Homing and Synchronizing Sequences

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   (c) Parallel algorithms
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Motivation for Homing Sequences: Testing [13]

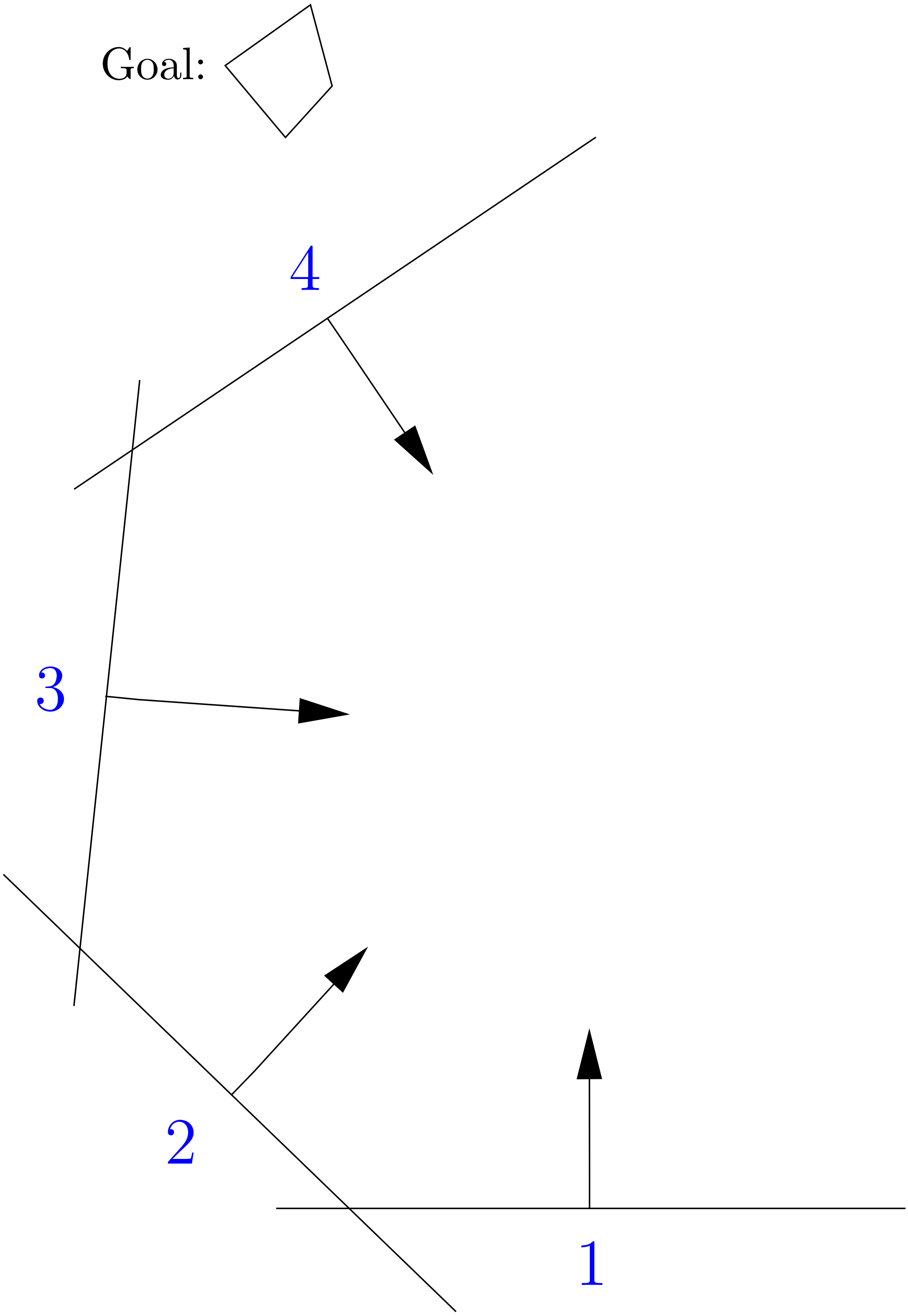
- Learning algorithms:

  experiment with a given a black box automaton
  until you learn the contents

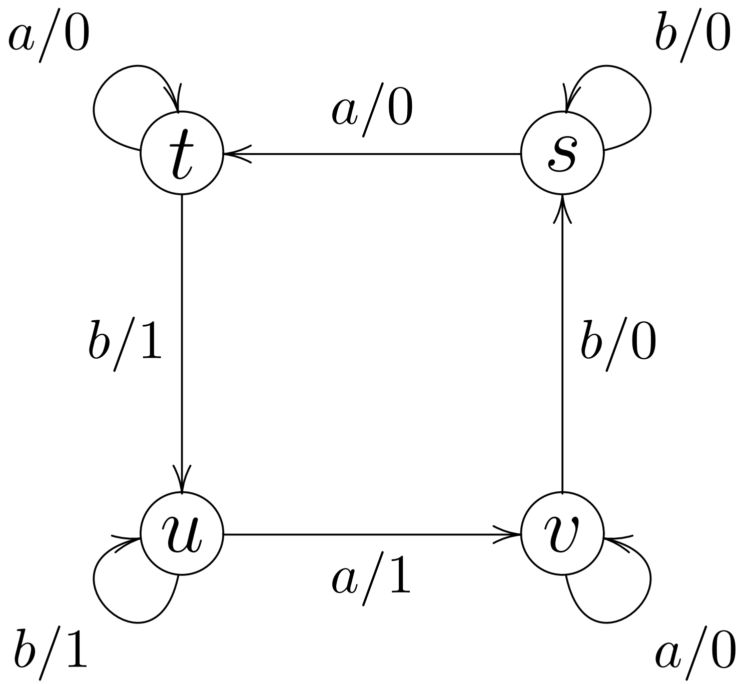
- Protocol verification

- Hardware fault-detection

Motivation for Synchronizing Sequences: Pushing Things [12]



Mealy Machines



Deterministic, *total*, finite state machine with outputs on transitions

Inputs: $I = \{a, b\}$

Outputs: $O = \{0, 1\}$

States: $S = \{s, t, u, v\}$

Mealy machine: $\mathcal{M} = \langle I, O, S, \delta, \lambda \rangle$

Inputs, I

Outputs, O

States, S

transition function ("arrows"), $\delta : S \times I \to S$

output function, $\lambda : S \times I \to O$

Synchronizing Sequences [12]

Intuitive Definition

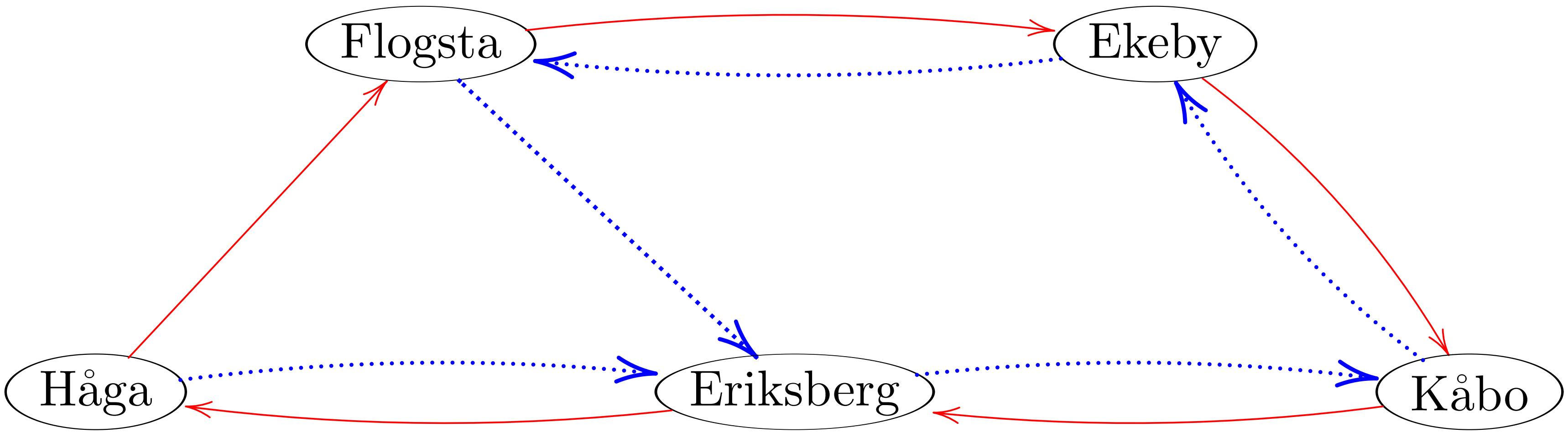
1. Initial state is unknown.
2. Apply a sequence $x \in I^*$ of inputs,
3. afterwards only one final state is possible

If this is possible, x is a **synchronizing sequence**

Formal Definition

$x \in I^*$ is synchronizing iff $|\delta(S, x)| = 1$

Example: Getting Home by Subway in Uppsala [14]



- Initial position is unknown
- There are no signs that reveal the current station
- Find your way to Flogsta, switching red and blue line as needed

Solution: brrbrrbrrbrr

Homing Sequences

Intuitive Definition

1. Initial state is unknown.
2. Apply a sequence $x \in I^*$ of inputs,
3. observe outputs,
4. conclude what the *final* state is

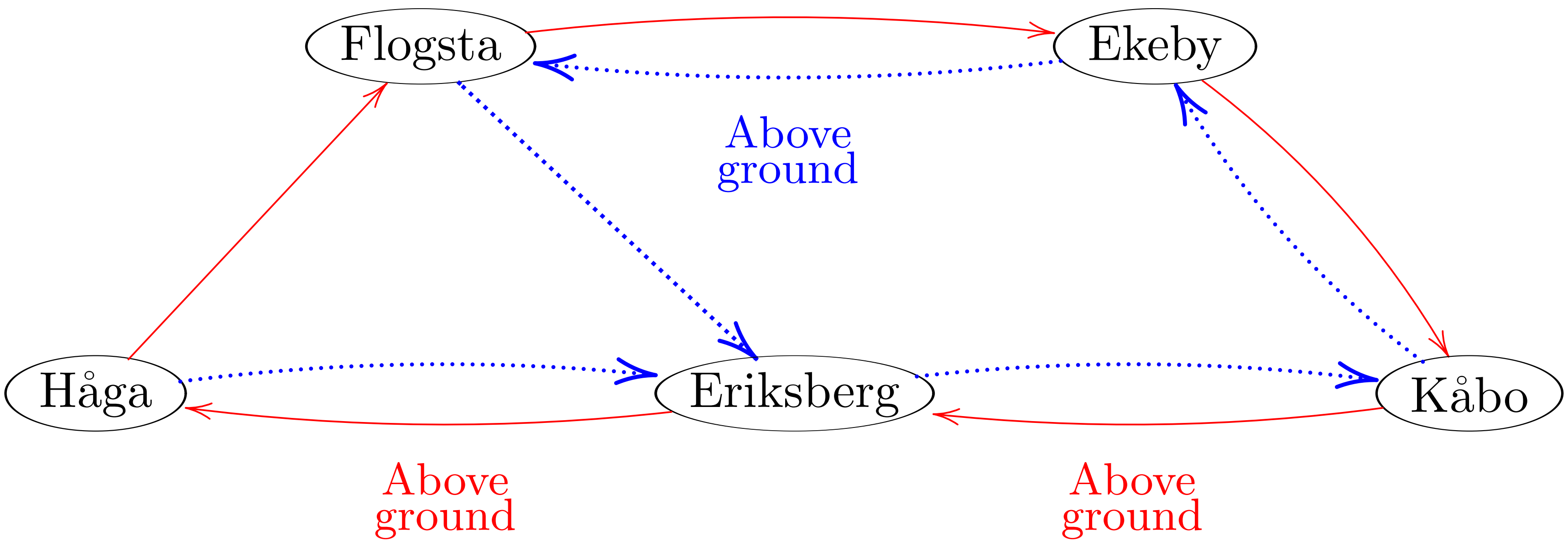
If this is possible, x is a **homing sequence**

Formal Definition

$x \in I^*$ is homing iff
for all states $s, t \in S$, $\quad \delta(s,x) \neq \delta(t,x) \implies \lambda(s,x) \neq \lambda(t,x)$

Homing Sequences: Example

- Homing sequences care about the output
- E.g., in Uppsala the subway sometimes goes above ground.
- Using this information, we can more efficiently figure out the final state.

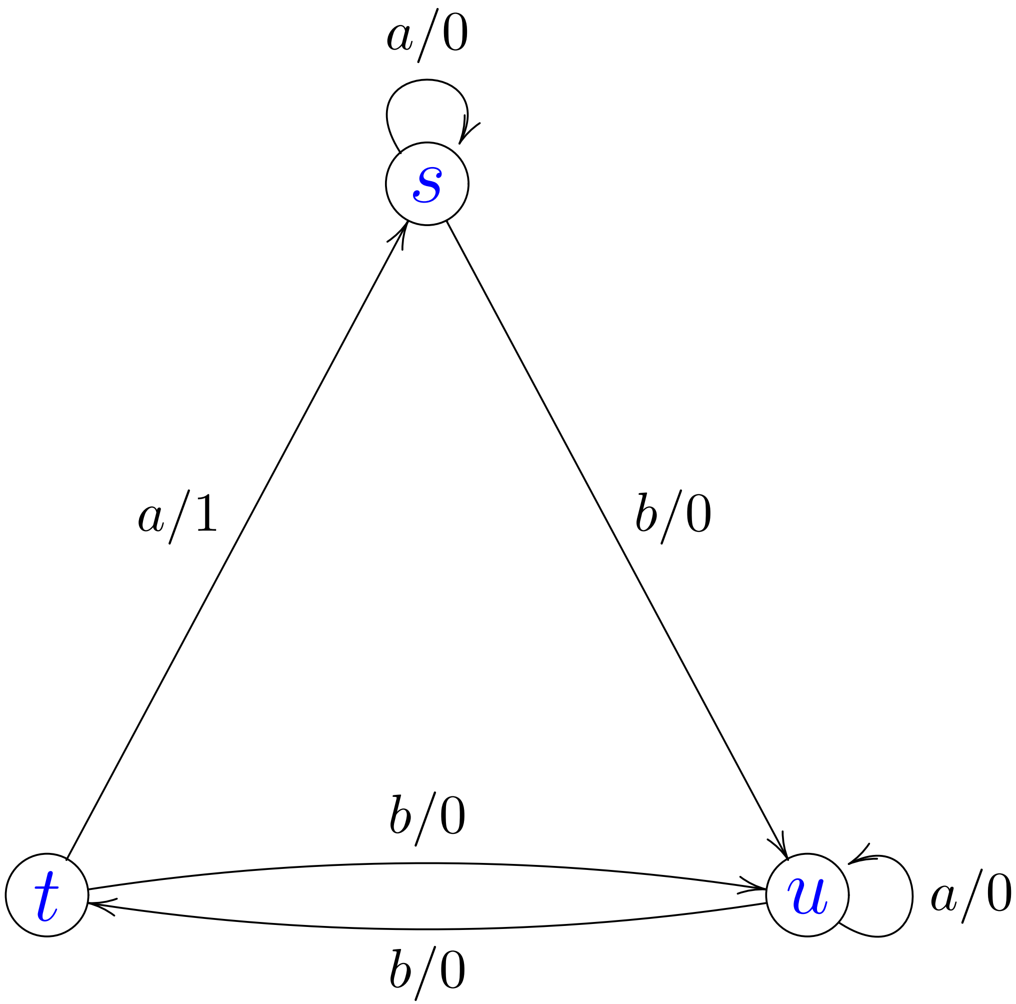


Solution: e.g., brr

Initial State Uncertainty [14]

- Data structure crucial in algorithms computing homing sequences
- The Initial State Uncertainty with respect to an input string
  "indicates for each output string the set of possible initial states"
- Formally, for an input string $x \in I^*$ it is the partition of states
  induced by the equivalence relation
  $s \equiv t \iff \lambda(s, x) = \lambda(t, x)$
  ("x produces the same output from s as from t")

Initial State Uncertainty: Example



Initial State Uncertainty: Example

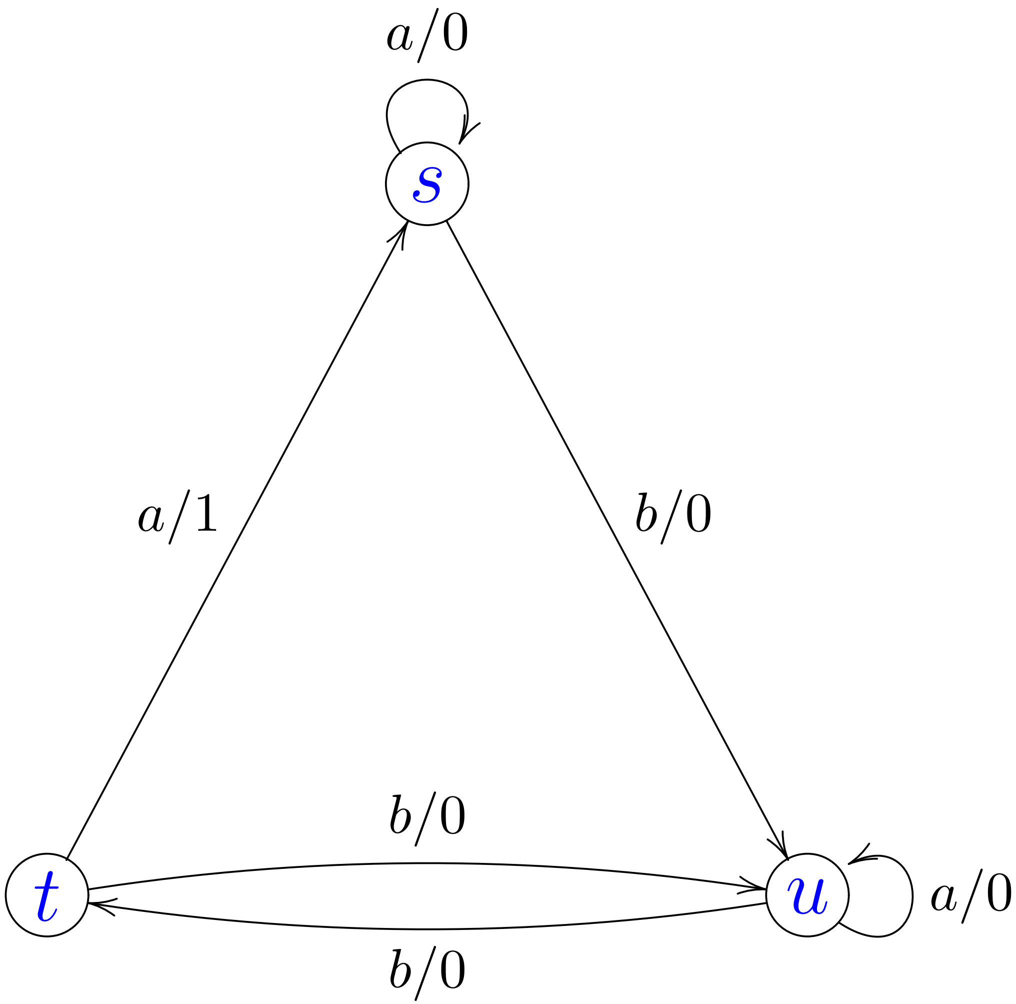
input string	initial state uncertainty
ε	$\{\{s, t, u\}\}$
a	$\{\{t\}_1, \{s, u\}_0\}$
ab	$\{\{t\}_{10}, \{s, u\}_{00}\}$
aba	$\{\{t\}_{100}, \{s\}_{000}, \{u\}_{001}\}$

(here, the output corresponding to a block is indicated in red)

Current State Uncertainty [15]

- Another data structure crucial in algorithms computing homing sequences
- The Current State Uncertainty with respect to an input string "indicates for each output string the set of possible final states"
- Formally, for an input string $x \in I^*$ it is the set $\sigma(x) \stackrel{\text{def}}{=} \{\delta(B, x) : B \text{ is a block of the initial state uncertainty w.r.t. } x\}$.
- **Important:** x is homing iff $\sigma(x)$ is a set of singletons

Current State Uncertainty: Example



Current State Uncertainty: Example

input string	initial state uncertainty	current state uncertainty
ε	$\{\{s,t,u\}\}$	$\{\{s,t,u\}\}$
a	$\{\{t\}_1, \{s,u\}_0\}$	$\{\{s\}_1, \{s,u\}_0\}$
ab	$\{\{t\}_{10}, \{s,u\}_{00}\}$	$\{\{u\}_{10}, \{u,t\}_{00}\}$
aba	$\{\{t\}_{100}, \{s\}_{000}, \{u\}_{001}\}$	$\{\{u\}_{100 \text{ or } 000}, \{s\}_{001}\}$

Computing Homing Sequences: Idea

Assume machine is minimized.

- Concatenate strings iteratively,
- in each step improving the current state uncertainty.
  ("$\sum_{B \in \sigma(x)} |B| - |\sigma(x)|$" decreases)
- Each string should be *separating* for two states in the same block:

  A *separating sequence* $x \in I^*$ for two states $s, t \in S$ gives different outputs: $\lambda(s, x) \neq \lambda(t, x)$

  Since the machine is minimized, separating sequences always exist

Computing Homing Sequences: Algorithm

function HOMING-FOR-MINIMIZED(Minimized Mealy machine $\mathcal{M}$)
    $x \leftarrow \varepsilon$
    while there is a block $X \in \sigma(x)$ with $|X| > 1$
        take two different states $s, t \in X$
        let y be a separating sequence for s and t
        $x \leftarrow xy$
    return x

Homing Sequences: Quality of Algorithm

(n = number of states, $|I|$ = number of input symbols)

- Time: $O(n^3 + n^2 \cdot |I|)$
- Space: $O(n)$
  (not counting the space needed by the output)
- Sequence length: $\leq n(n-1)/2$
  Some machines require $\geq n(n-1)/2$

Computing Synchronizing Sequences: Idea **[17]**

Very similar to algorithm for homing sequences:

- Concatenate strings iteratively,
- in each step decrease $|\delta(S, x)|$.
- Each string should be *merging* for two states in $\delta(S, x)$:
  - A *merging sequence* $y \in I^*$ for two states $s, t \in S$ takes them to the same final state: $\delta(s, y) = \delta(t, y)$
  - This guarantees that $|\delta(S, xy)| < |\delta(S, x)|$
  - Merging sequences exist for all states $\iff$ there is a synchronizing sequence

Computing Synchronizing Sequences: Algorithm

[18]

Very similar to algorithm for homing sequences:

function SYNCHRONIZING(Mealy machine $\mathcal{M}$)
    $x \leftarrow \varepsilon$
    while $|\delta(S, x)| > 1$
        take two different states $s, t \in \delta(S, x)$
        let y be a merging sequence for s and t
          (if none exists, **return** FAILURE)
        $x \leftarrow xy$
    return x

Synchronizing Sequences: Quality of Algorithm

[19–20]

- Time: $O(n^3 + n^2 \cdot |I|)$
- Space: $O(n^2 + n \cdot |I|)$
  (not counting the space needed by the output)
- Sequence length: $\leq (n^3 - n)/6$

  Černý's conjecture: length $\leq (n-1)^2$
  (true in special cases, open in general)

  Some machines require length $\geq (n-1)^2$

Homing Sequences for General Machines [20–21]

- We don't need to assume the machine is minimized
- A different algorithm solves this more general problem, but less efficiently

  Combines ideas from algorithms for homing and synchronizing sequences
- Often possible to assume the machine is minimized

Adaptive Homing Sequences [21–22]

- Apply the sequence as it is being computed,
- and let current input depend on previous outputs

- Can use modified version of the usual homing sequence algorithm

- May result in shorter sequence,
- but equally long in the worst case: $(n-1)^2$

Finding the Shortest Sequence [24–26]

- It is important to minimize the length of sequences:
  - recall pushing things
  - in testing, a machine may be remote or very slow
- Exponential algorithms have been used
- Unfortunately, the problems are NP-complete
- Even impossible to approximate unless P=NP
  (follows from NP-completeness proof)

Related Problems are PSPACE-complete [26–28]

1. Nondeterministic transition system
   (instead of deterministic)
2. The initial state is in a subset $X \subseteq S$
   (instead of S)
3. The final state may be in a subset $X \subseteq S$
   (instead of any single state in S)

Parallel Algorithms [29]

Homing Sequences

- Randomized algorithm uses $\log^2 n$ time, $O(n^7)$ processors.
  Hence, the problem belongs to RNC.
- Deterministic algorithm uses $O(\sqrt{n}\log^2 n)$ time
  Impractical due to high communication cost
- There is also a practical randomized algorithm

Synchronizing Sequences

- No known parallel algorithm
- Except one for *monotonic automata*

Conclusion

Homing sequences

- Problem is more or less solved (optimal and polynomial algorithm is known)
- Apparently more used for testing than synchronizing sequences

Synchronizing sequences

- Open question:
  Narrow the gap between upper bound $O(n^3)$ and lower bound $\Omega(n^2)$
  for the length of sequences
- Interesting algebraic properties and other applications,
  but less used for testing