Push Down Automata

Sipser pages 109 - 114

Push Down Automata

Push Down Automata (PDAs) are ε-NFAs with stack memory.

Transitions are labeled by an input symbol together with a pair of the form X/α

.

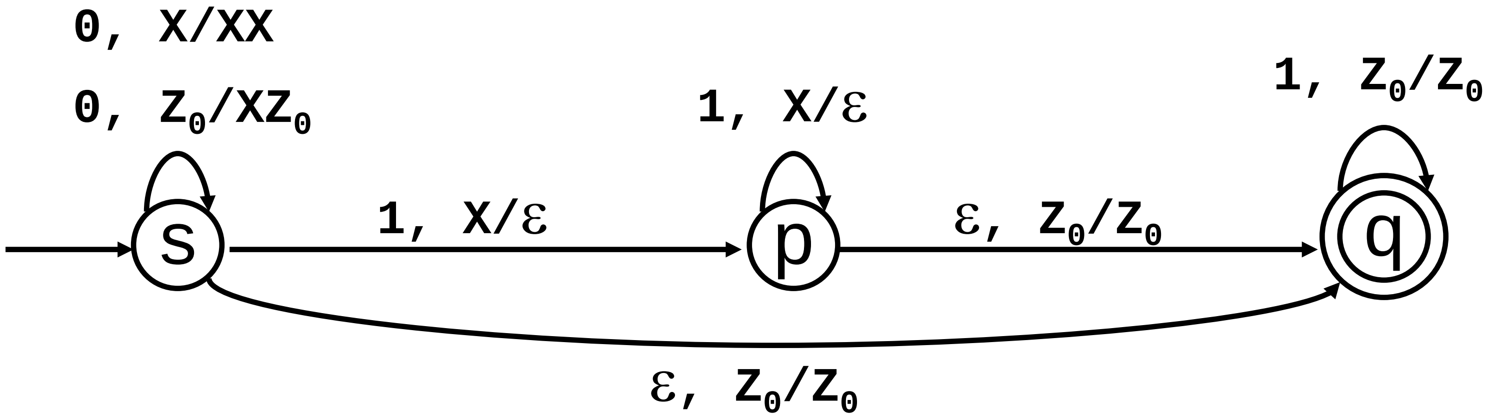
The transition is possible only if the top of the stack contains the symbol X

After the transition, the stack is changed by replacing the top symbol X with the string of symbols α. (Pop X, then push symbols of α.)

Example

PDAs can accept languages that are not regular. The following one accepts:

$$L=\{0^i1^j \mid 0 \leq i \leq j\}$$



Definition

A PDA is a 6-tuple $P=(Q,\Sigma,\Gamma,\delta,q_0,F)$ where Q, Σ, q_0, F are as in NFAs, and

- Γ is the *stack alphabet*. It is assumed that initially the stack is empty.

- δ: $Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \longrightarrow P(Q \times \Gamma_\varepsilon{}^*)$ is the *transition function*: given a state, an input symbol (or ε), and a stack symbol, Γ_ε, it gives us a finite number of pairs (q,α), where q is the next state and α is the string of stack symbols that will replace X on top of the stack.

- Recall $\Sigma_\varepsilon=(\Sigma\cup\{\varepsilon\})$ $\quad\Gamma_\varepsilon=(\Gamma\cup\{\varepsilon\})$

In our example, the transition from s to s labeled $(0, Z_0/XZ_0)$ corresponds to the fact $(s, XZ_0) \in \delta(s, 0, Z_0)$. A complete description of the transition function in this example is given by

$\delta(s,0,Z_0) = \{(s,XZ_0)\}$

$\delta(s,0,X) = \{(s,XX)\}$

$\delta(s,\varepsilon,Z_0) = \{(q,Z_0)\}$

$\delta(s,1,X) = \{(p,\varepsilon)\}$

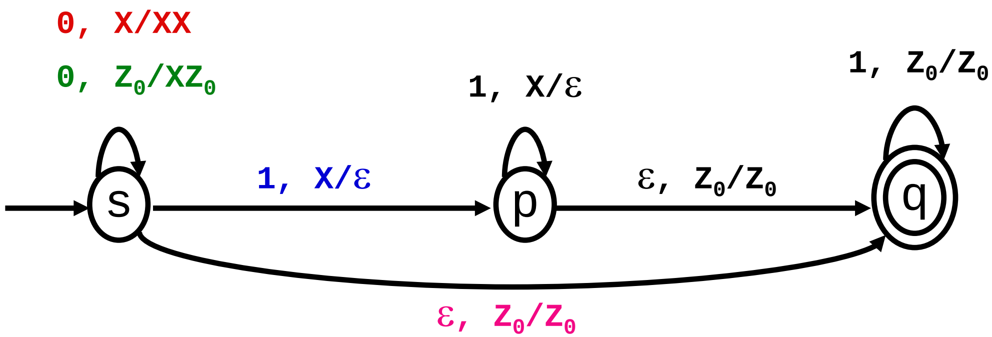
$\delta(p,1,X) = \{(p,\varepsilon)\}$

$\delta(p,\varepsilon,Z_0) = \{(q,Z_0)\}$

$\delta(q,1,Z_0) = \{(q,Z_0)\}$

and

$\delta(q,a,Y)=\emptyset$ for all other possibilities.



Sipser style acceptance

- Suppose a string w can be written: $w_1 \, w_2 \, \ldots \, w_m$
  - $W_i \in \Sigma_\varepsilon$ Some of the w_i are allowed to be ε
  - I.e. One may write "abc" as $a \, \varepsilon \, b \, c \, \varepsilon$
- If there exist two sequences
  - $s_0 \, r_1 \, \ldots \, r_m \in Q$
  - $s_0 \, s_1 \, \ldots \, s_m \in \Gamma^*$ (The s_i represent the stack contents at step i)

1. $r_0 = q_0$ and $s_0 = \varepsilon$
2. $(r_i+1, b) \in \delta(r_i, w_i+1, a)$
   $s_i = at \quad s_{i+1} = bt$
3. $R_m \in F$

Instantaneous Descriptions and Moves of PDAs

IDs (also called *configurations*) describe the execution of a PDA at each instant. An ID is a triple (q, w, α), with this intended meaning:

- q is the current state
- w is the remaining part of the input
- α is the current content of the stack, with top of the stack on the left.

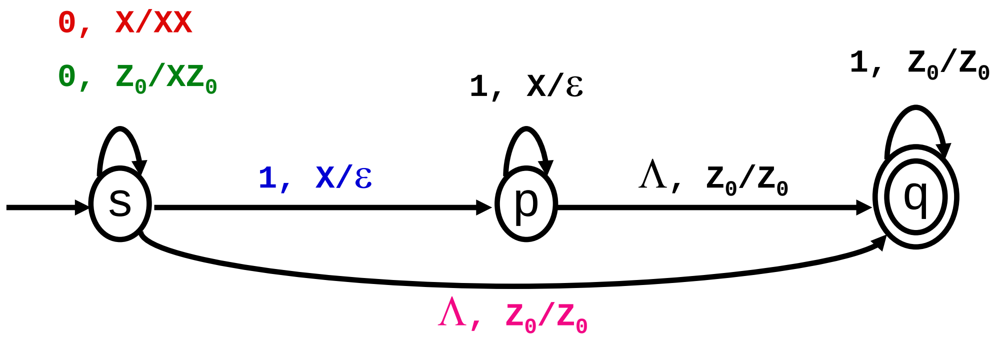
The relation |- describes possible moves from one ID to another during execution of a PDA. If $\delta(q,a,X)$ contains (p,α), then

$$(q,aw,X\beta) \ \text{|-} \ (p,w,\alpha\beta)$$

is true for every w and β.

The relation |-^* is the reflexive-transitive closure of |-

We have (q,w,a) |-^* (q',w',a') when (q,w,a) leads through a sequence (possibly empty) of moves to (q',w',a')



(s,011,z) |- (s,11,xz) |- (P,1,z)|-(q,1,z) |- (q,””,Z)

(s,011,z) |- (q,011,Z)

Properties of |-

Property 1.

If $(q,x,\alpha) \mid\!-^* (p,y,\beta)$

Then $(q,xw,\alpha\gamma) \mid\!-^* (p,yw,\beta\gamma)$

If you only need some prefix of the input (x) and stack (α) to make a series of transitions, you can make the same transitions for any longer input and stack.

Property 2.

If $(q,xw,\alpha) \mid\!-^* (p,yw,\beta)$

Then $(q,x,\alpha) \mid\!-^* (p,y,\beta)$

It is ok to remove unused input, since a PDA cannot add input back on once consumed.

Another notion of acceptance

A PDA as above *accepts* the string w iff

$(q_0, w, \varepsilon) \vdash^* (p, \varepsilon, \alpha)$ is true for some final state p and some α. (We don't care what's on the stack at the end of input.)

The *language* L(P) of the PDA P is the set of all strings accepted by P.

Here is the chain of IDs showing that the string `001111` is accepted by our example PDA:

$(s,001111,\varepsilon)$

$\vdash (s,01111,XZ_0)$

$\vdash (s, 1111,XXZ_0)$

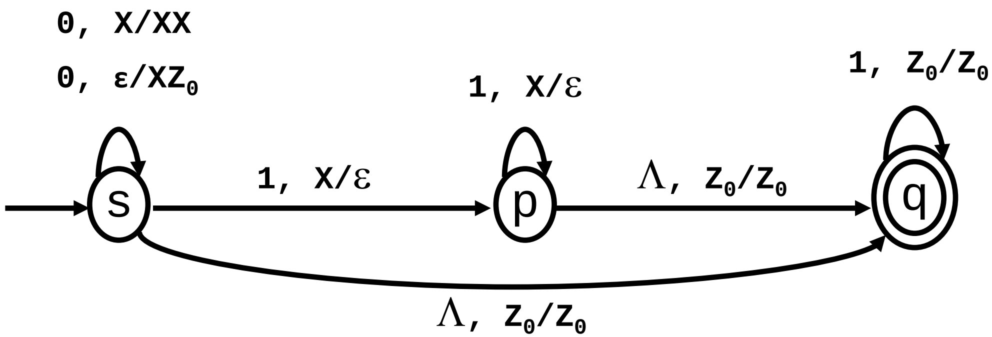
$\vdash (p,111,XZ_0)$

$\vdash (p,11,Z_0)$

$\vdash (q,11,Z_0)$

$\vdash (q,1,Z_0)$

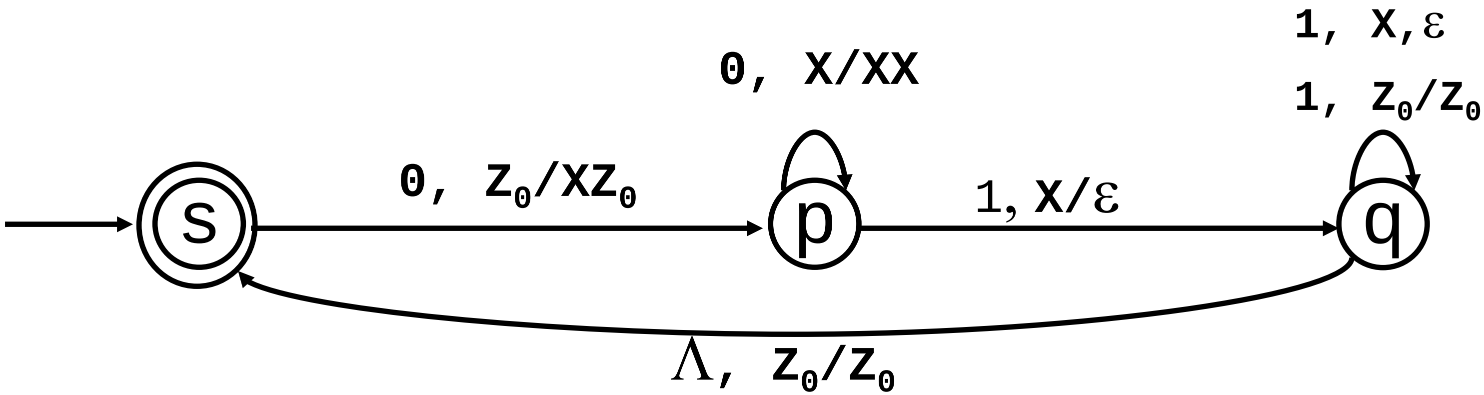
$\vdash (q,\varepsilon,Z_0)$



The language of the following PDA is

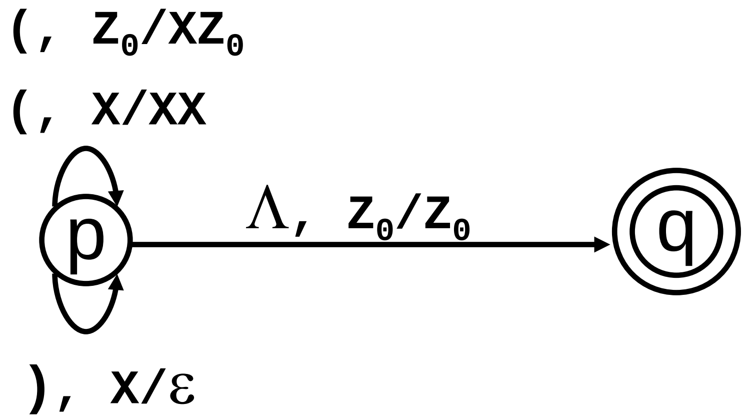
$\{0^i1^j \mid 0 < i \leq j\}^*$.

How can we prove this?



Example

A PDA for the language of balanced parentheses:



Acceptance by Empty Stack

Define N(P) to be the set of all strings *w* such that

$(q_0, w, \varepsilon) \vdash^* (q, \varepsilon, \varepsilon)$

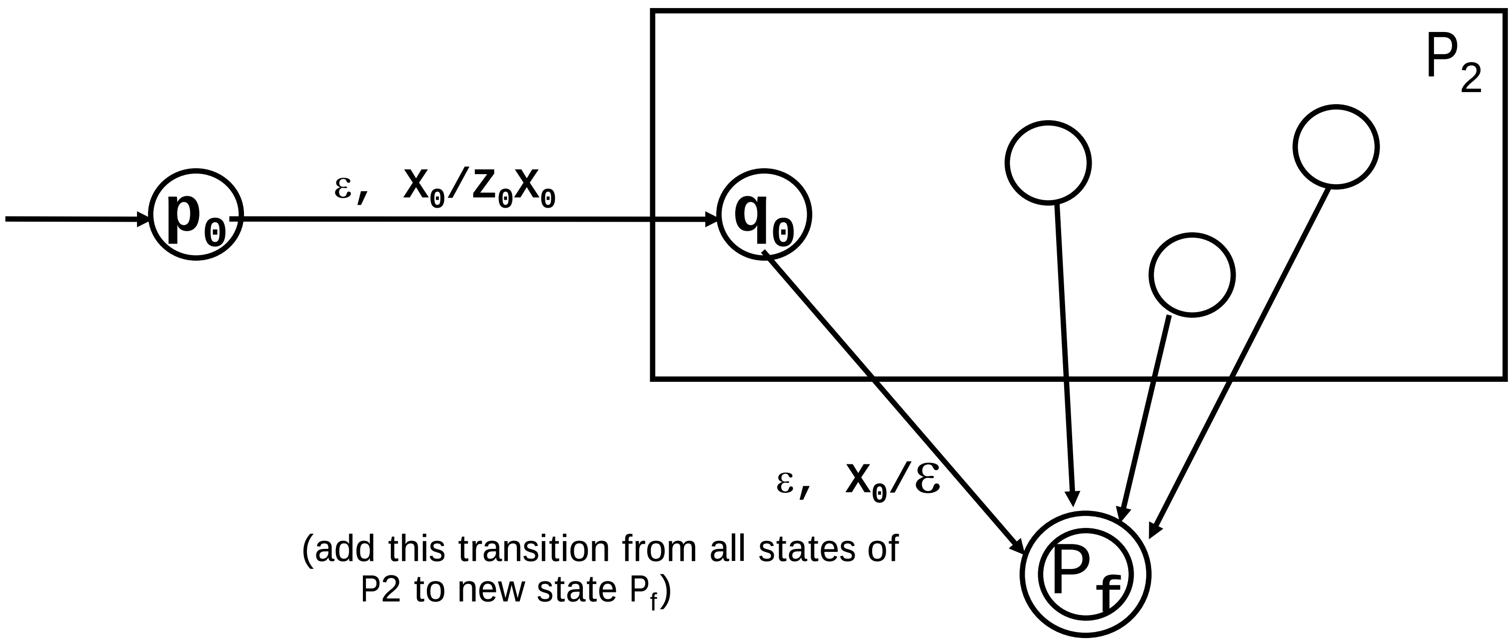
for some state q. These are the strings P *accepts by empty stack*. Note that the set of final states plays no role in this definition.

Theorem. A language is $L(P_1)$ for some PDA P_1 if and only if it is $N(P_2)$ for some PDA P_2.

Proof 1

1. From empty stack to final state.

Given P_2 that accepts by empty stack, get P_1 by adding a new start state and a new final state as in the picture below. We also add a new stack symbol X_0 and make it the start symbol for P_1's stack.



(add this transition from all states of P2 to new state P_f)

Proof 2

2. From final state to empty stack.

Given P_1, we get P_2 again by adding a new start state, final state and start stack symbol. New transitions are seen in the picture.