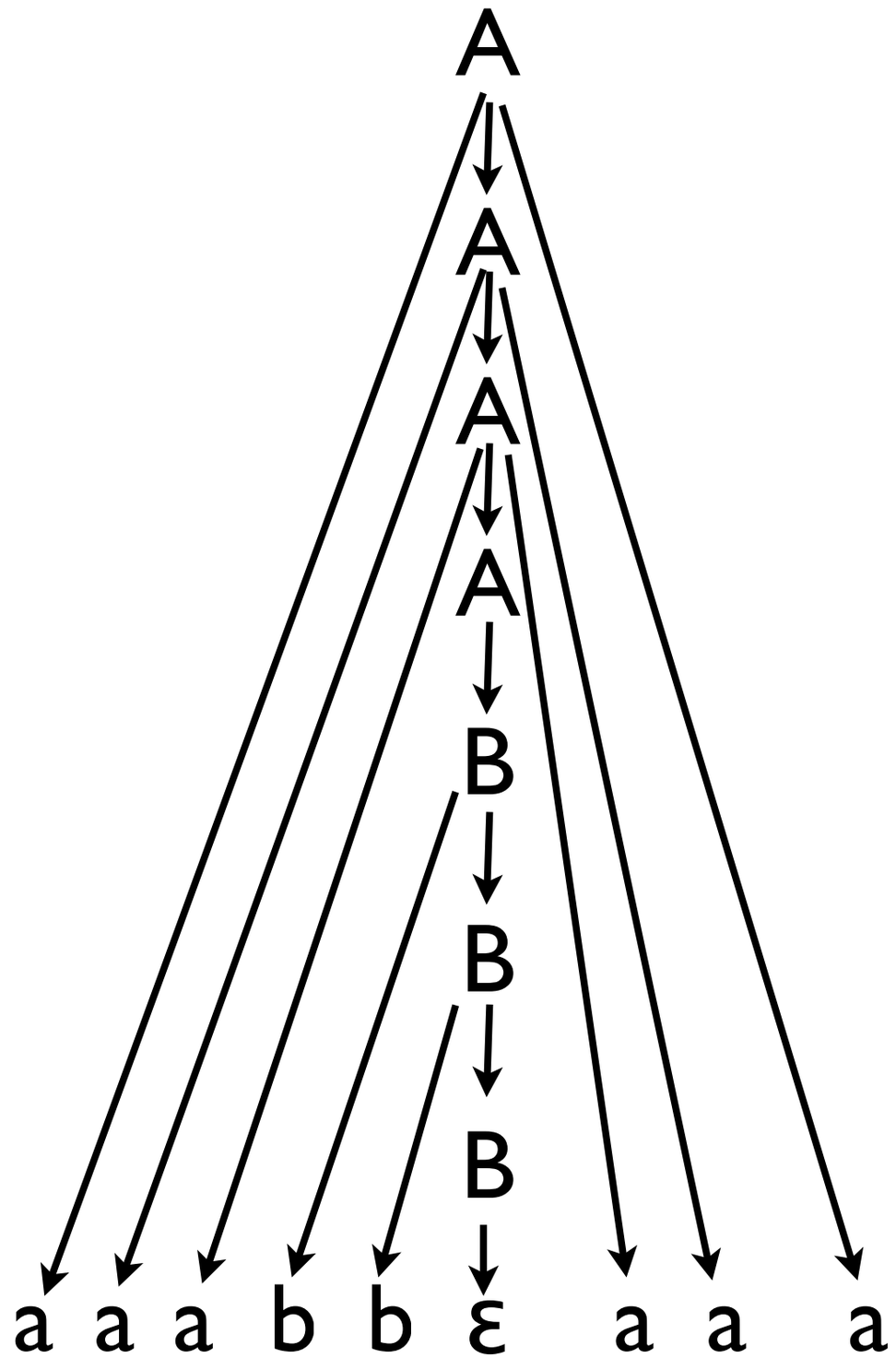
- Sample Derivation:

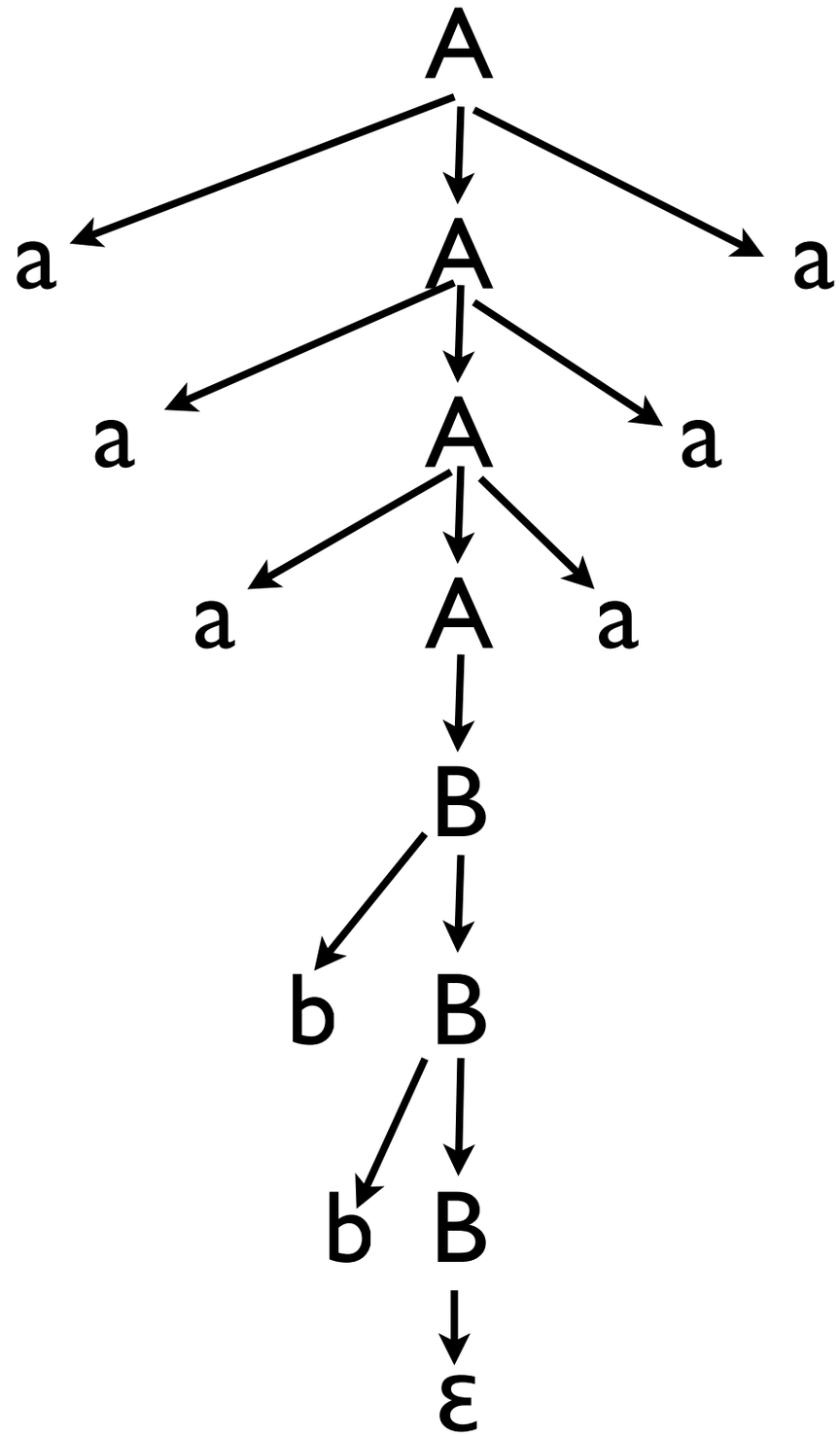
  $\underline{A} \Rightarrow a\underline{A}a \Rightarrow aa\underline{A}aa \Rightarrow aaa\underline{A}aaa \Rightarrow aaa\underline{B}aaa$

  $\Rightarrow aaab\underline{B}aaa \Rightarrow aaabb\underline{B}aaa \Rightarrow aaabbaaa$

- Language?

Derivations in Tree Form





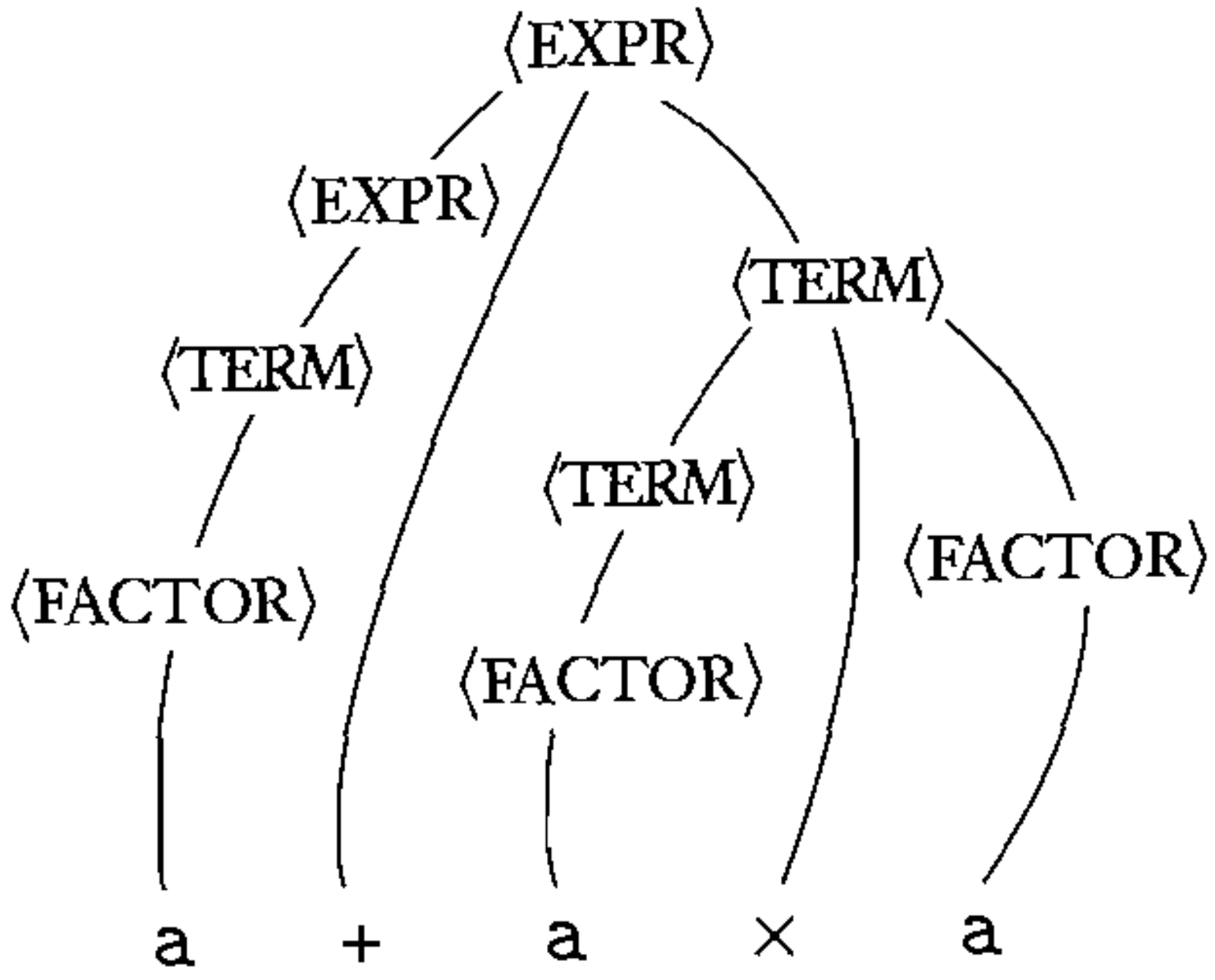
Arithmetic expressions in a programming language

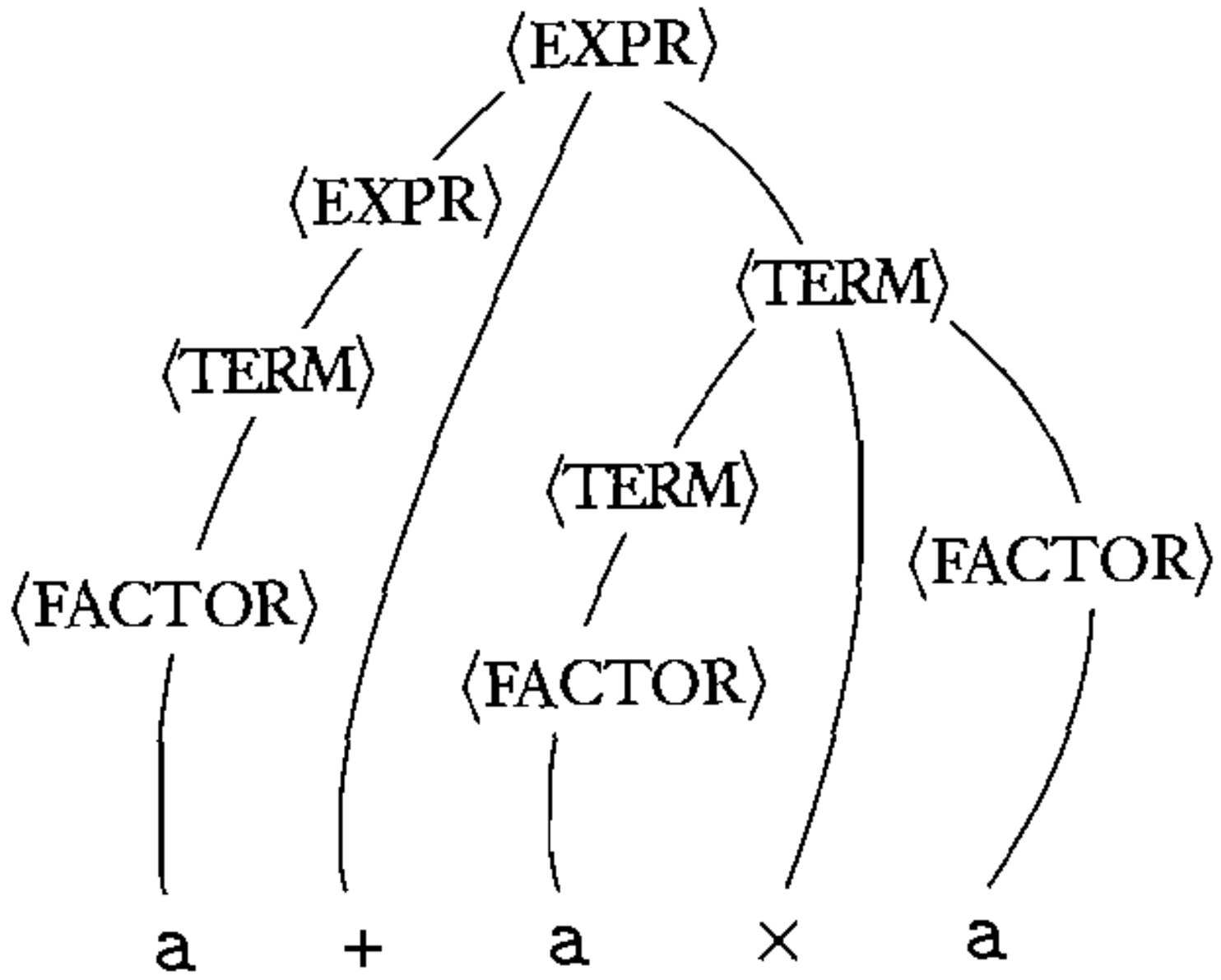
Consider grammar $G_4 = (V, \Sigma, R, \langle \text{EXPR} \rangle)$.

V is $\{\langle \text{EXPR} \rangle, \langle \text{TERM} \rangle, \langle \text{FACTOR} \rangle\}$ and Σ is $\{\texttt{a}, \texttt{+}, \texttt{x}, \texttt{(}, \texttt{)}\}$. The rules are

$$\langle \text{EXPR} \rangle \rightarrow \langle \text{EXPR} \rangle \texttt{+} \langle \text{TERM} \rangle \mid \langle \text{TERM} \rangle$$
$$\langle \text{TERM} \rangle \rightarrow \langle \text{TERM} \rangle \texttt{x} \langle \text{FACTOR} \rangle \mid \langle \text{FACTOR} \rangle$$
$$\langle \text{FACTOR} \rangle \rightarrow \texttt{(} \langle \text{EXPR} \rangle \texttt{)} \mid \texttt{a}$$

- Derive: a + a × a


⟨EXPR⟩
⟨EXPR⟩
⟨TERM⟩
⟨TERM⟩
⟨TERM⟩
⟨FACTOR⟩
⟨FACTOR⟩
⟨FACTOR⟩
a
+
a
×
a



Notice how the grammar gives the meaning a + (a×a)

Grammars in real computing

- CFG's are universally used to describe the syntax of programming languages
  - Perfectly suited to describing **recursive** syntax of expressions and statements
  - Tools like compilers must **parse** programs; parsers can be generated automatically from CFG's
  - Real languages usually have a few non-CF bits
- CFG's are also used in XML DTD's

Formal definition of CFG

- A Context-free grammar is a 4-tuple (V, Σ, R, S) where

1. V is a finite set called the **variables** (non-terminals)
2. Σ is a finite set (disjoint from V) called the **terminals**,
3. R is a finite set of **rules**, where each rule maps a variable to a string $s \in (V \cup \Sigma)^*$
4. $S \in V$ is the start symbol

Definition of Derivation

- Let u, v and w be strings in $(V \cup \Sigma)^*$, and let $A \rightarrow w$ be a rule in R,
- then $uAv \Rightarrow uwv$ (read: uAv **yields** uwv)
- We say that $u \overset{*}{\Rightarrow} v$ (read: u **derives** v) if

  u = v or there is a sequence $u_1, u_2, \ldots u_k$, $k \geq 0$, s.t. $u \Rightarrow u_1 \Rightarrow u_2 \Rightarrow \ldots \Rightarrow u_k \Rightarrow v$

- The **language** of the grammar is

  $\{w \in \Sigma^* \mid S \overset{*}{\Rightarrow} w\}$

Derivations ⇔ Parse Trees

- Each derivation can be viewed as a **parse tree** with variables at the internal nodes and terminals at the leaves
  - Start symbol is at the root
  - Each yield step $uAv \Rightarrow uwv$ where $w=w_1w_2...w_n$ corresponds to a node labeled A with children $w_1,w_2,...,w_n$.
  - The final result in Σ^* can be seen by reading the leaves left-to-right

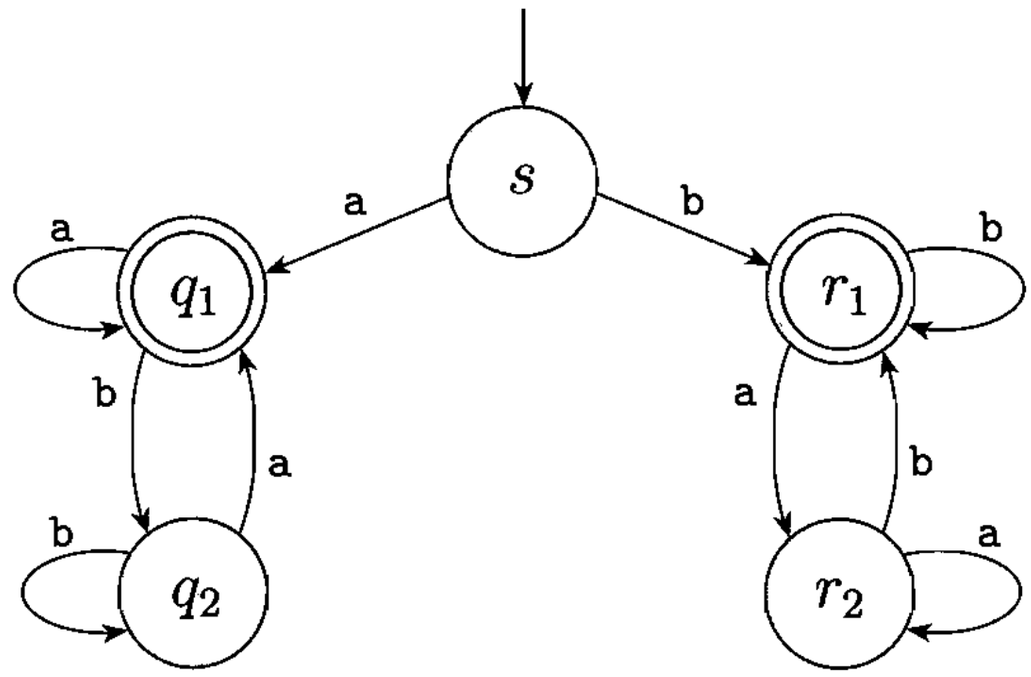
Simple CFG examples

- Find grammars for:
  - L = {w ∈ {a,b}* l w begins and ends with the same symbol}
  - L = {w ∈ {a,b}* l w contains an odd number of a's}
  - $\mathscr{L}\,[(\varepsilon + 1)(01)^*(\varepsilon + 0)]$
- Draw example derivations

All regular languages have context free grammars

- Proof:
  - Regular language is accepted by an NFA.
  - We can generate a regular grammar from the NFA (Lecture 6, Hein Alg. 11.11)
  - Any regular grammar is also a CFG. (Immediate from the definition of the grammars).

Example



- $S \rightarrow aQ_1$ $S \rightarrow bR_1$
- $Q_1 \rightarrow aQ_1$ $Q_1 \rightarrow bQ_2$
- $Q_2 \rightarrow aQ_1$ $Q_2 \rightarrow bQ_2$
- $R_1 \rightarrow aR_2$ $R_1 \rightarrow bR_1$
- $R_2 \rightarrow aR_2$ $R_2 \rightarrow bR_1$
- $Q_1 \rightarrow \varepsilon$ $R_1 \rightarrow \varepsilon$
  - Resulting grammar may be quite different from one we designed by hand.

Some CFG's generate non-regular languages

- Find grammars for the following languages
  - $L = \{a^n b^m a^n \mid a,b \geq 0\}$
  - $L = \{w \in \{a,b\}^* \mid w$ contains equal numbers of a's and b's$\}$
  - $L = \{ww^R \mid w \in \{a,b\}^*\}$
- Draw example derivations

Ambiguity

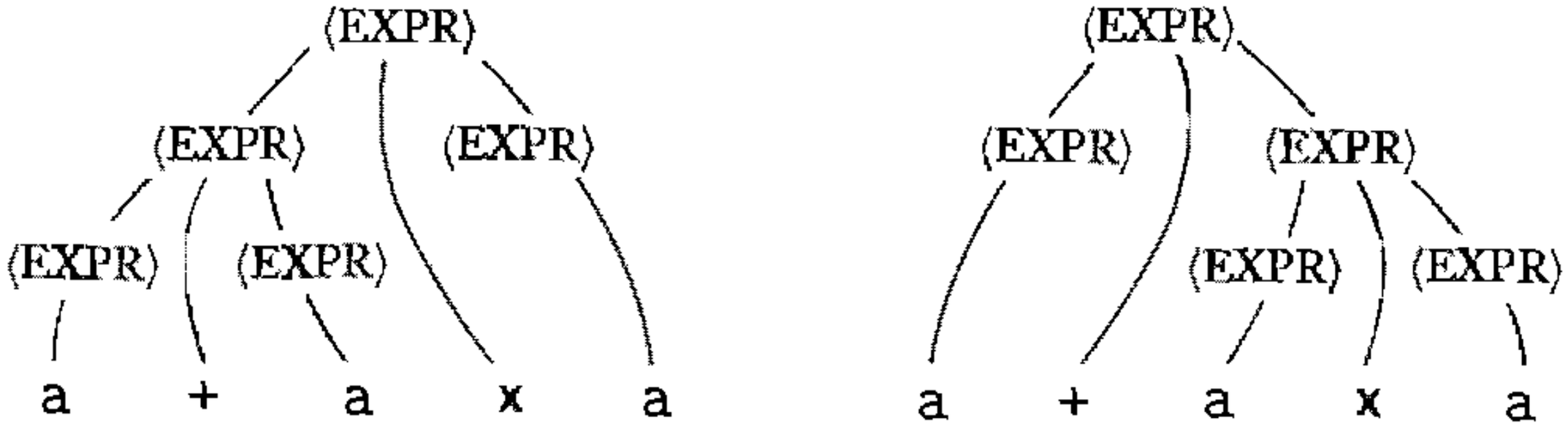
- A grammar in which the same string can be given more than one parse *tree* is **ambiguous**.

- Example: another grammar for arithmetic expressions

‹EXPR› → ‹EXPR› + ‹EXPR› |
‹EXPR› × ‹EXPR› |
(‹EXPR›)
| a

- Derive: a + a × a

- This grammar is ambiguous: there are two *different* parse trees for a + a × a



- Ambiguity is a bad thing if we're interested in the structure of the parse
  - Ambiguity doesn't matter if we're interested only in *defining* a language.

Leftmost Derivations

- In general, in any step of a derivation, there might be several variables that can be reduced by rules of the grammar.
- In a leftmost derivation, we choose to always reduce the leftmost variable.

- Example: given grammar $S \rightarrow aSb \mid SS \mid \varepsilon$
  - A left-most derivation:

    $\underline{S} \Rightarrow a\underline{S}b \Rightarrow a\underline{S}Sb \Rightarrow aa\underline{S}bSb \Rightarrow aab\underline{S}b \Rightarrow aabb$

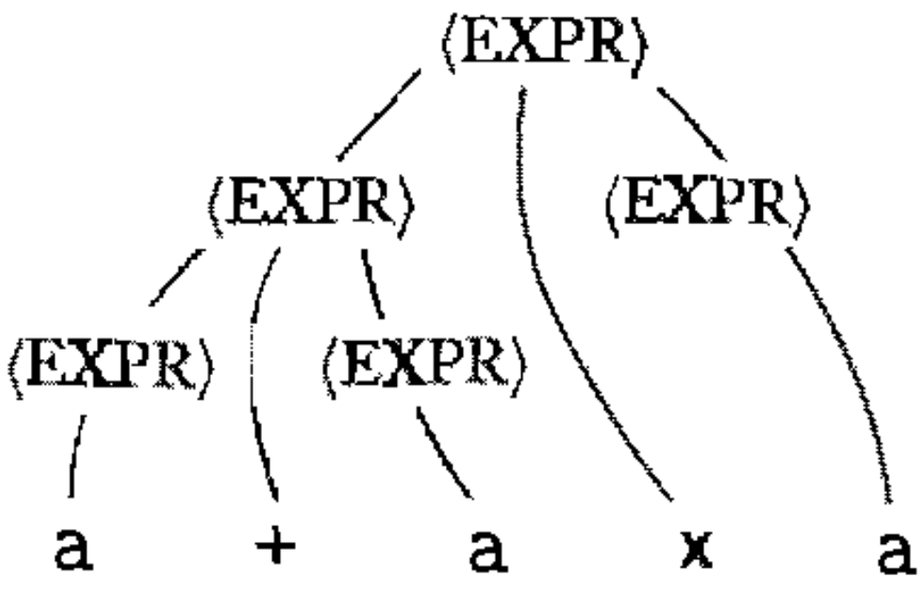
  - A non-left-most derivation:

    $\underline{S} \Rightarrow a\underline{S}b \Rightarrow aS\underline{S}b \Rightarrow a\underline{S}b \Rightarrow aa\underline{S}bb \Rightarrow aabb$

Ambiguity *via* left-most derivations

- Every parse tree corresponds to a unique left-most derivation
- So if a grammar has more than one left-most derivation for some string, the grammar is ambiguous
  - Note: merely having two derivations (not necessarily left-most) for one string is **not** enough to show ambiguity

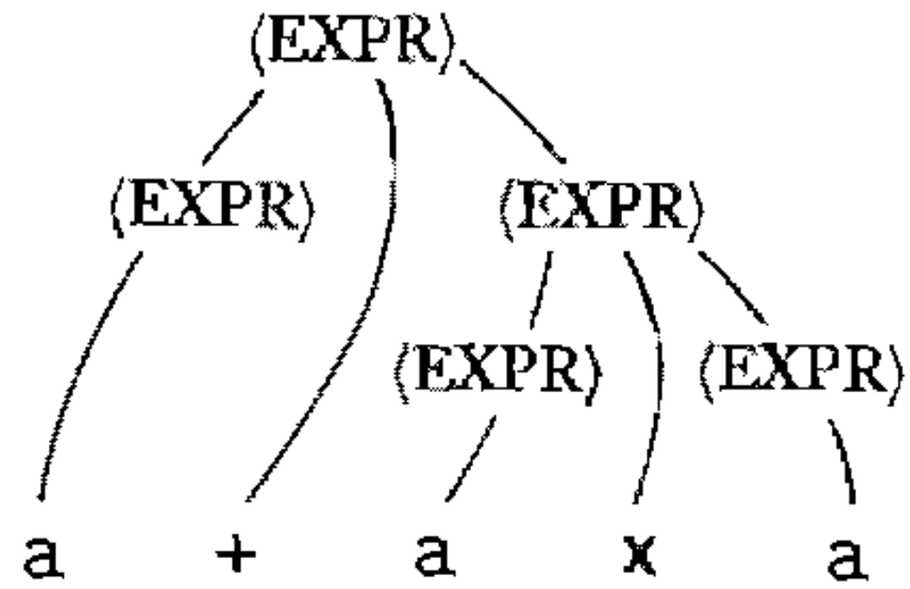
Ambiguity



$\underline{E} \Rightarrow \underline{E} \times E \Rightarrow \underline{E} + E \times E$

$\Rightarrow a + \underline{E} \times E \Rightarrow a + a \times \underline{E}$

$\Rightarrow a + a \times a$



$\underline{E} \Rightarrow \underline{E} + E \Rightarrow a + \underline{E} \Rightarrow$

$a + \underline{E} \times E \Rightarrow a + a \times \underline{E} \Rightarrow$

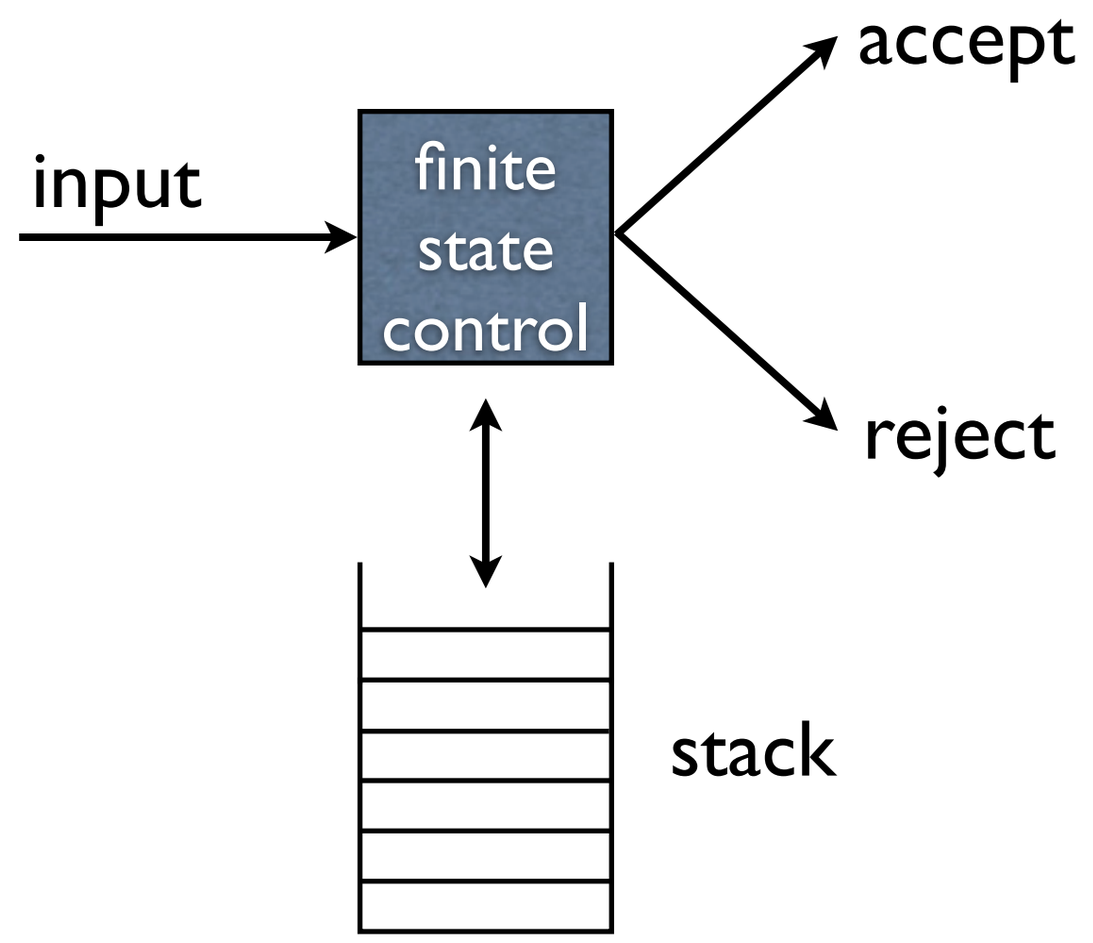
$a + a \times a$

Context-free languages

- Closed under $\cup$, $*$ and $\cdot$, and under $\cap$ with a regular language
  - How do we prove these properties?
- *Not* closed under intersection, complement or difference
- Recognizable by pushdown automata
  - A pushdown automaton is a generalization of a finite-state automaton

Pushdown Automata

- Why can't a FSA recognize $a^n b^n$?
  - "storage" is finite
- How can we fix the problem?
  - add unbounded storage
- What's the simplest kind of unbounded storage
  - a pushdown stack



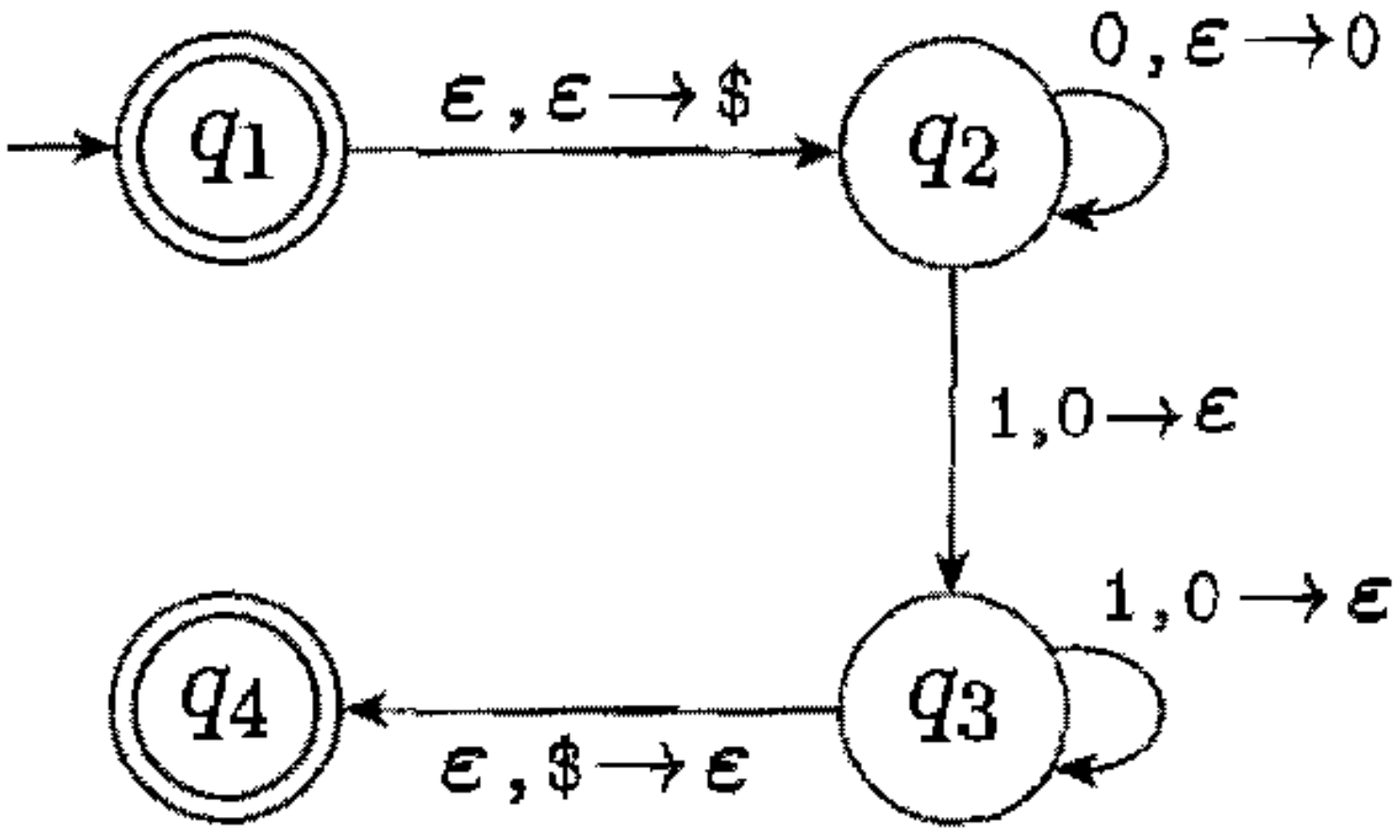
History

- PDAs independently invented by Oettinger [1961] and Schutzenberger [1963]
- Equivalence between PDAs and CFG known to Chomsky in 1961; first published by Evey [1963].

Executing a PDA

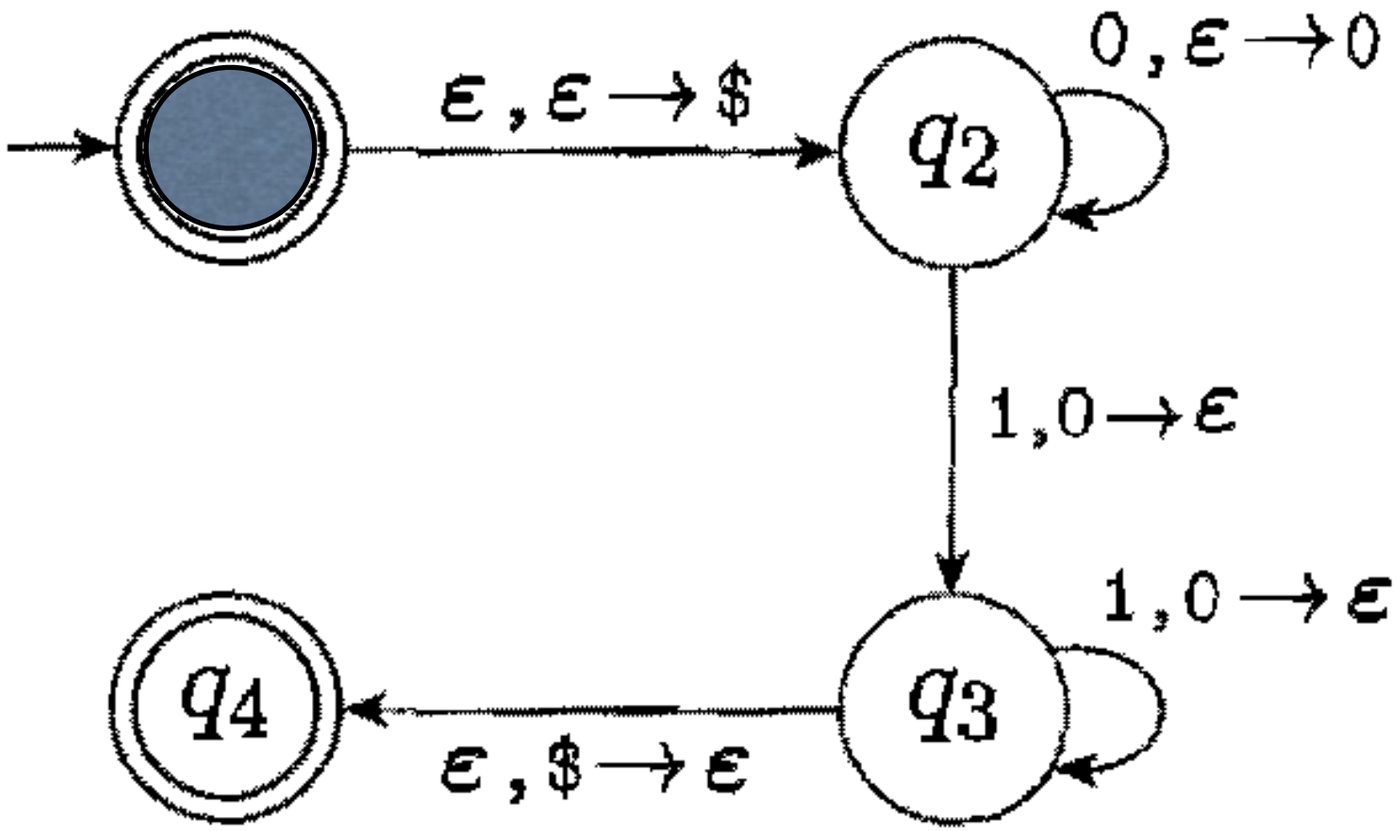
- The behavior of a PDA at any point depends on ⟨state,stack,unread input⟩
- PDA begins in start state with stated symbol on the stack
- On each step it optionally reads an input character, pops a stack symbol, and non-deterministically chooses a new state and optionally pushes one or more stack symbols
- PDA accepts if it is in a final state and there is no more input.

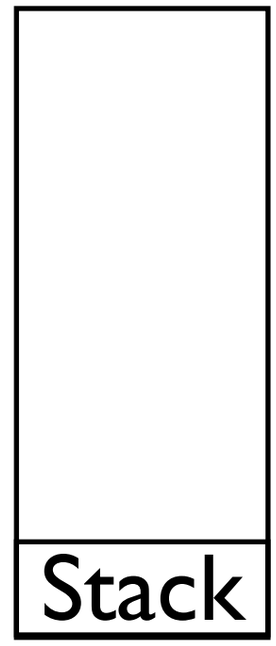
Example PDA


q_1
$\varepsilon, \varepsilon \to \$$
q_2
$0, \varepsilon \to 0$
$1, 0 \to \varepsilon$
q_3
$1, 0 \to \varepsilon$
q_4
$\varepsilon, \$ \to \varepsilon$

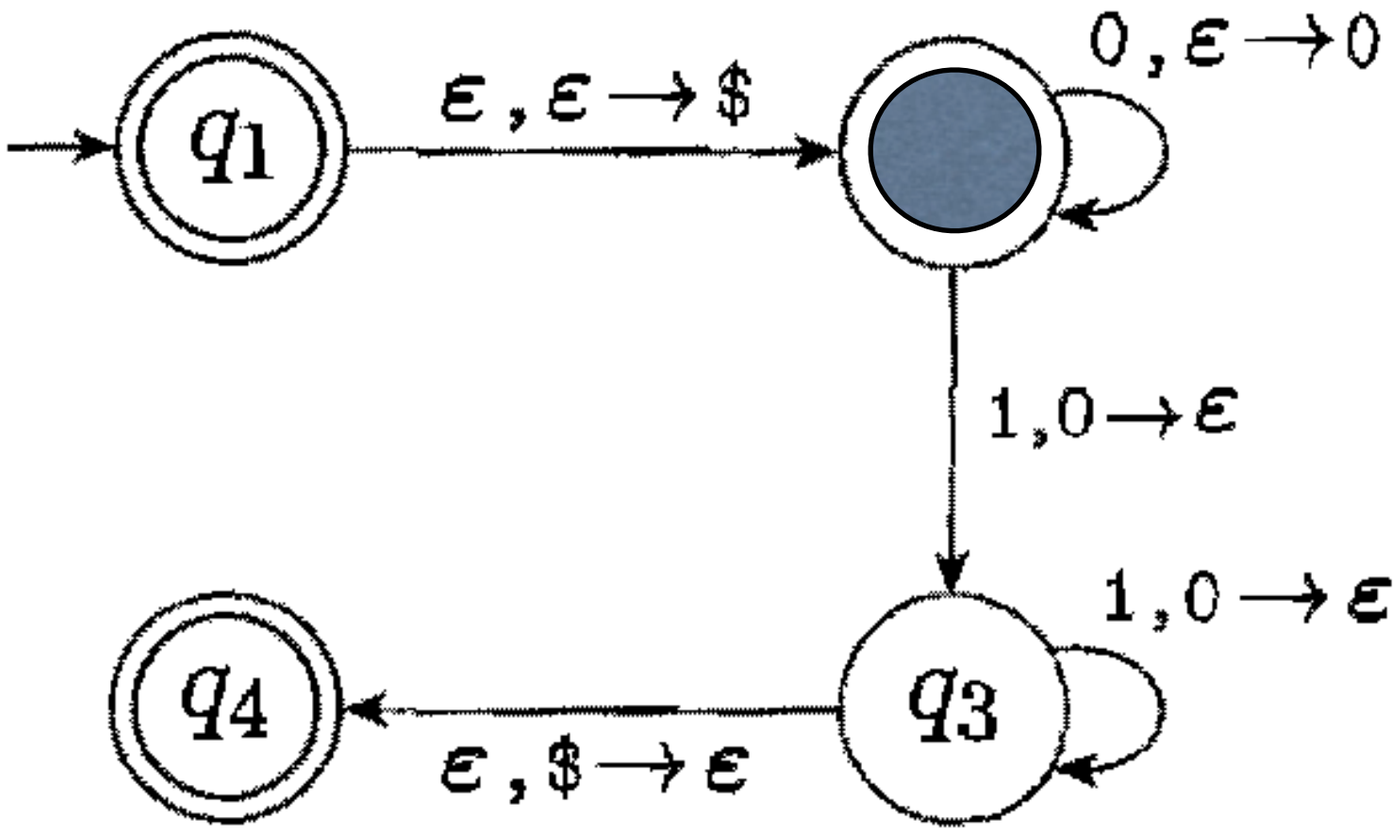
Example Execution: 0011

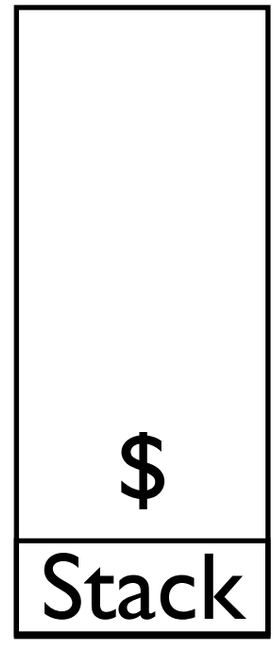
Begin in initial state
with empty stack





Example Execution: 0011





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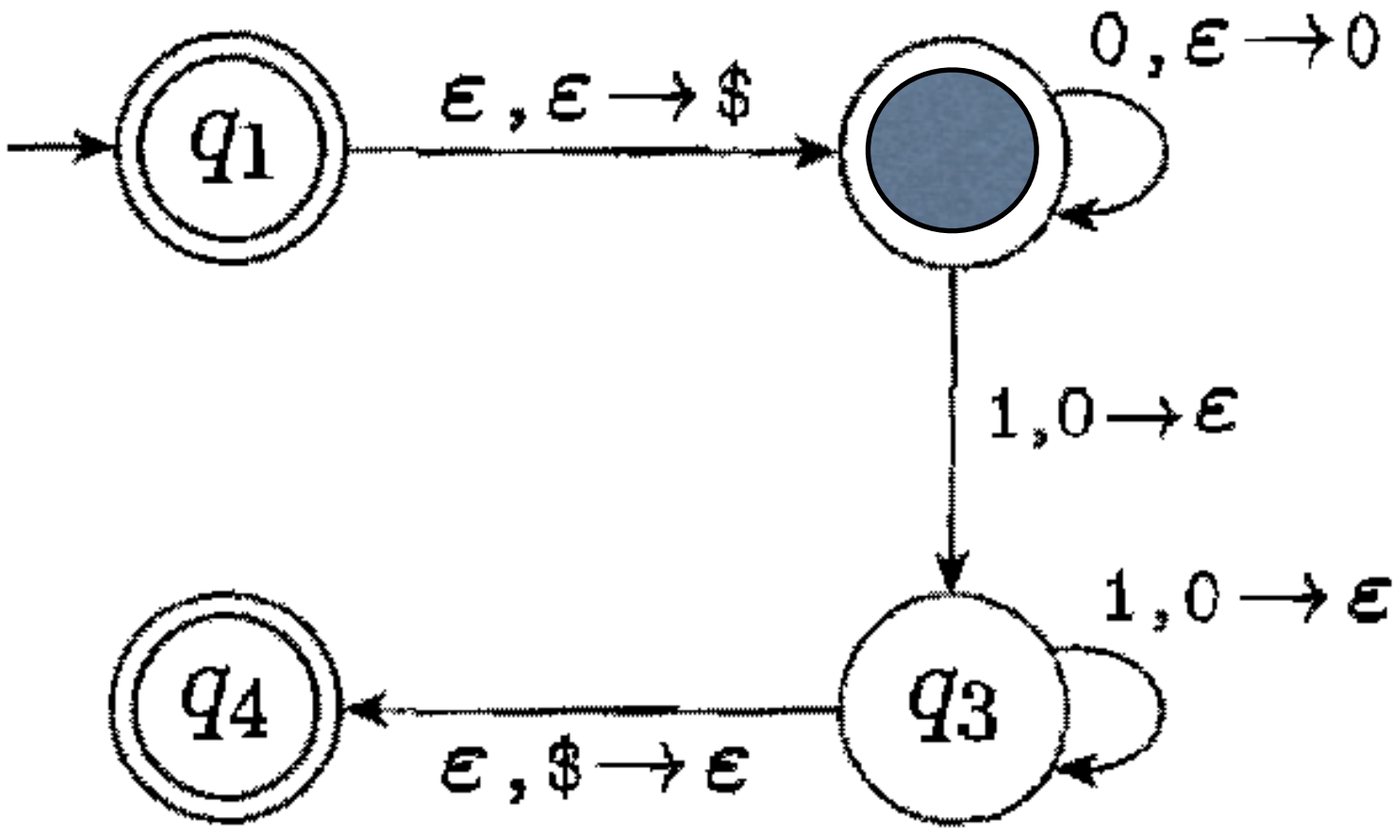

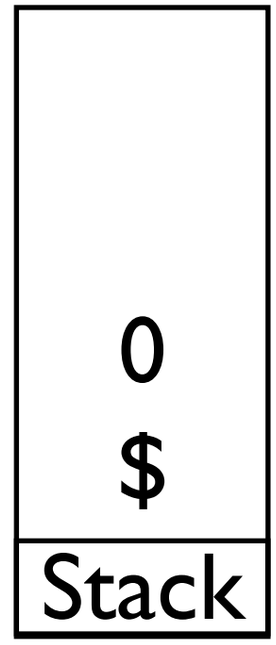

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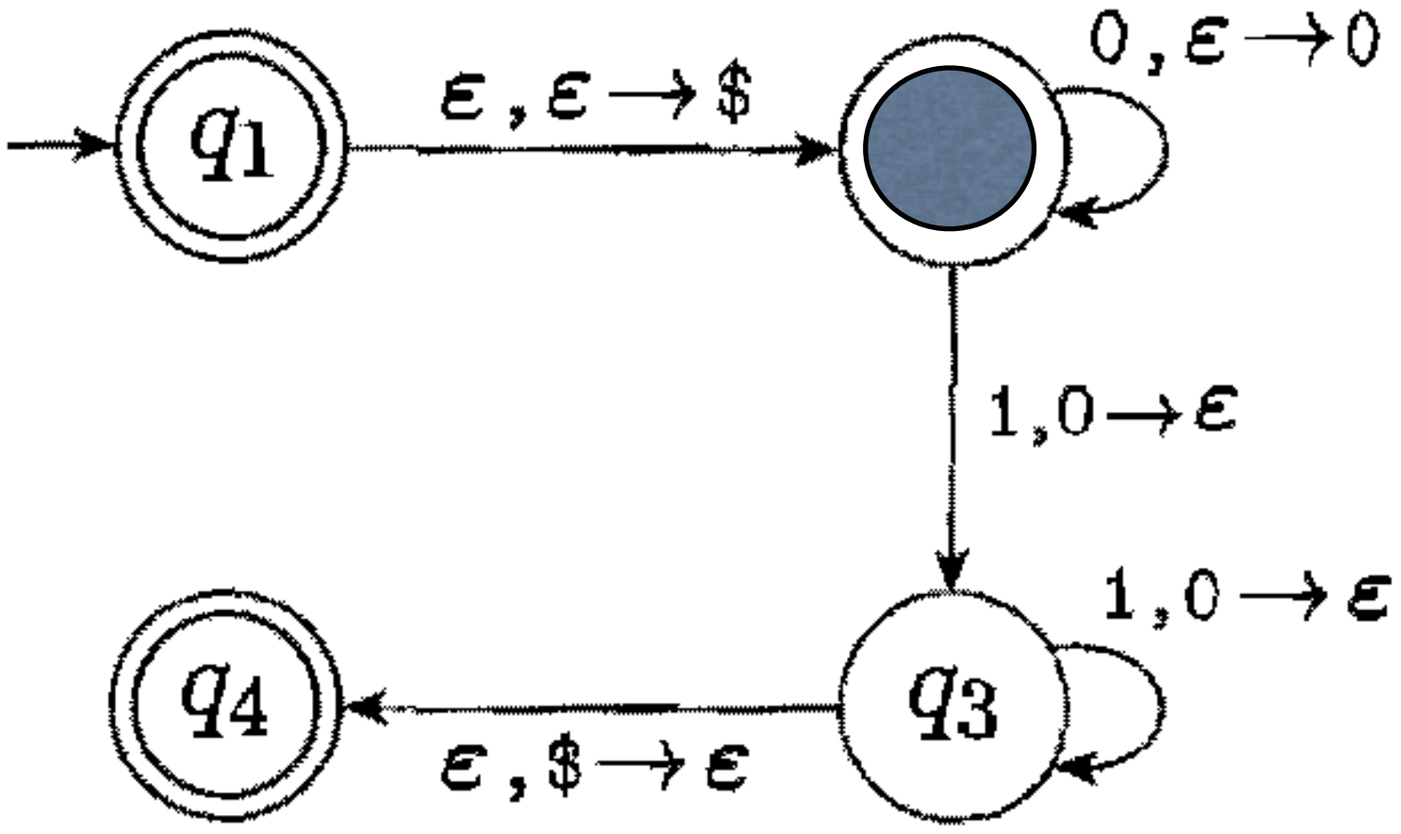

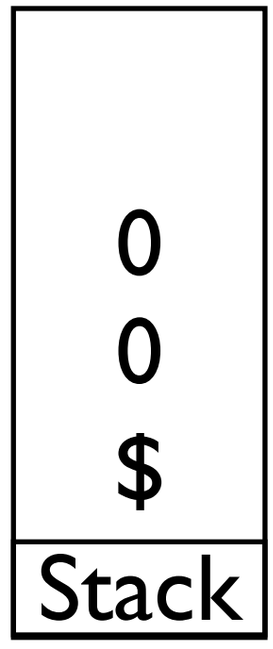

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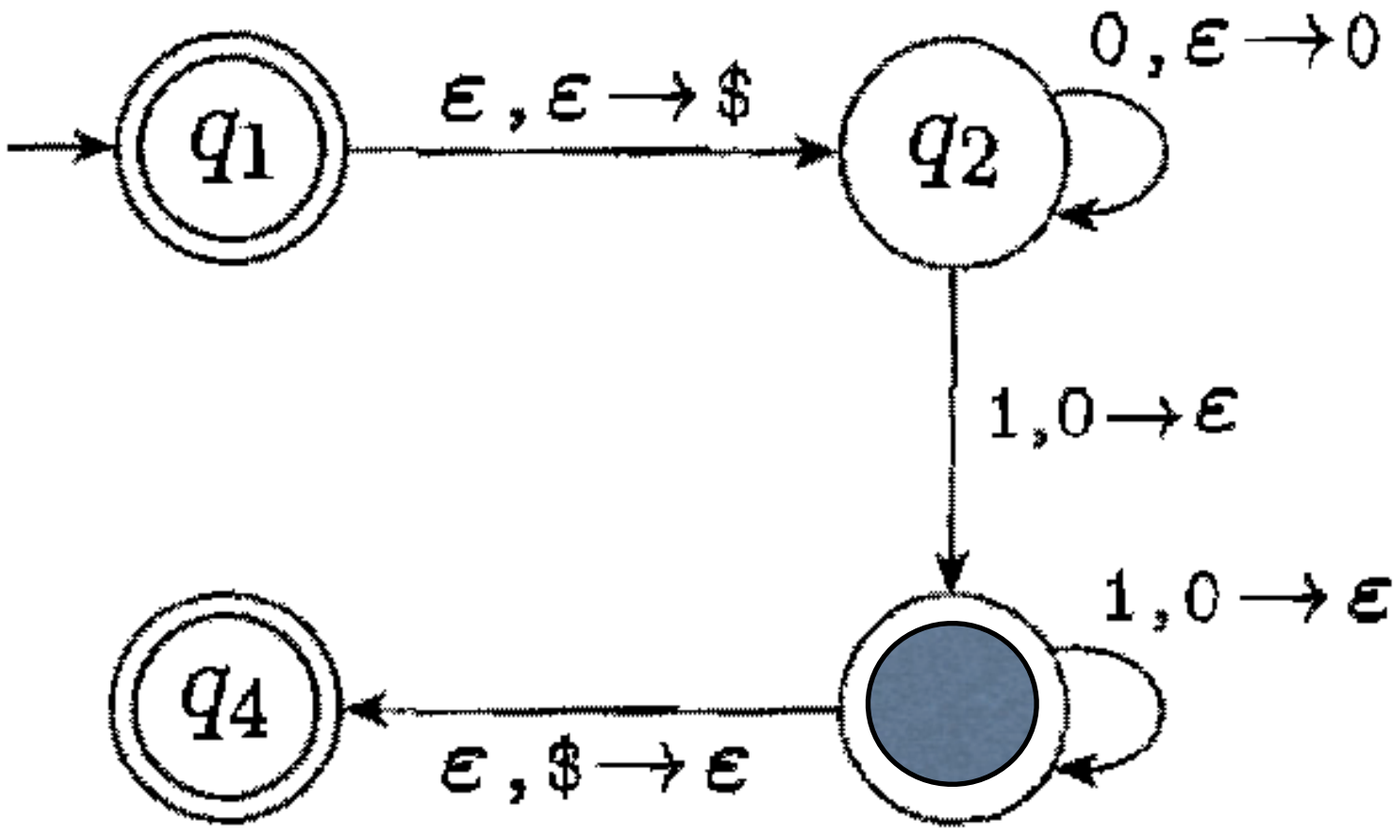

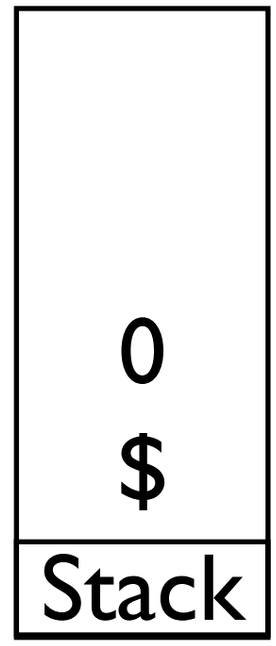

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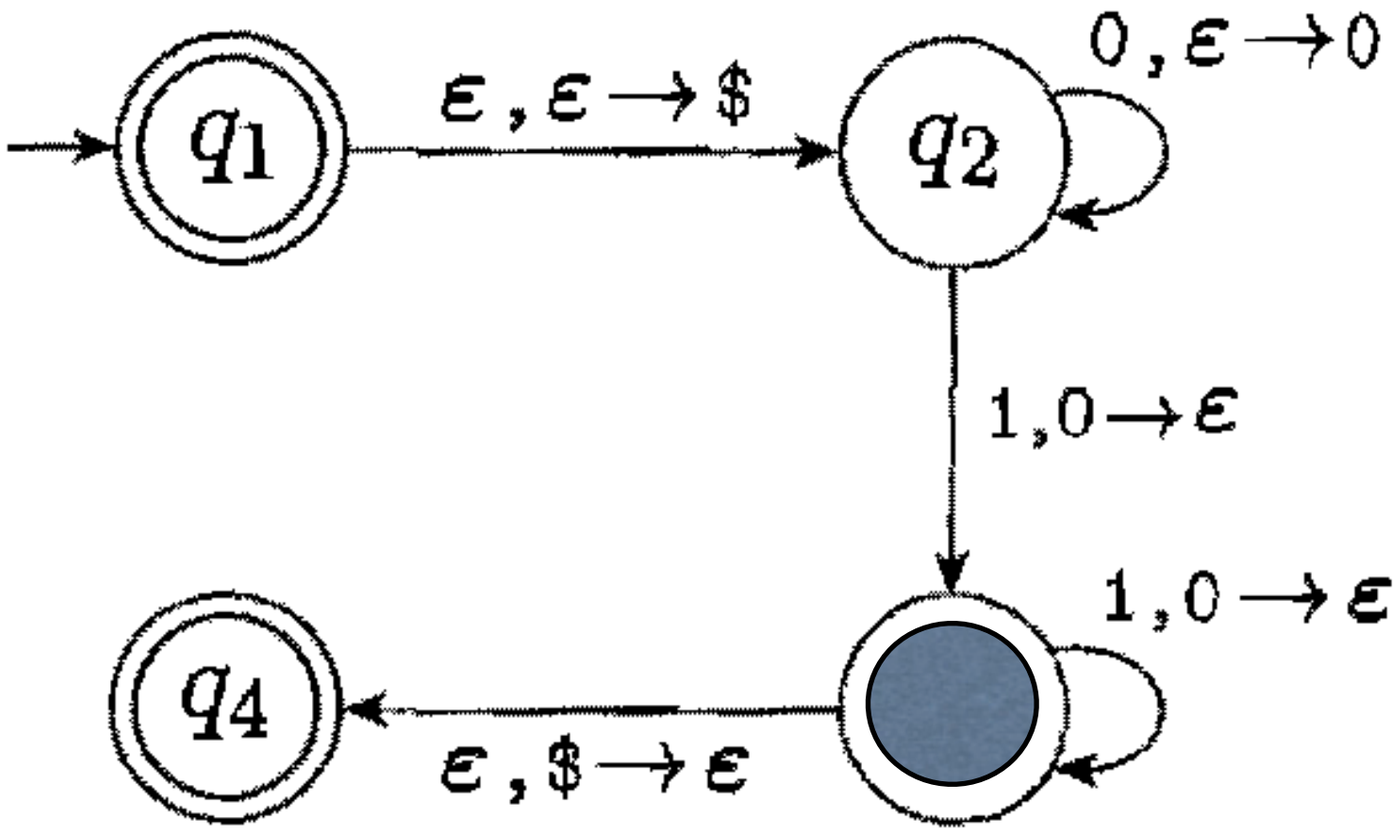

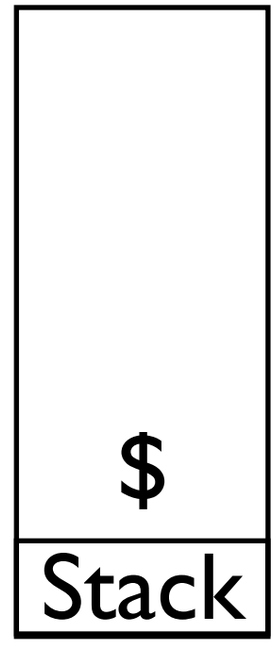

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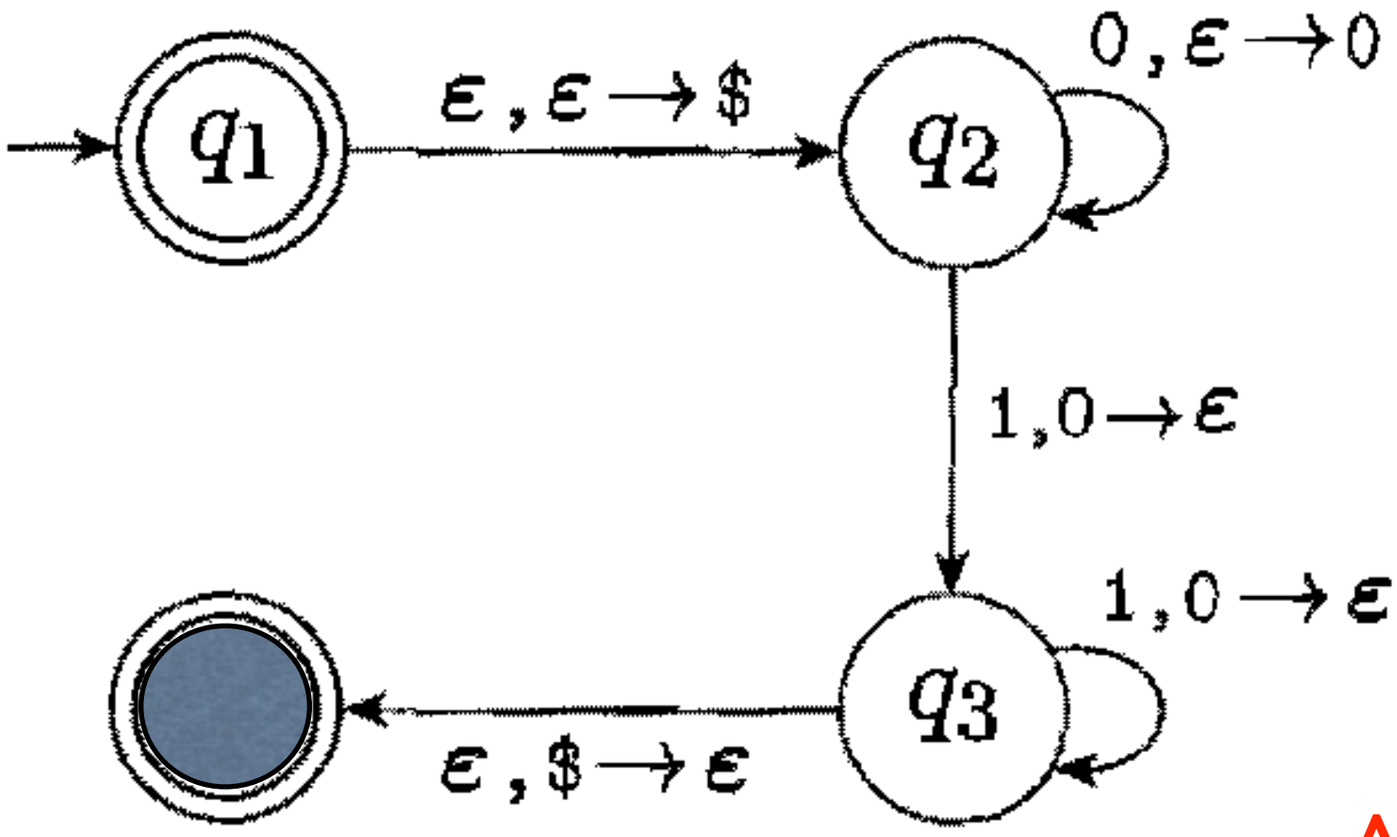
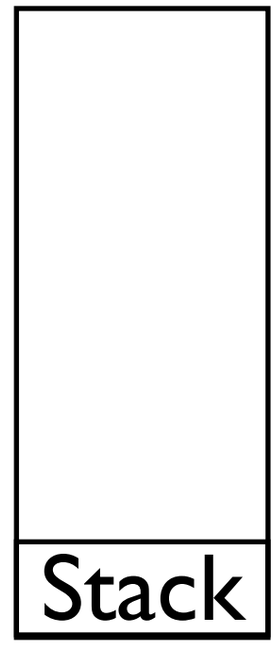




Accept!

PDA's can keep count!

- This PDA can recognize $\{0^n1^n \mid n \geq 0\}$ by
  - First, pushing a symbol on the stack for each 0
  - Then, popping a symbol off the stack for each 1
  - Accepting iff the stack is empty when the end of input is reached (and not before)
- The size of the stack is unbounded.
  - That is, no matter how big the stack grows, it is always possible to push another symbol on it.
  - So PDA's can use the stack to count arbitrarily high

Pushdown Automata (PDA)

- A pushdown automaton M is defined as a 7-tuple: $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$, where:
  - Q is a set of states, $q_0 \in Q$ is the start state
  - Σ is the input alphabet,
  - Γ is the stack alphabet, $Z_0 \in \Gamma$ is the initial stack symbol
  - $\delta : (Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon) \rightarrow \mathcal{P}\{Q \times \Gamma^*\}$ is the transition function
  - $F \subseteq Q$ is a set of final states, and
  - $X_\varepsilon = X \cup \{\varepsilon\}$, the set X augmented with ε

Executing a PDA

- The *configuration* (or PDA + Hopcroft's *instantaneous description*, ID) of a PDA consists of
  1. The PDA,
  2. The current state of the PDA,
  3. the remaining input, and
  4. The contents of its stack.

Transitions

- We defined $\delta : (Q \times A_\varepsilon \times \Gamma_\varepsilon) \rightarrow \mathcal{P}\{Q \times \Gamma^*\}$
  - The transitions $\delta(q, a, \gamma)$ are applicable iff
    - q is the current state,
    - $a = \varepsilon$, or the next character on the input tape is a , and
    - $\gamma = \varepsilon$, or the top of the stack is γ
  - If you select a transition (q', ω) , then
    - The new state is q'
    - if $\gamma \neq \varepsilon$, γ is popped off of the stack, and
    - the (possibly empty) sequence of symbols ω is pushed onto the stack

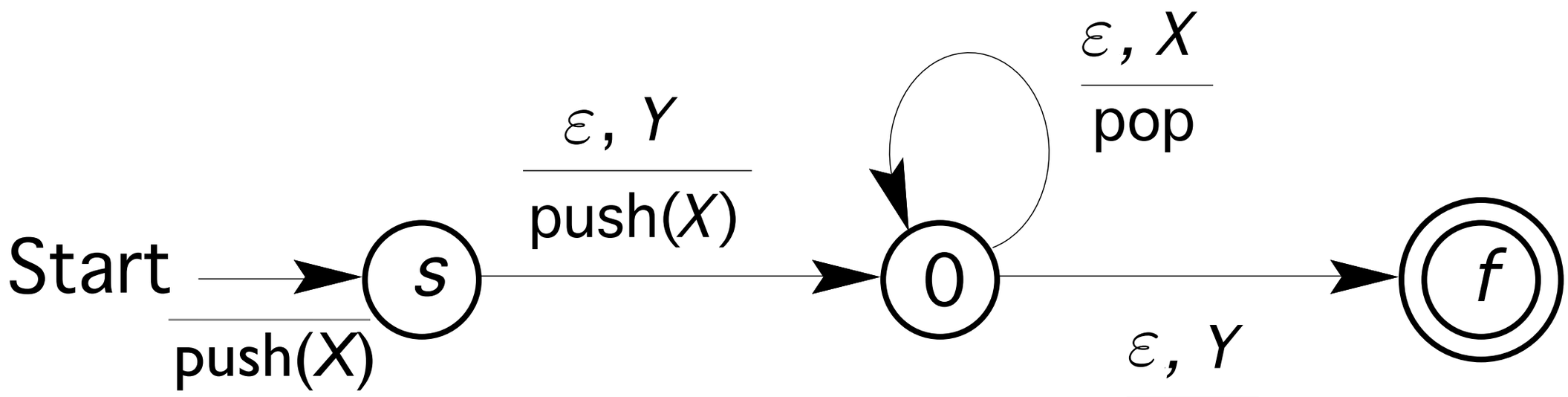
PDAs are non-deterministic

For two reasons:

- The transition function δ answers a *set*
- We have the choice of taking an ε–transition, or reading an input symbol

Trivial Example

Hein Example 12.4



- $\delta(s, \varepsilon, Y) = \{(0, XY)\}$
- $\delta(0, \varepsilon, X) = \{(0, \varepsilon)\}$
- $\delta(0, \varepsilon, Y) = \{(f, Y)\}$

Accepting a String

- There are two ways in which a PDA can *accept* a string:
  1. final state is in the set F (implies F non-empty)
  2. if F is empty (there are no final states), then the PDA accepts when the stack is empty
- The previous example accepts by …

Acceptance by Final State

A run of PDA $M = (Q, A, \Gamma, \delta, q_0, \gamma_0, F)$ is a sequence

$$(q_0, \gamma_0) \stackrel{a_0}{\to} (q_1, s_1) \stackrel{a_1}{\to} \cdots \stackrel{a_{n-1}}{\to} (q_n, s_n)$$

with $q_0, \ldots, q_n \in Q$, $s_1, \ldots, s_n \in \Gamma^*$, and $a_0, \ldots, a_{n-1} \in A$ such that:

for all $i \in [0 \,..\, n-1]$. $(q_{i+1}, \gamma_{i+1}) \in \delta(q_i, a_i, \gamma_i)$ and

$s_i = \gamma_i t_i$ and $s_{i+1} = \gamma_{i+1} t_i$ for some $t_i \in \Gamma^*$, and

$w = a_0 a_1 a_2 \ldots a_{n-1}$ is the input.

The run accepts w if $q_n \in F$.

The language of M, $L(M)$ is given by

$$L(M) = \{w \in A^* | \; w \text{ is accepted by some run of } M\}$$

Acceptance by Empty Stack

- There is an alternative definition of acceptance:
  - If the set of final states is empty, then
  - the PDA accepts an input string if it can read all of the characters of the string and end up in a state in which the stack is empty.

Acceptance by Empty Stack

A run of PDA $M = (Q, A, \Gamma, \delta, q_0, \gamma_0, \emptyset)$ is a sequence

$$(q_0, \gamma_0) \stackrel{a_0}{\to} (q_1, s_1) \stackrel{a_1}{\to} \cdots \stackrel{a_{n-1}}{\to} (q_n, s_n)$$

with $q_0, \ldots, q_n \in Q$, $s_1, \ldots, s_n \in \Gamma^*$, and $a_0, \ldots, a_{n-1} \in A$ such that:

for all $i \in [0 \,..\, n-1]$. $(q_{i+1}, \gamma_{i+1}) \in \delta(q_i, a_i, \gamma_i)$ and

$s_i = \gamma_i t_i$ and $s_{i+1} = \gamma_{i+1} t_i$ for some $t_i \in \Gamma^*$, and

$w = a_0 a_1 a_2 \ldots a_{n-1}$ is the input.

The run accepts w if $s_n = \varepsilon$.

The language of M, $L(M)$ is given by

$$L(M) = \{w \in A^* | \; w \text{ is accepted by some run of } M\}$$

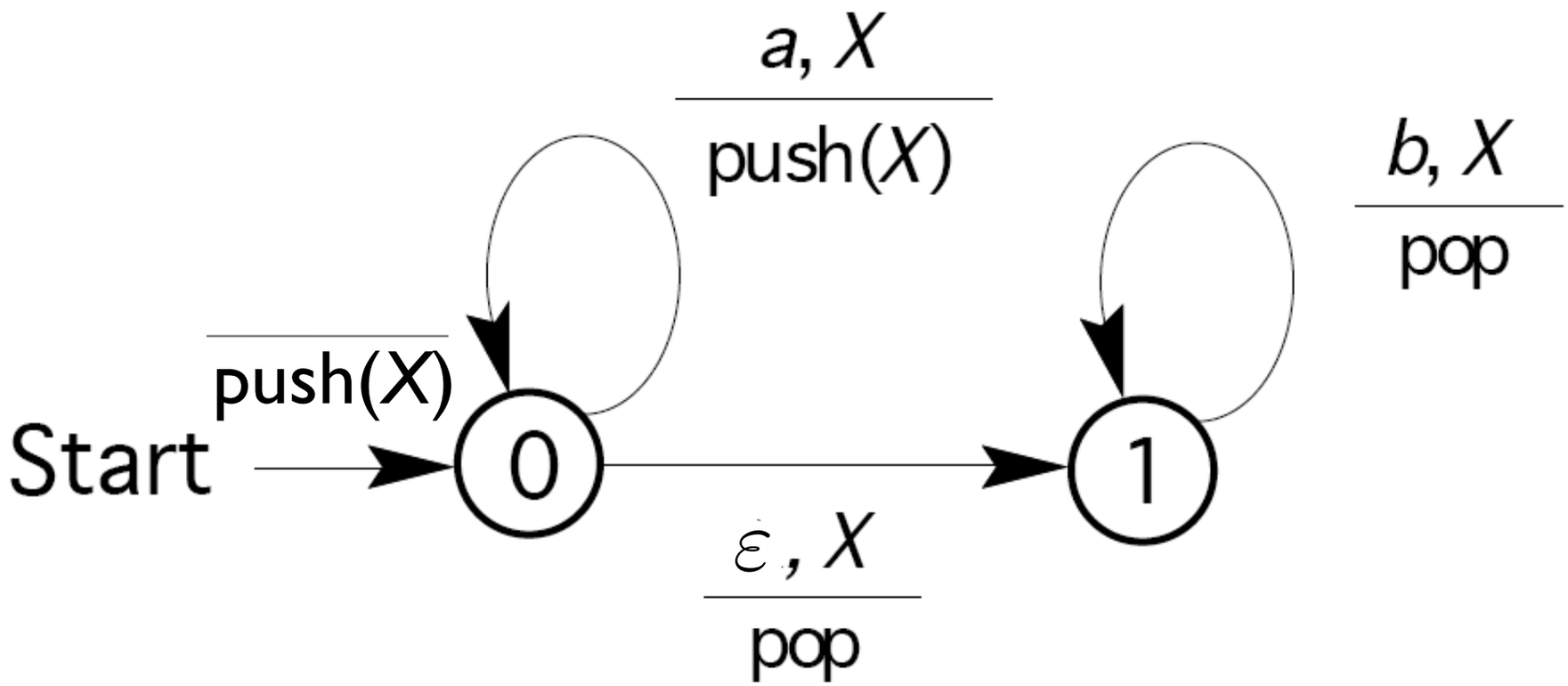
It doesn't matter!

- PDAs that accept by empty stack and PDAs that accept by final state have *equivalent power*

- What does this mean?
  - That given either one, we can build the other
  - Consequently, the class of languages that they accept are the same

Example

(Hein 12.2)

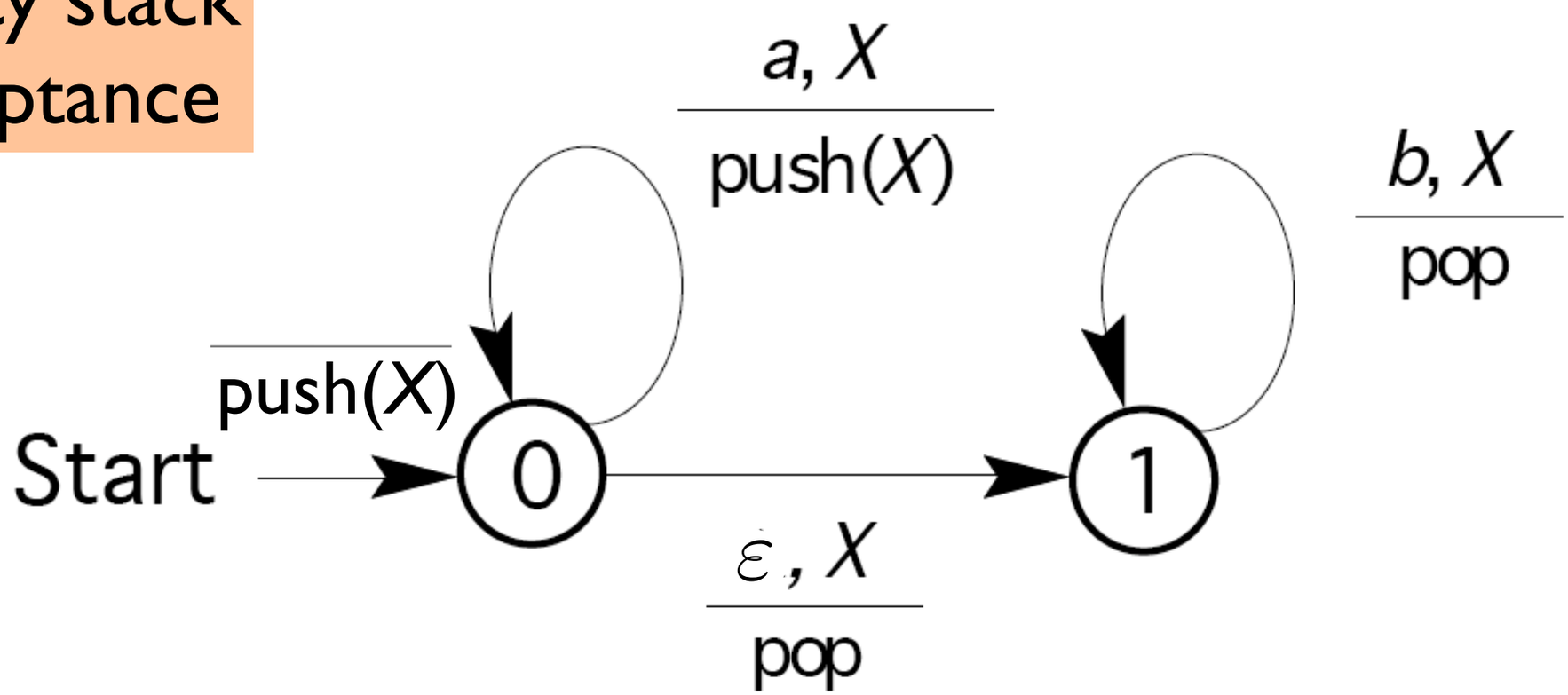


$\frac{a, X}{push(X)}$ means that (in state 0), with input a and with X on the top of the stack, we can transition into state 0, and put an additional X onto the stack.

Example

(Hein 12.2)

Empty stack acceptance



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(Hein 12.2)

M = ({0,1}, {a, b},
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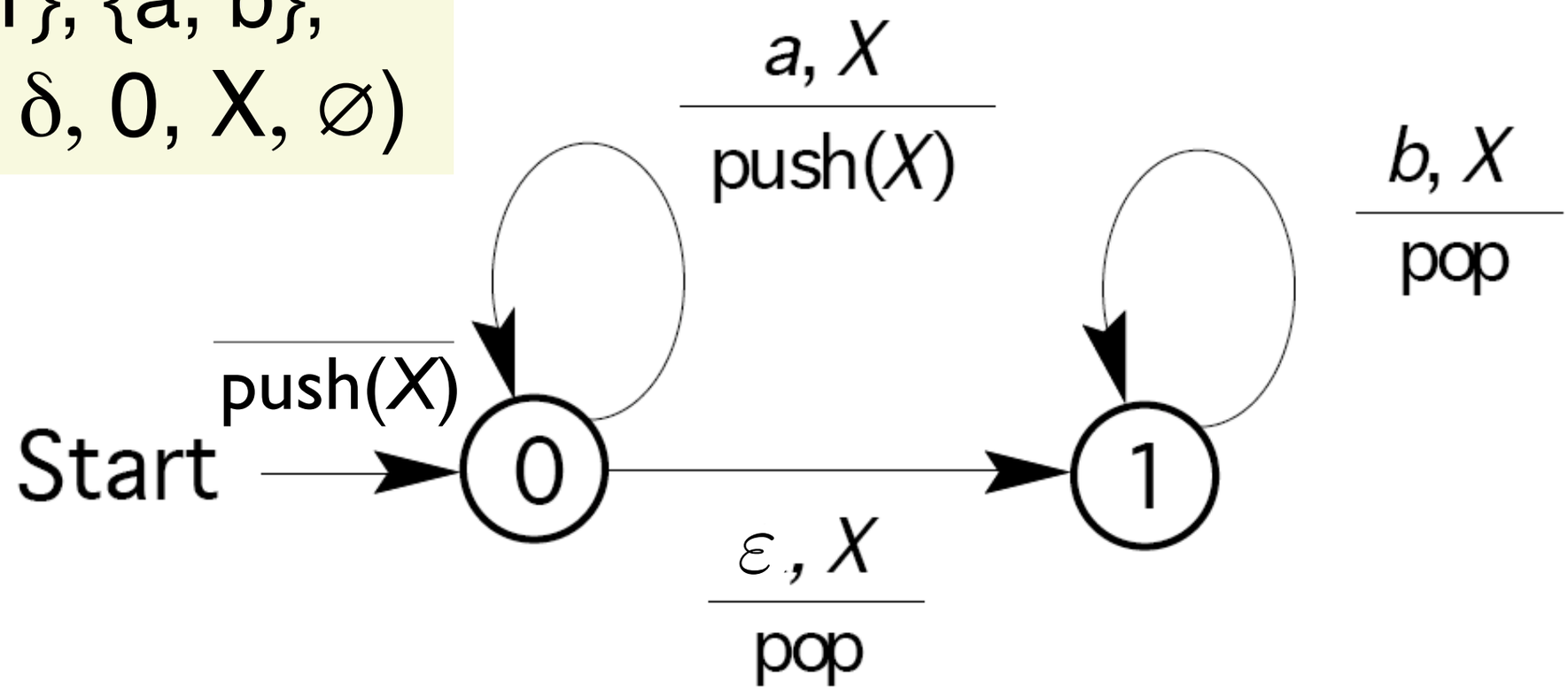

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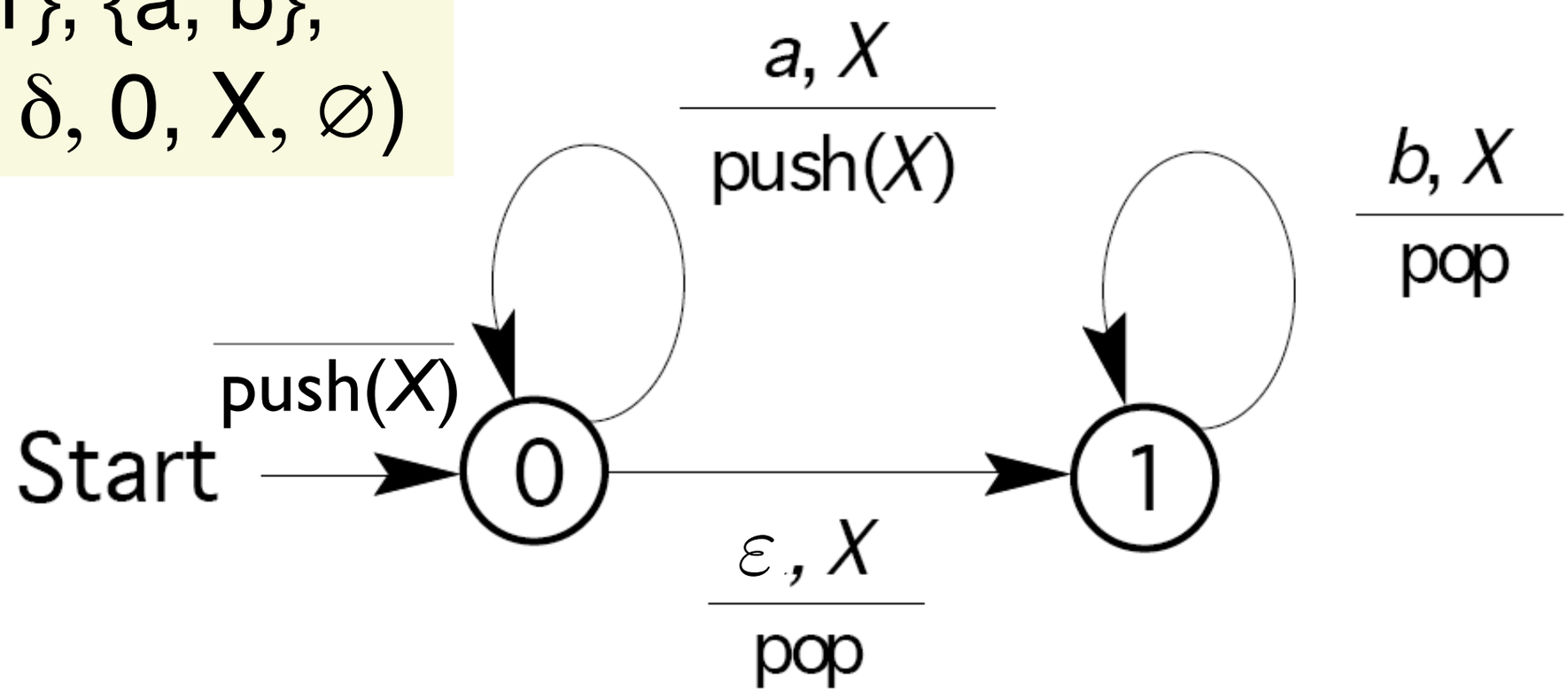
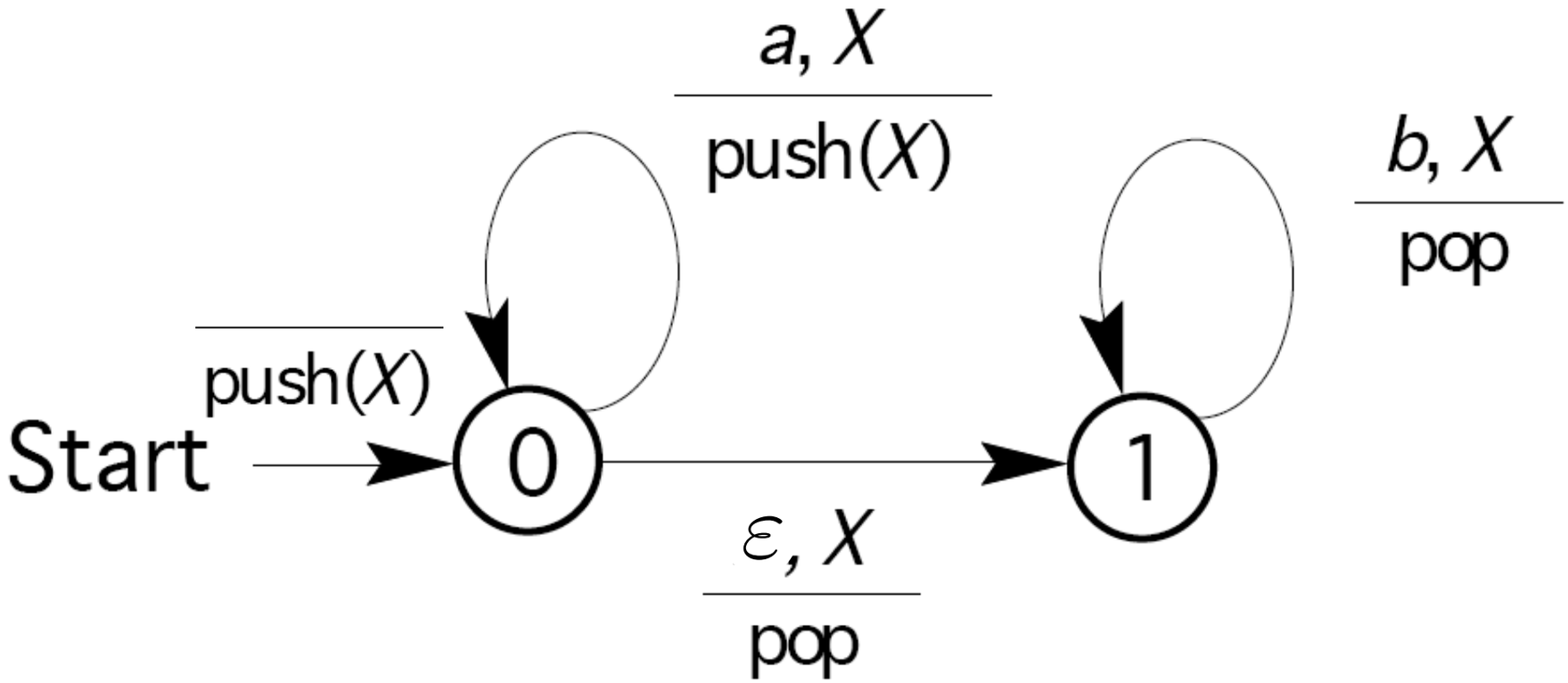


- δ(0, a, X) = {(0, XX)}
- δ(0, ε, X) = {(1, ε)}
- δ(1, b, X) = {(1, ε)}

Transformation to a Final State PDA



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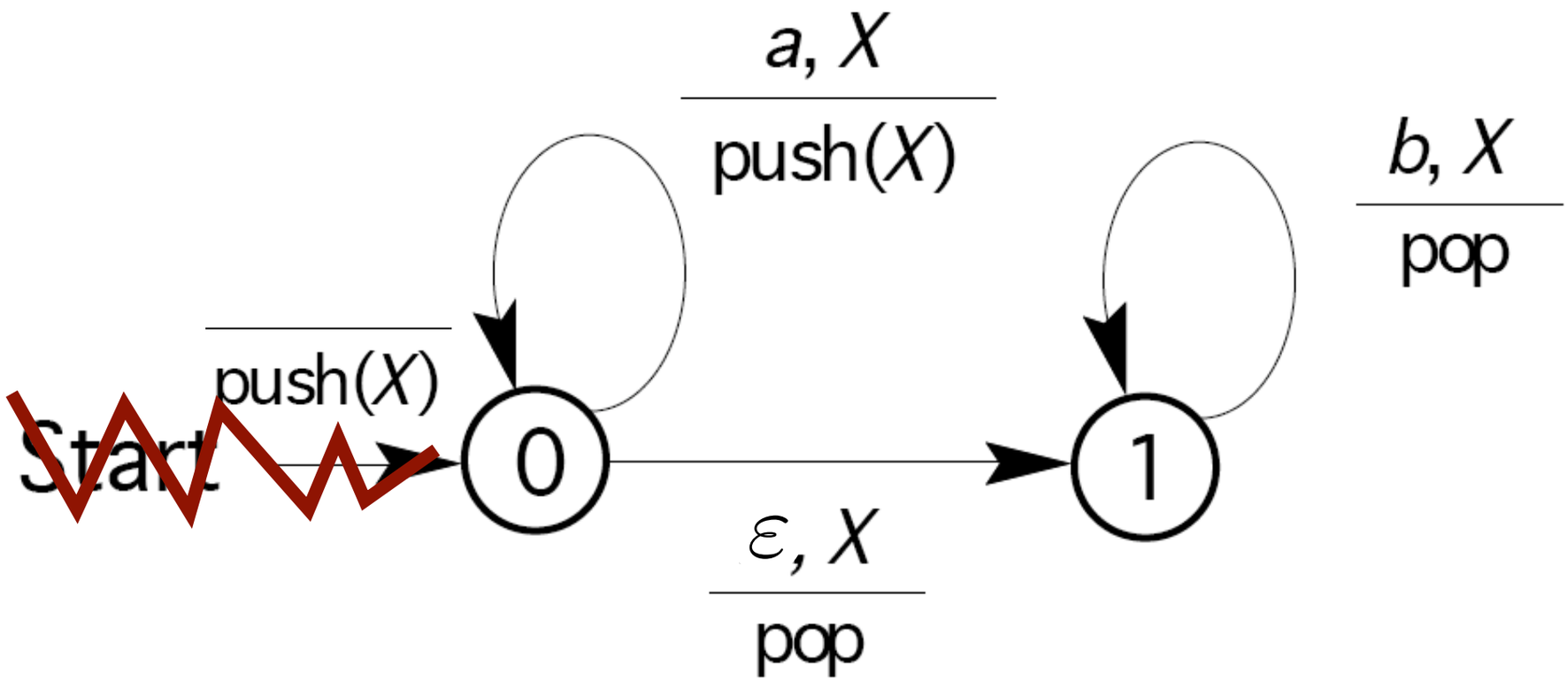

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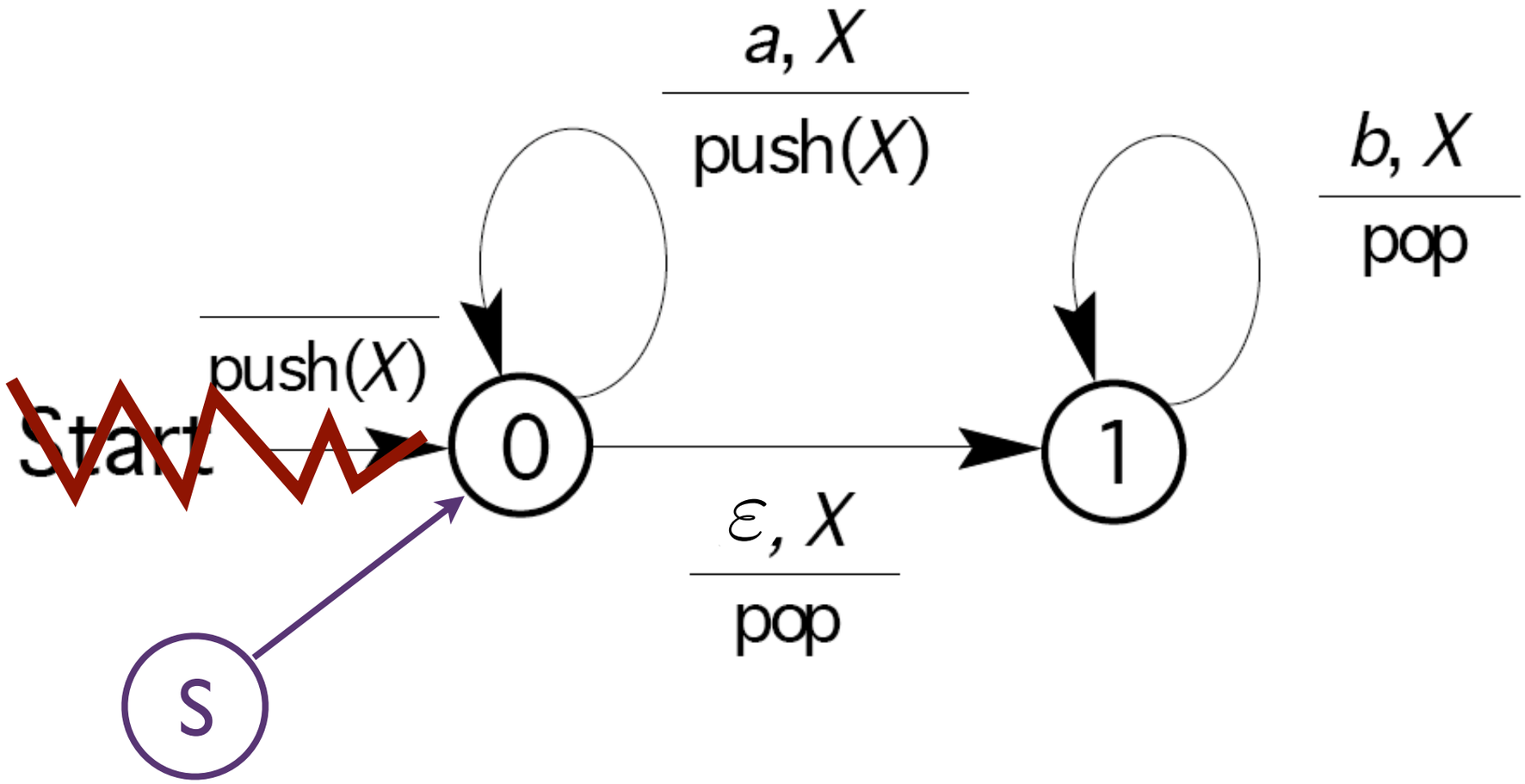

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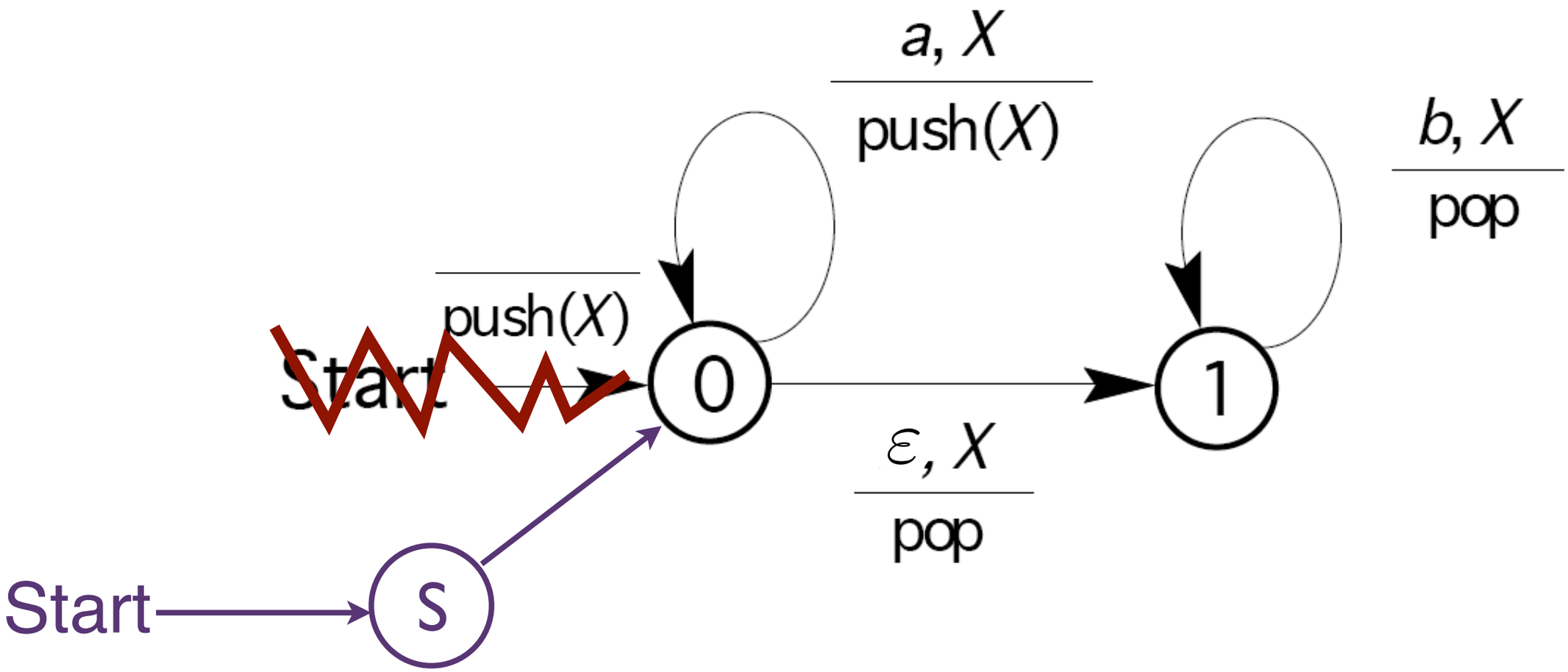

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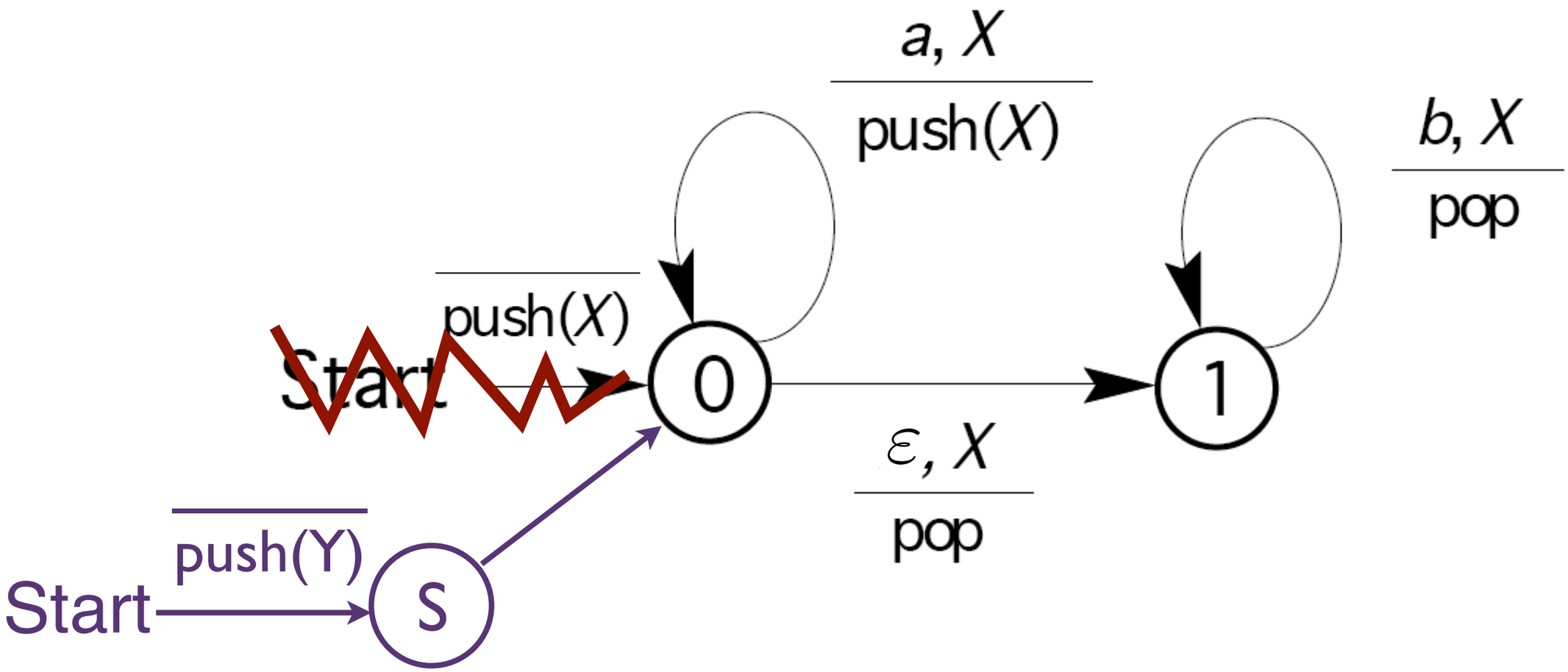

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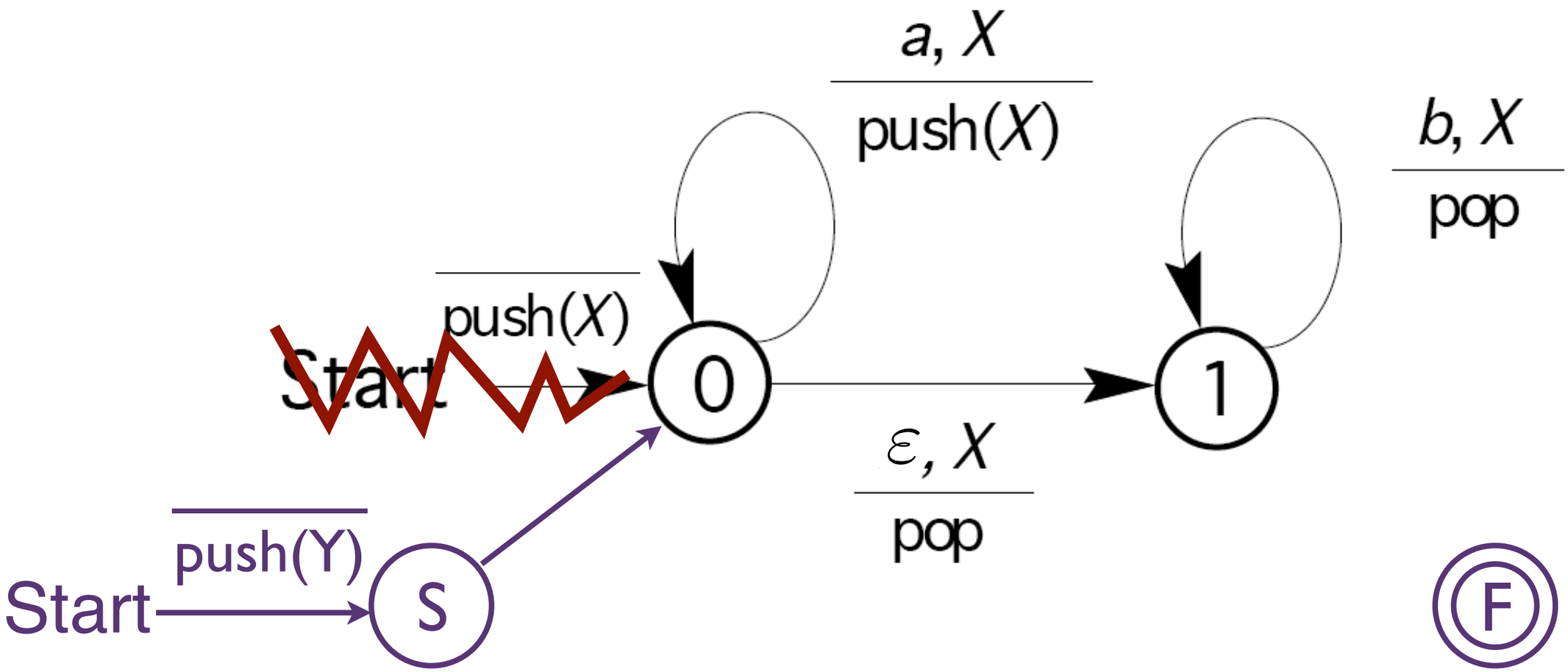

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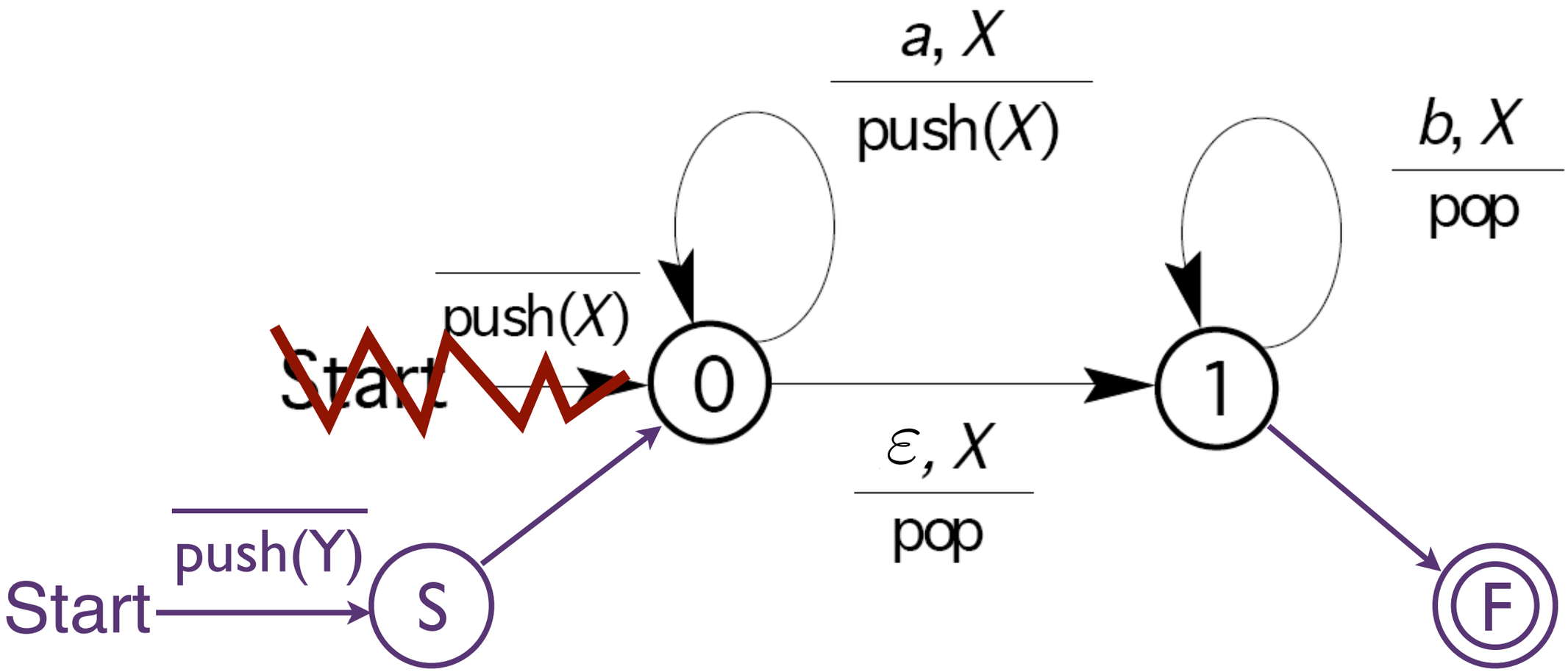

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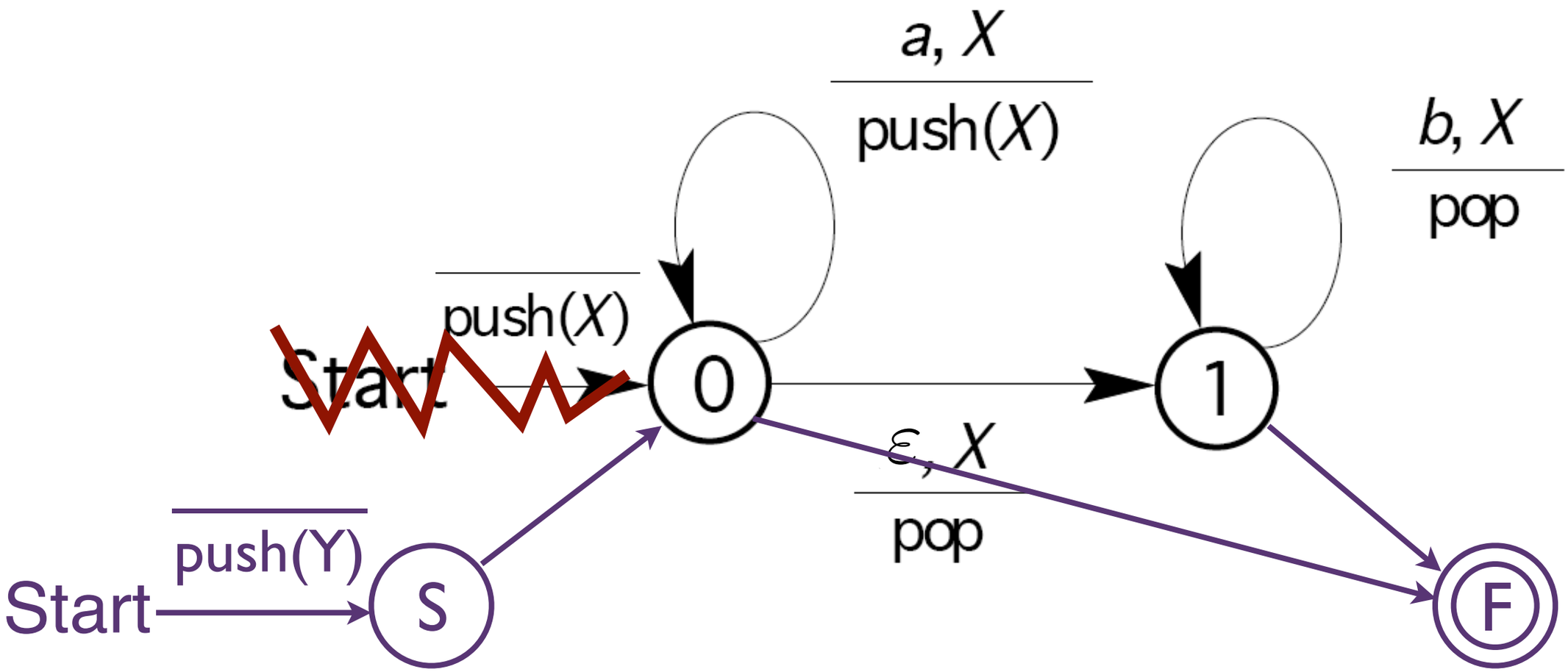

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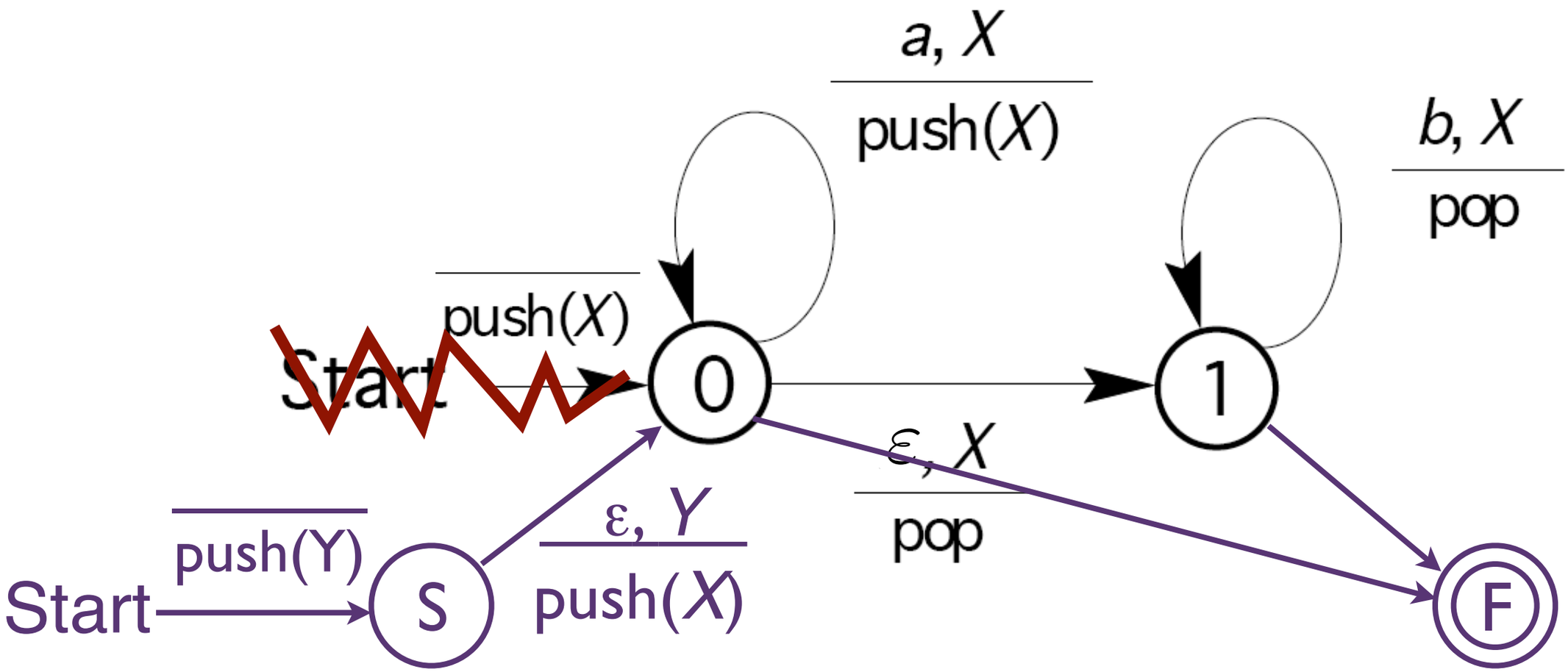

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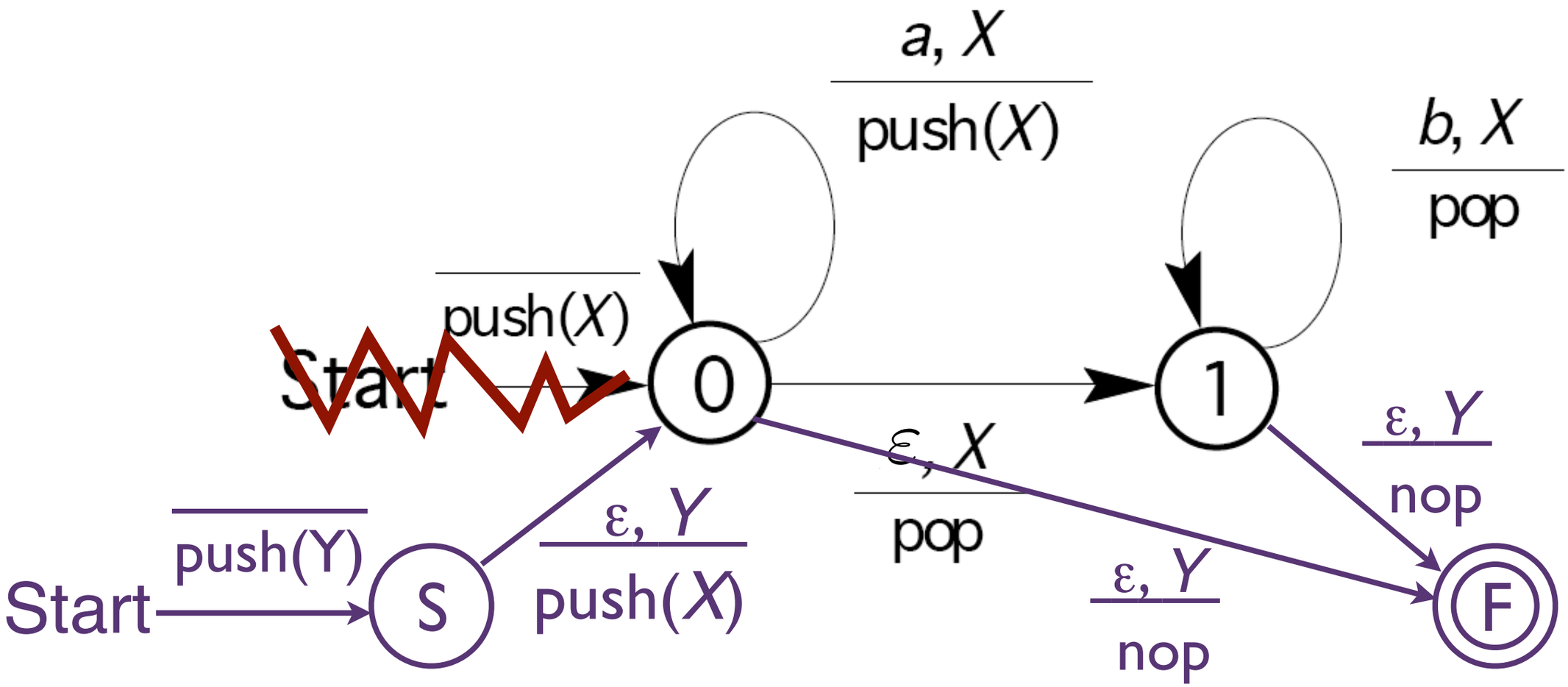

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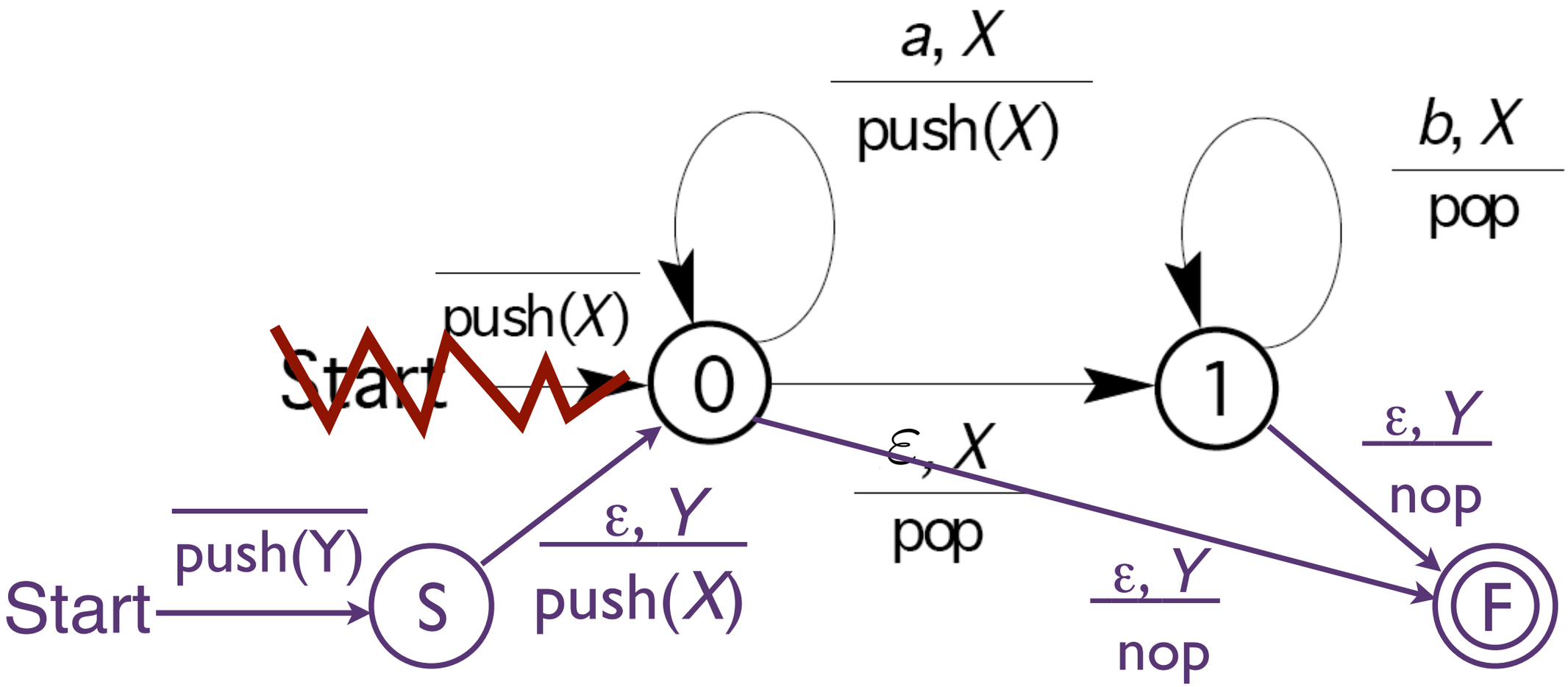
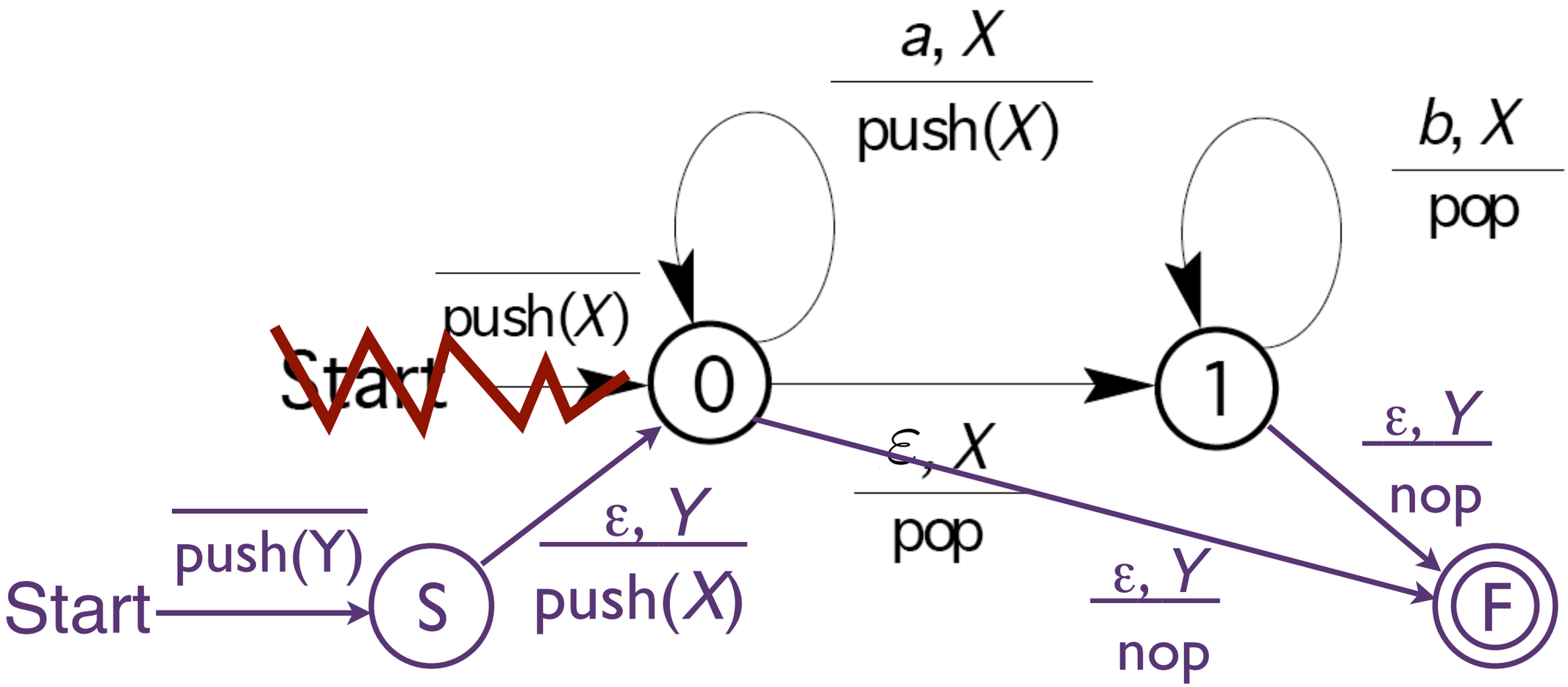


- N = ({0,1, S, F}, {a, b}, {X,Y}, δ′, S, Y, {F})

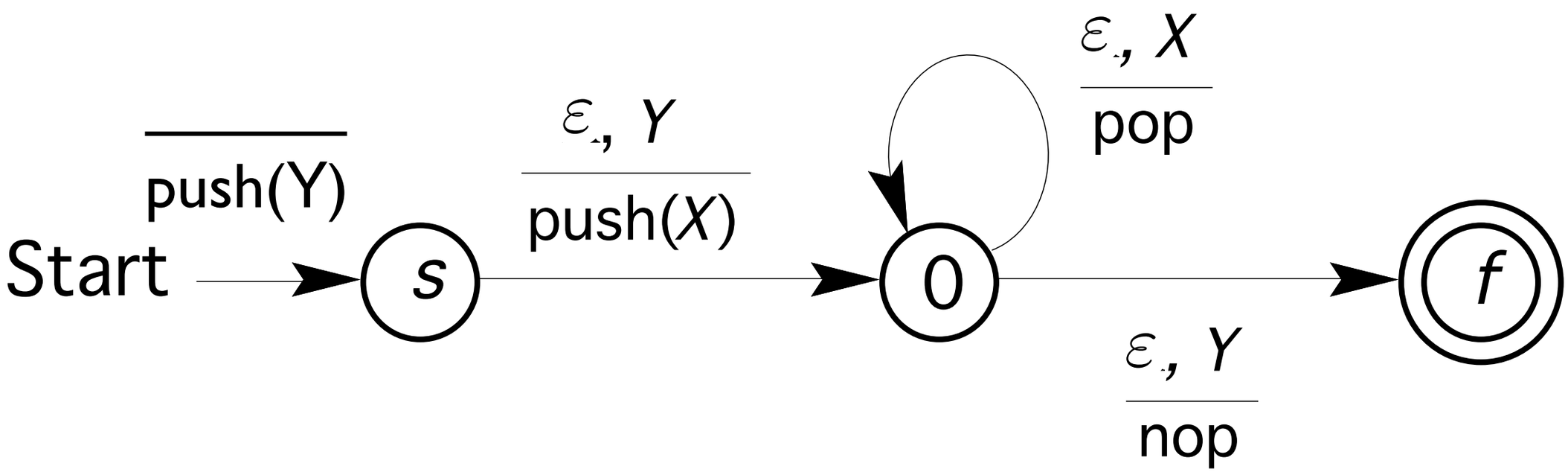
Transformation to a Final State PDA



- N = ({0,1, S, F}, {a, b}, {X,Y}, δ′, S, Y, {F})
- δ′ = δ + δ′(S, ε, Y) = {(0, XY)} +

  δ′(0, ε, Y) = {(F, ε)} + δ′(1, ε, Y) = {(F, ε)}

Transformation to an EmptyStack PDA



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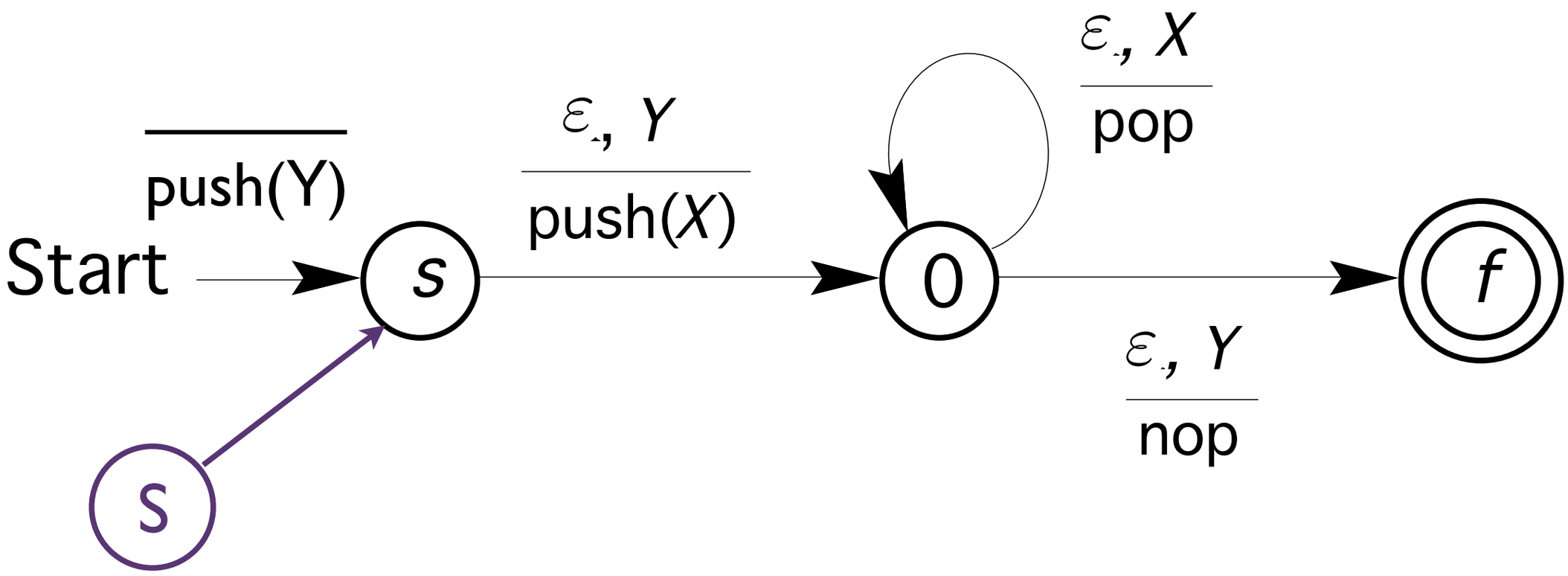

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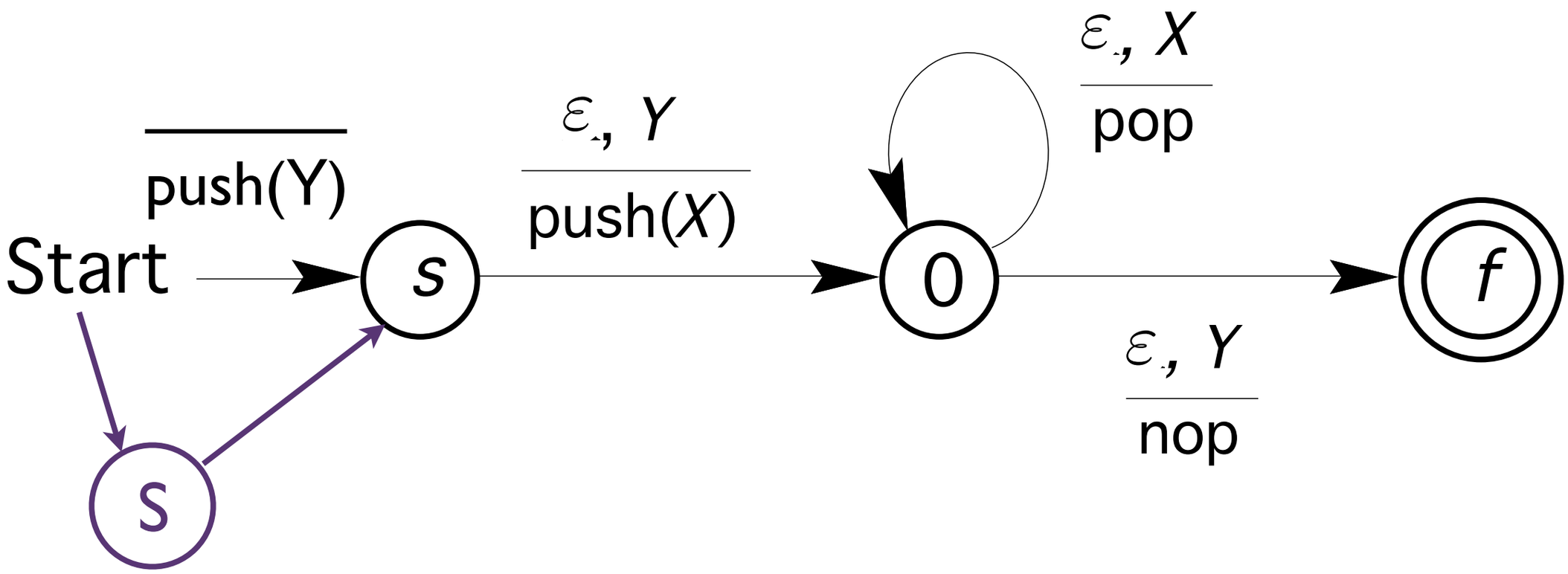

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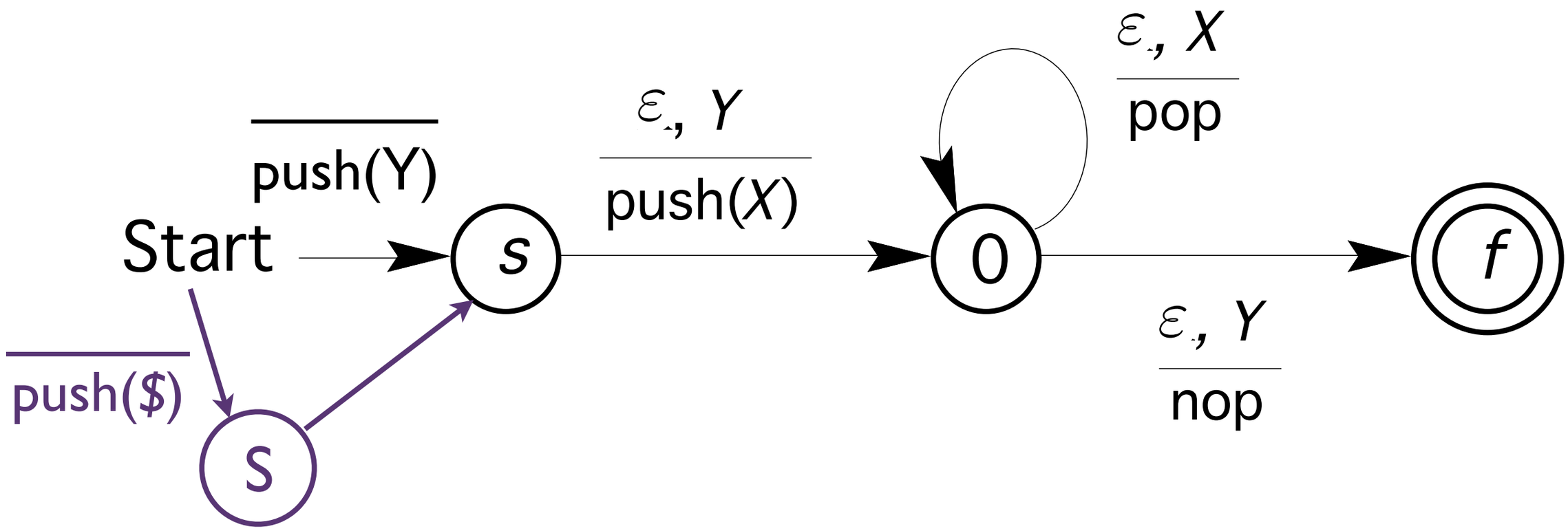

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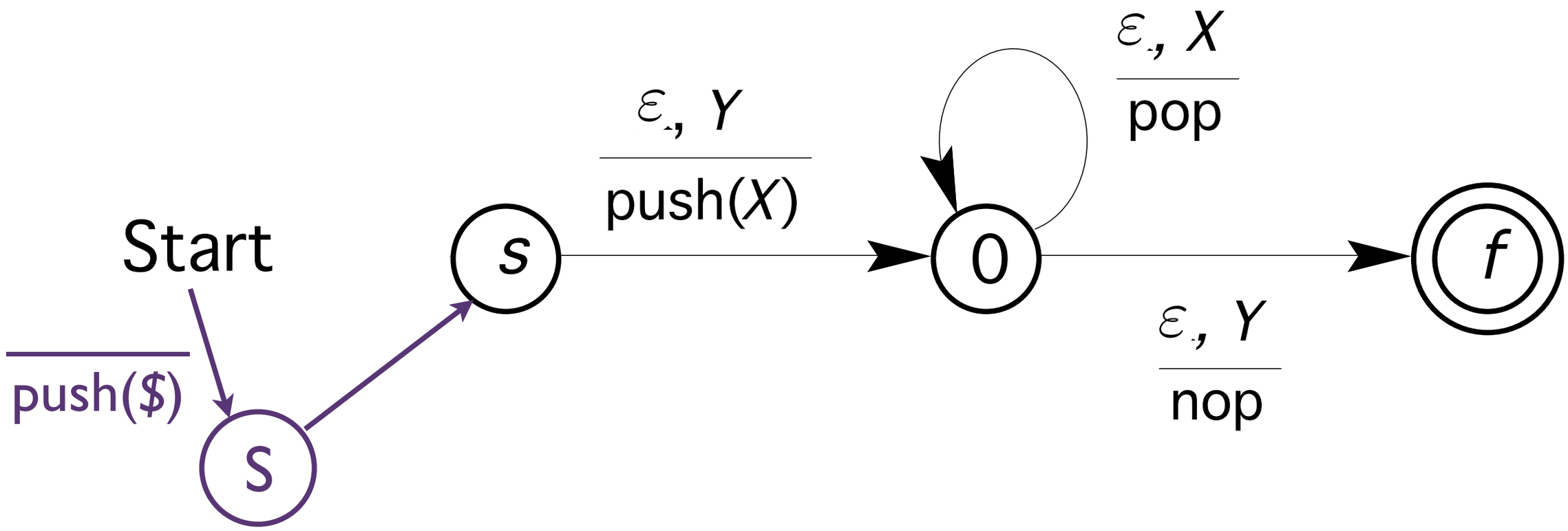

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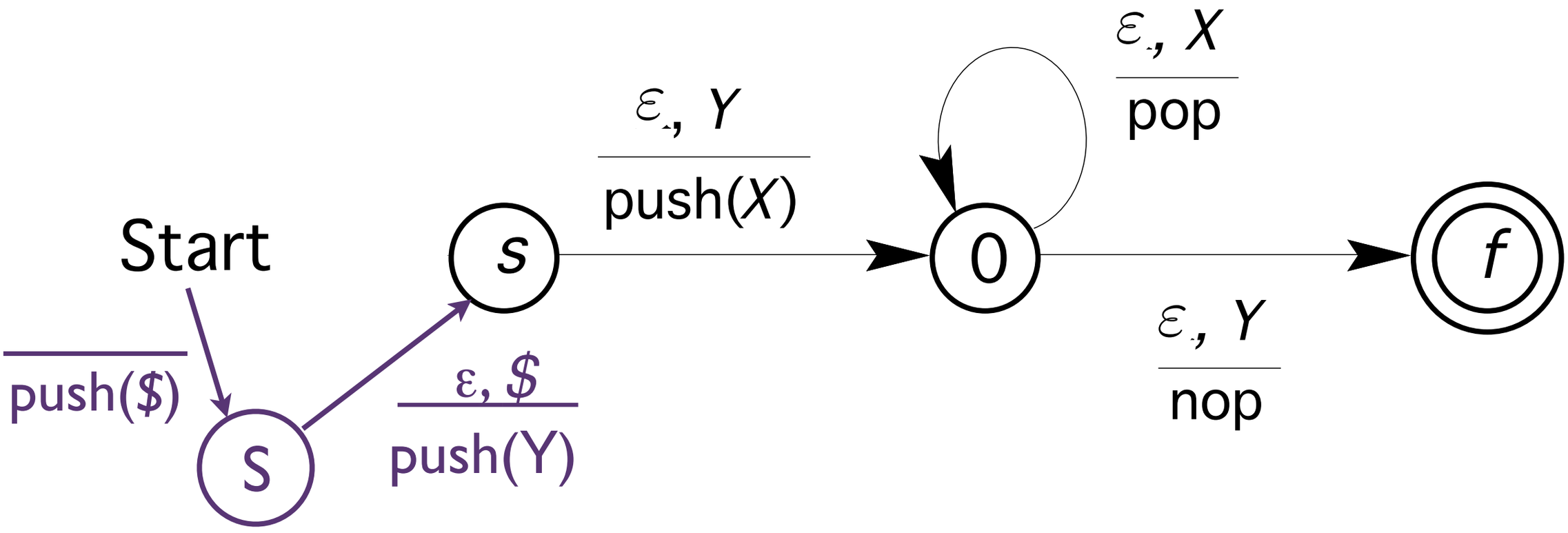

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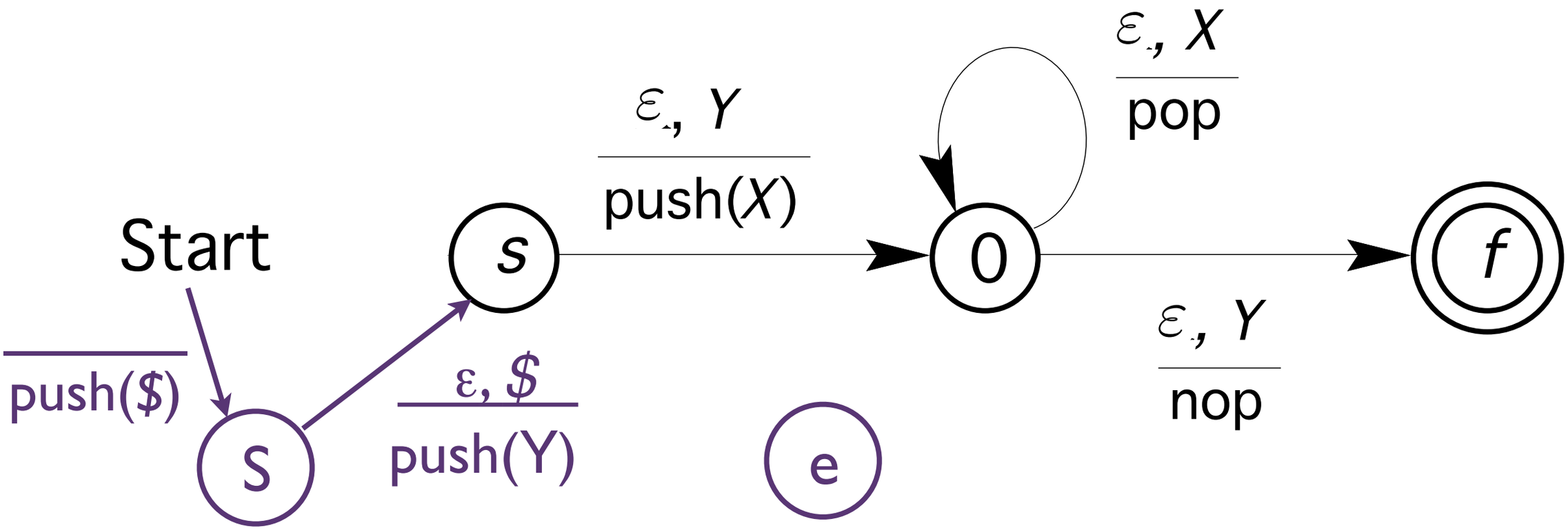

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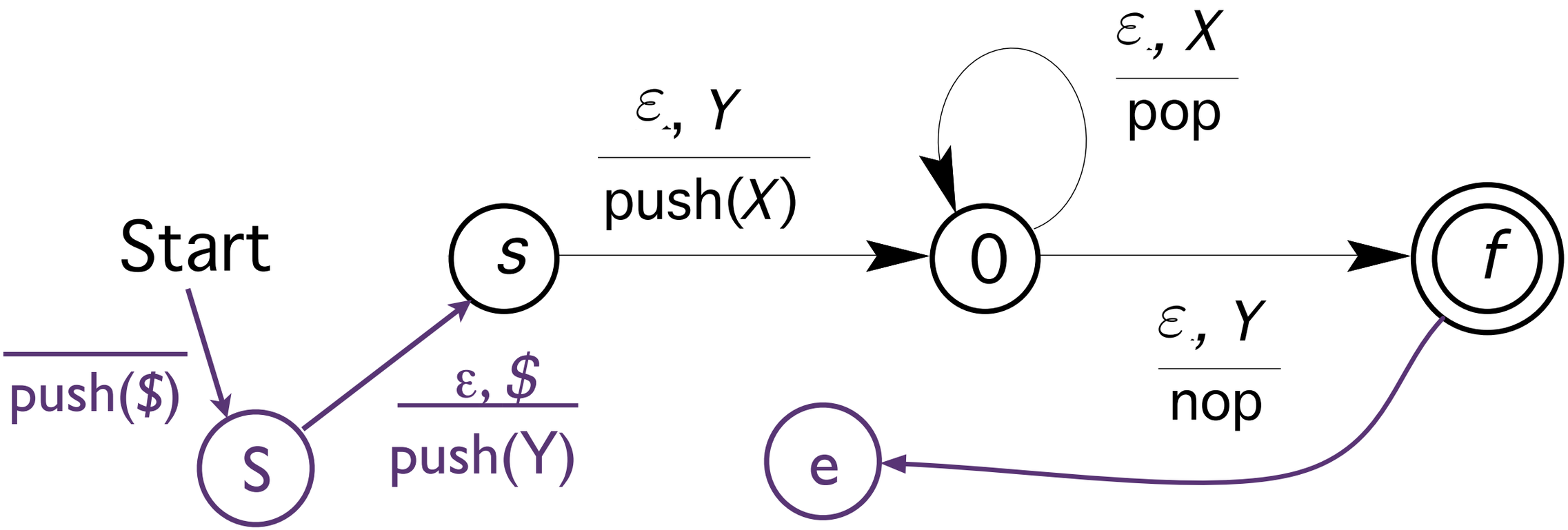

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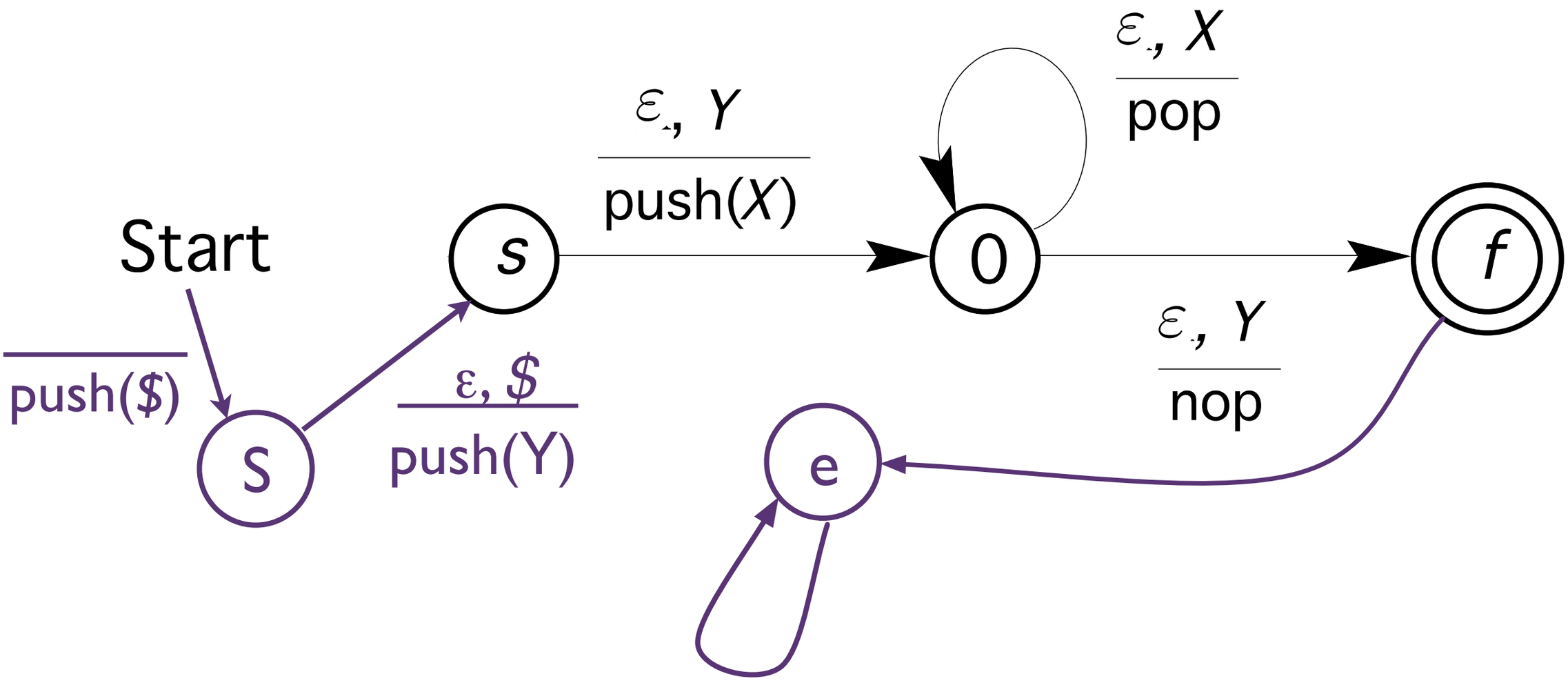

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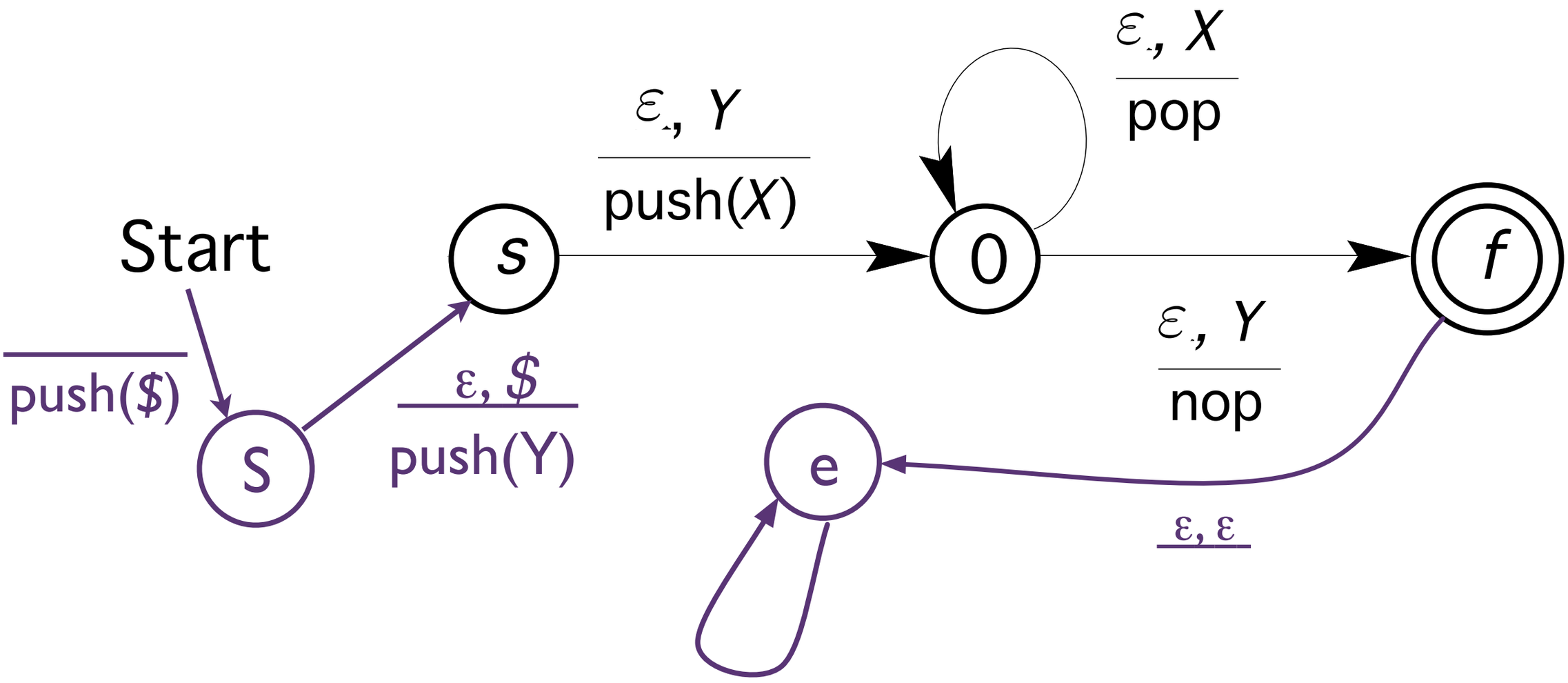

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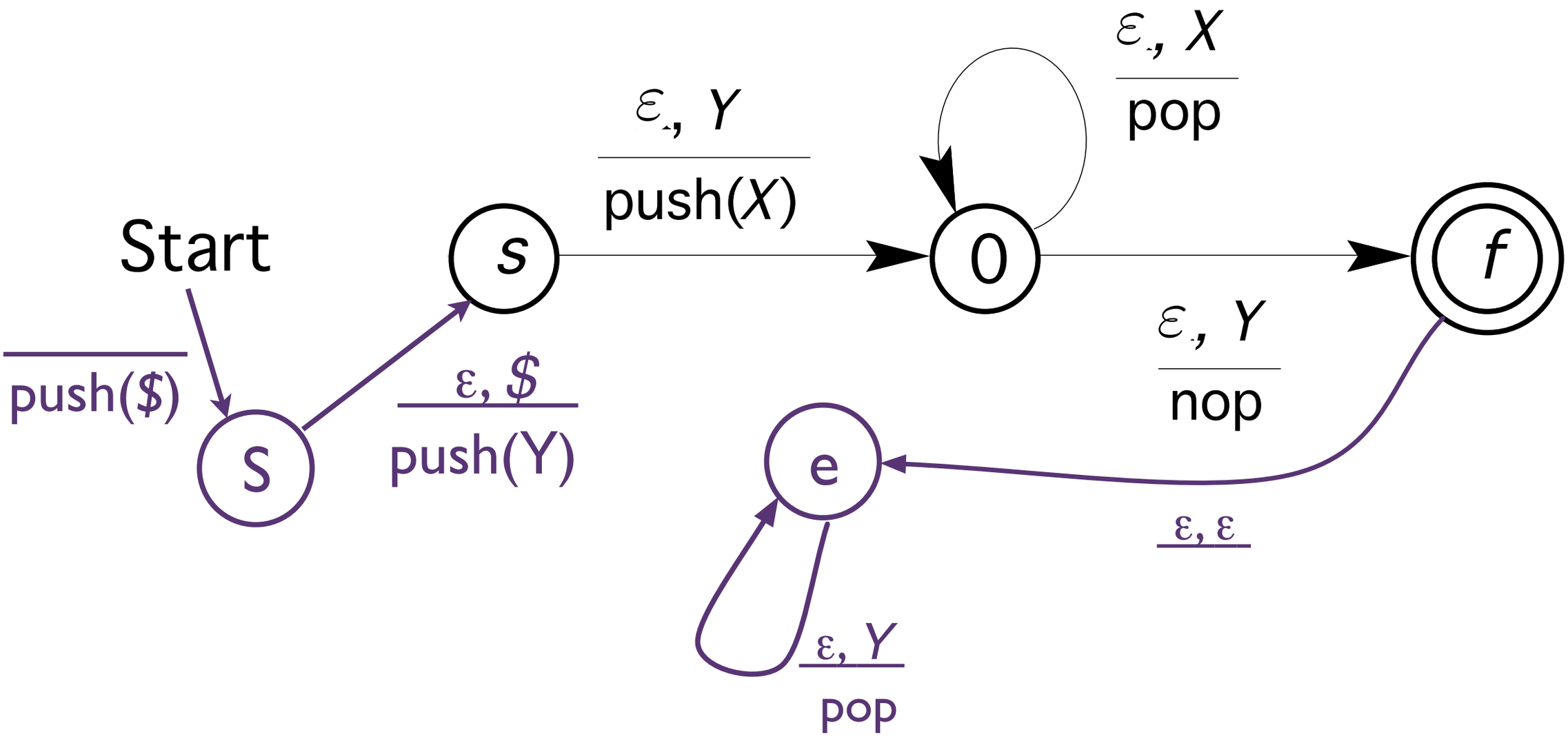

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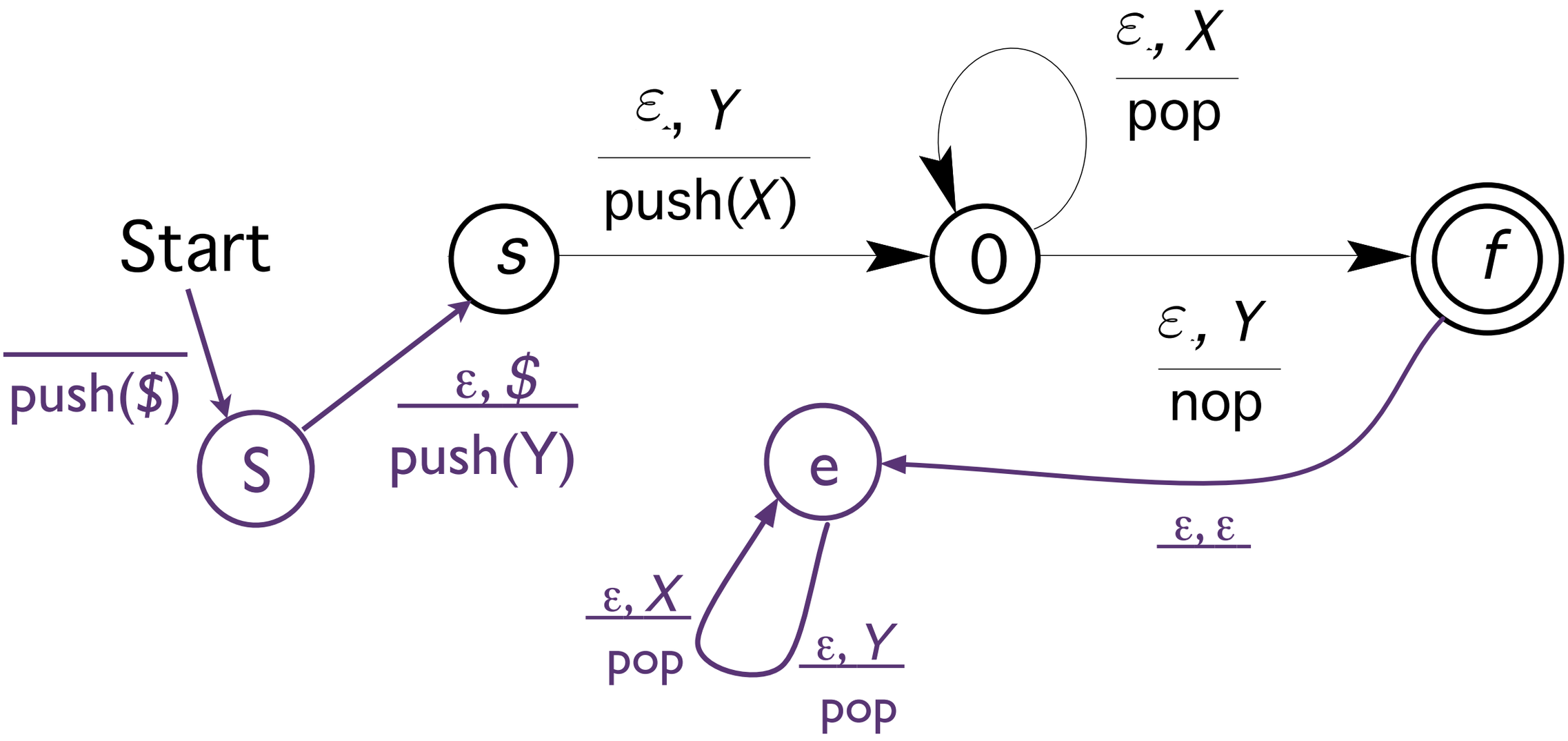

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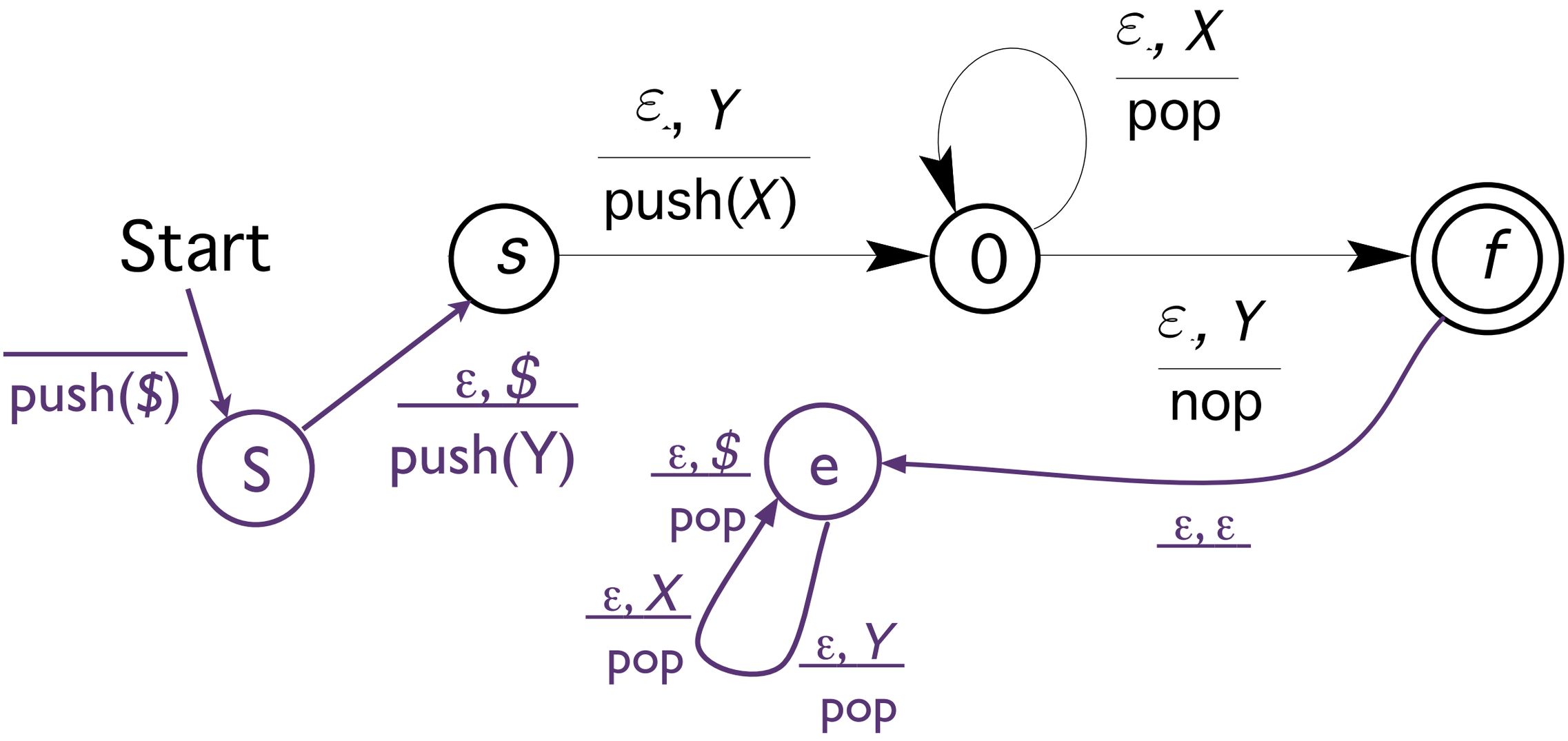

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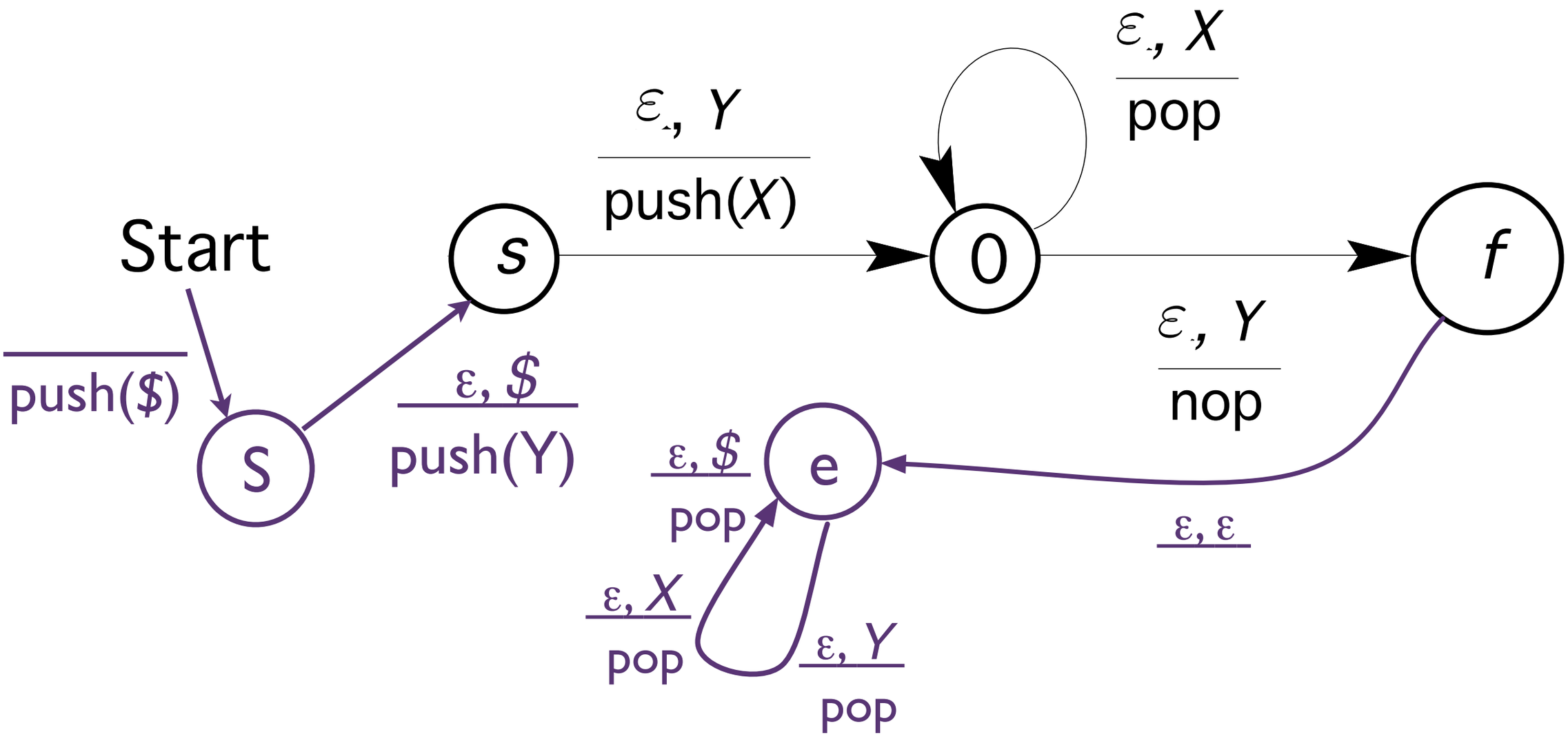

Other Variations

- Hein says: “the execution of a PDA always begins with one symbol on the stack”
- Other authors say “the stack is initially empty”
- Does it matter?

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Other Variations

- Hein says: “the execution of a PDA always begins with one symbol on the stack”
- Other authors say “the stack is initially empty”
- Does it matter?

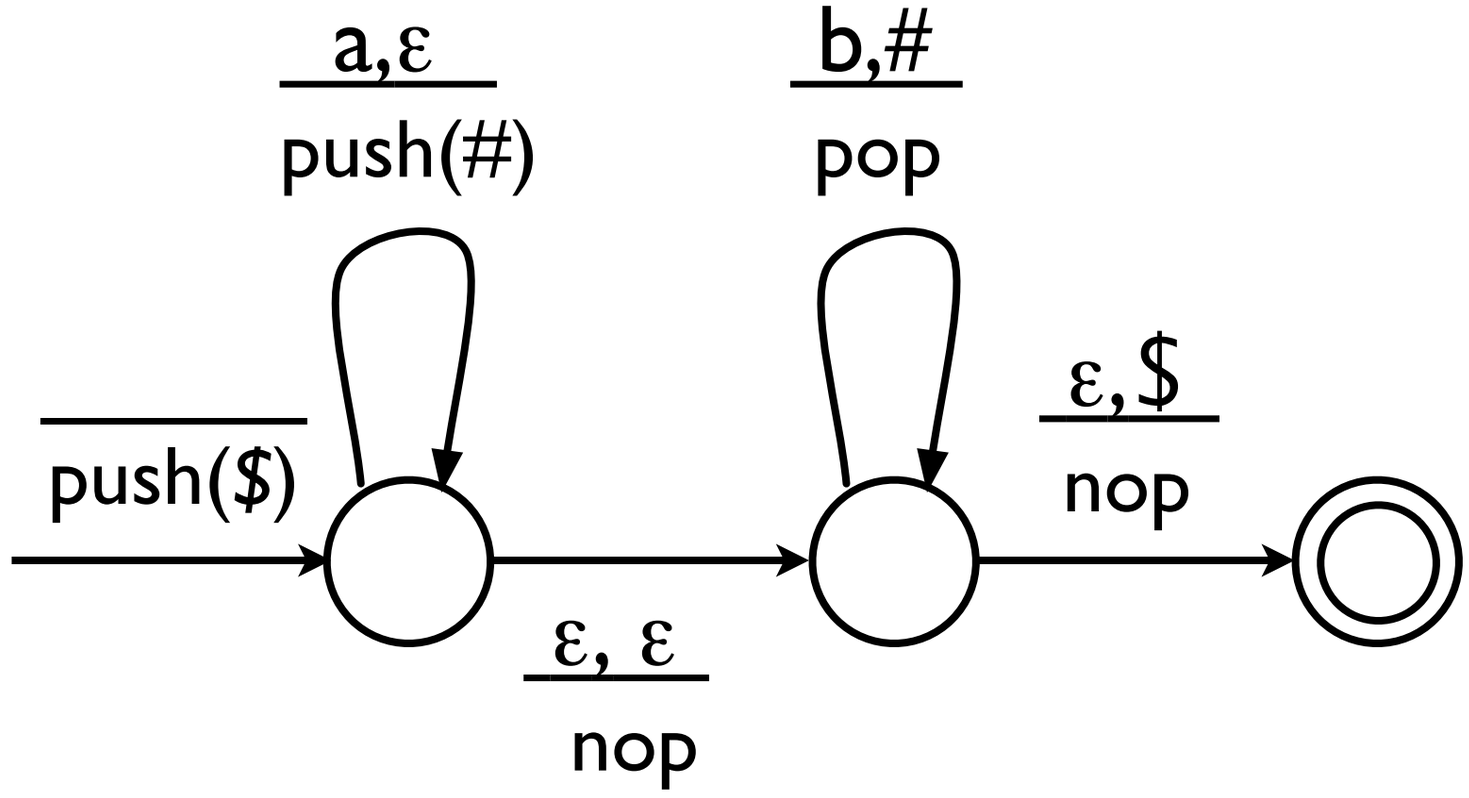
- Can you always convert an initially empty PDA to one with an initial symbol on the stack?
- Can you always convert a PDA with an initial symbol to one whose stack is initially empty?

Stack Operations

- Hopcroft *et al.* says:
  - on each transition, to look at the top of the stack you must pop it, but
  - you can push on as many stack symbols as you like, including the one that you just popped.
- Other authors say
  - you can push *zero or one* symbols onto the stack
- Hein says:
  - on each transition, you can **push**, **pop** or **nop** the stack, but you can't do more than one thing
- Does it matter?

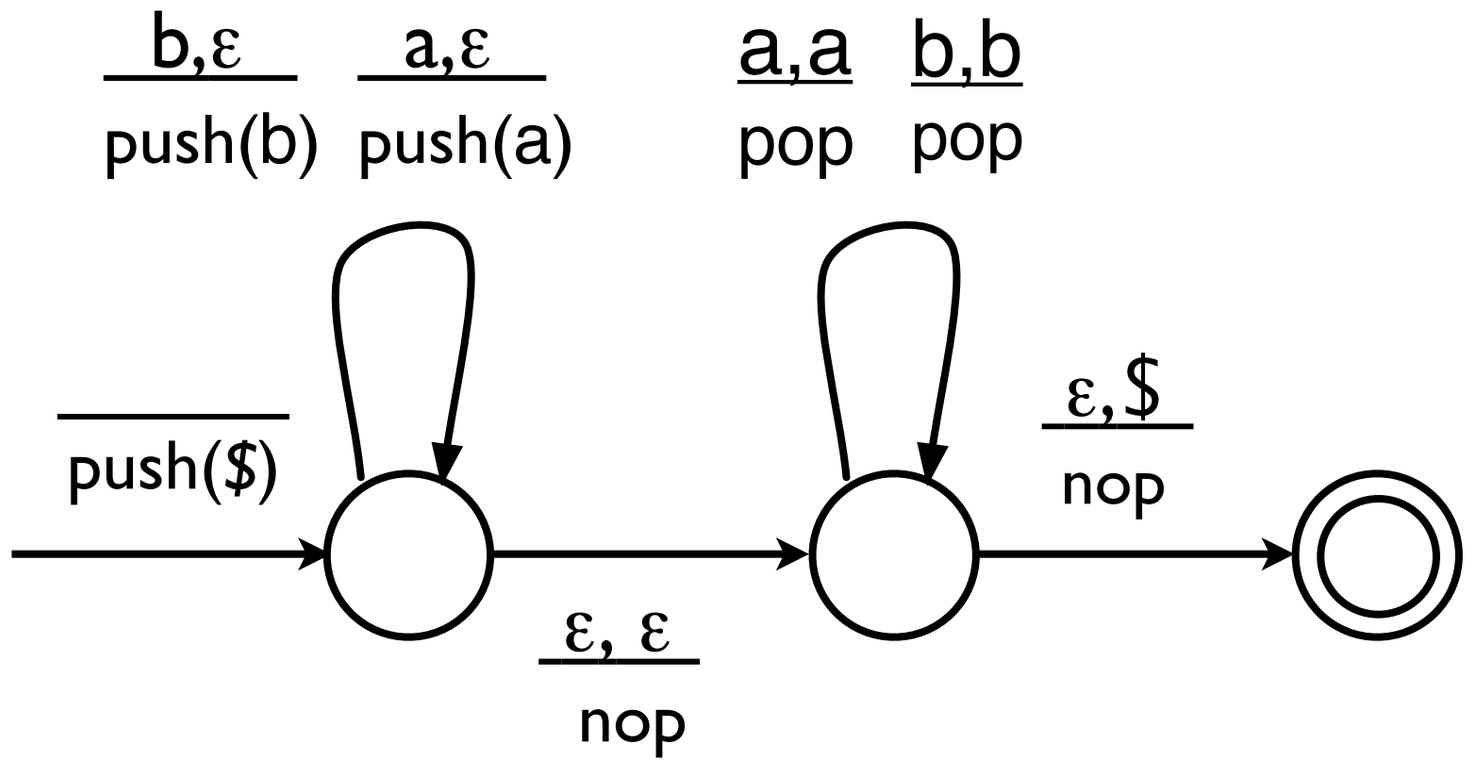
More Examples

$L_1 = \{a^n b^n : n \geq 0\}$



Palindromes of even length

$L_2 = \{ww^R \mid w \in \{a, b\}^*\}$



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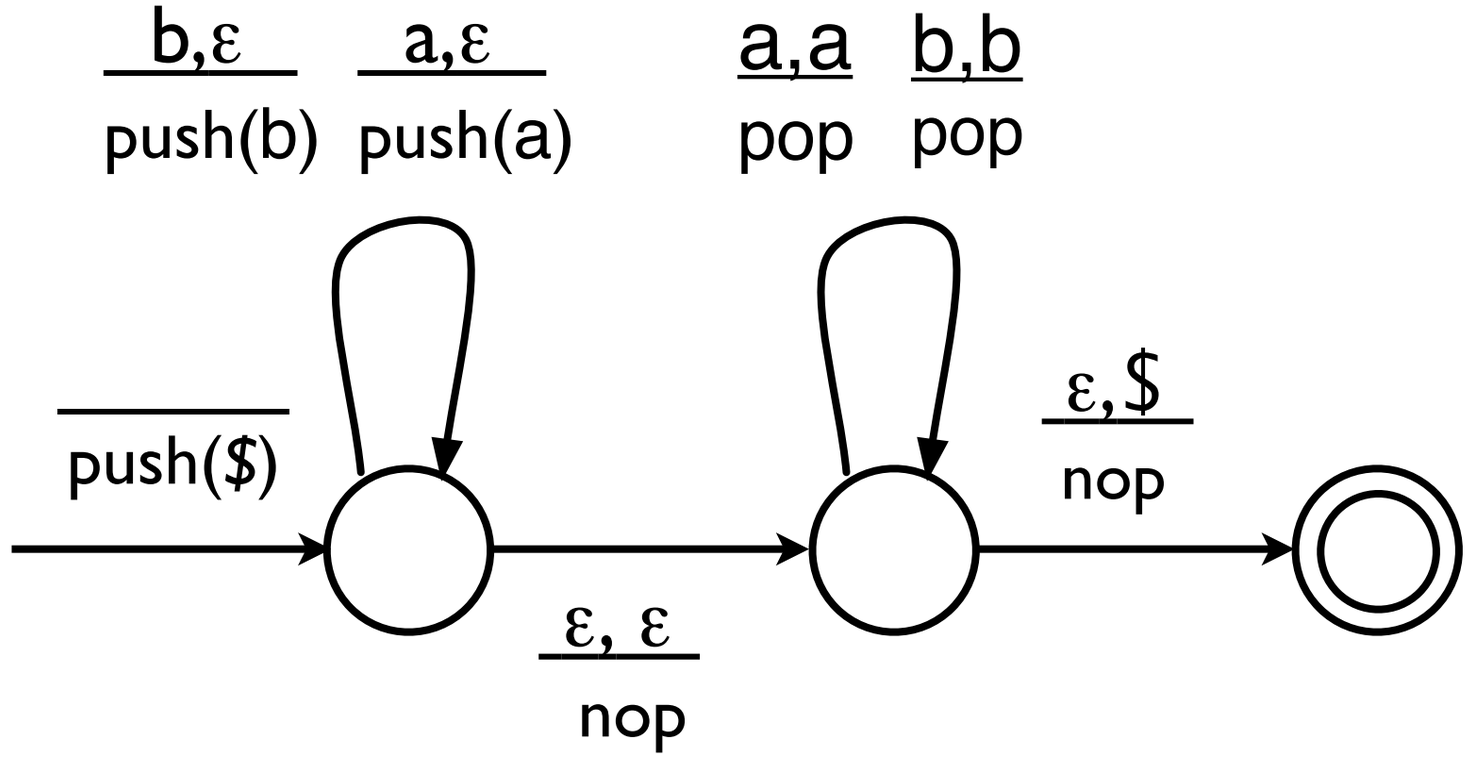

- This machine is non-deterministic. (Why?)

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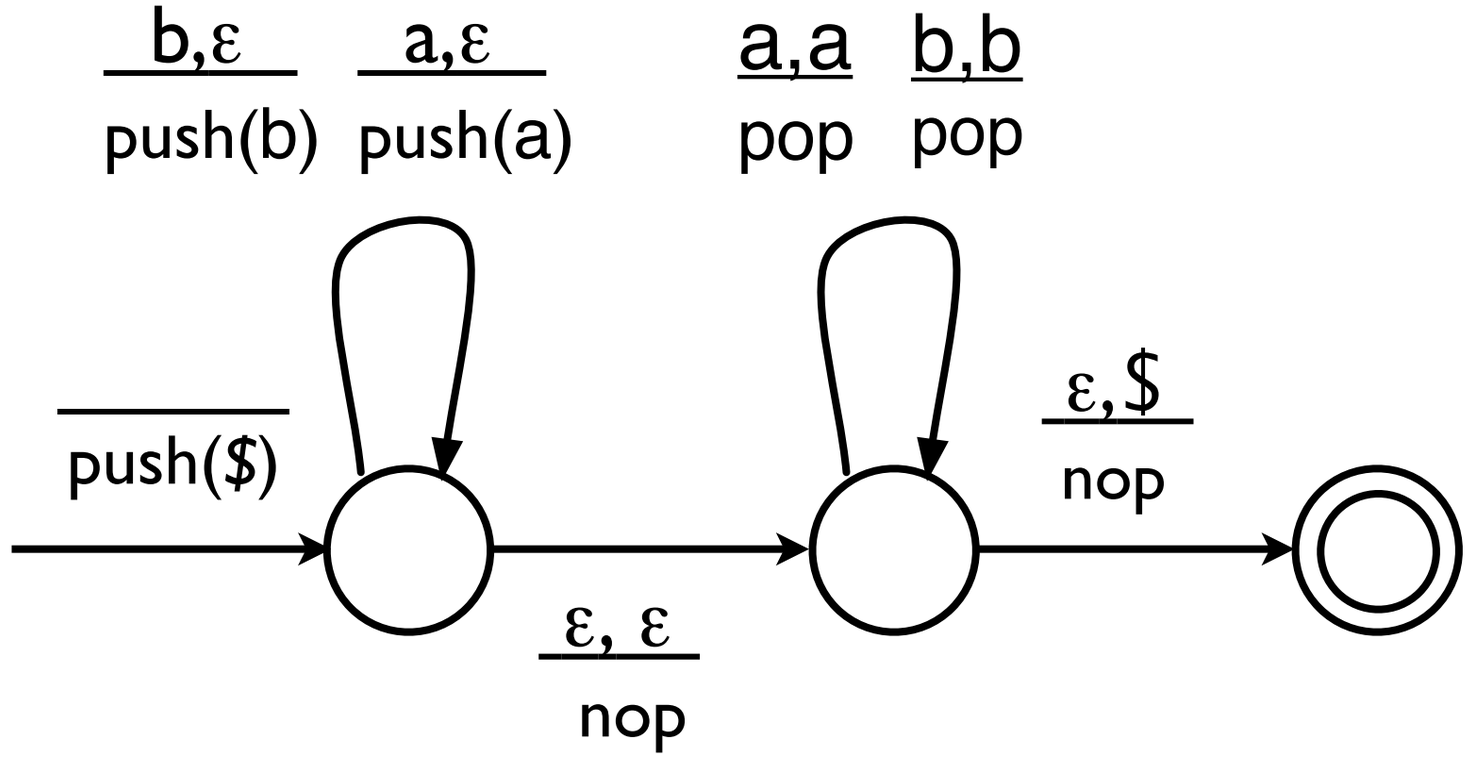
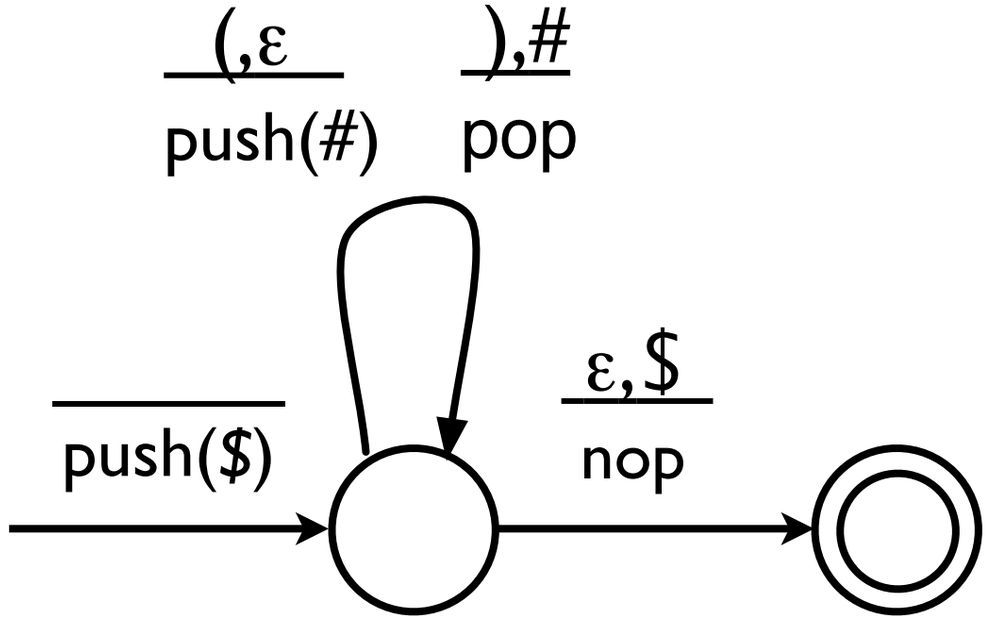


- This machine is non-deterministic. (Why?)
- Can you design a *deterministic* PDA for L_2?

Balanced Parenthesis

$L_3 = \{w \in \{(,)\}^* : w \text{ is a balanced string of parenthesis}\}$



Equal numbers of *a*s and *b*s