Quantum algorithm for a generalized hidden shift problem

Andrew Childs
Caltech

Wim van Dam
UC Santa Barbara

What is the power of quantum computers?

Quantum mechanical computers can efficiently solve problems that classical computers (apparently) cannot.

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- Hallgren 2005, Schmidt-Vollmer 2005: Finding unit/class groups of number fields

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Questions:

- What is the computational power of quantum mechanics?
- Is public-key cryptography possible in a quantum world?
  Shor's algorithm breaks RSA, elliptic curve cryptosystems, Diffie-Hellman key exchange, etc.
  What about, e.g., lattice cryptosystems?

Generalized hidden shift problem

Given: $f(b,x) : \{0,1,\ldots,M-1\} \times \mathbb{Z}_N \to S$

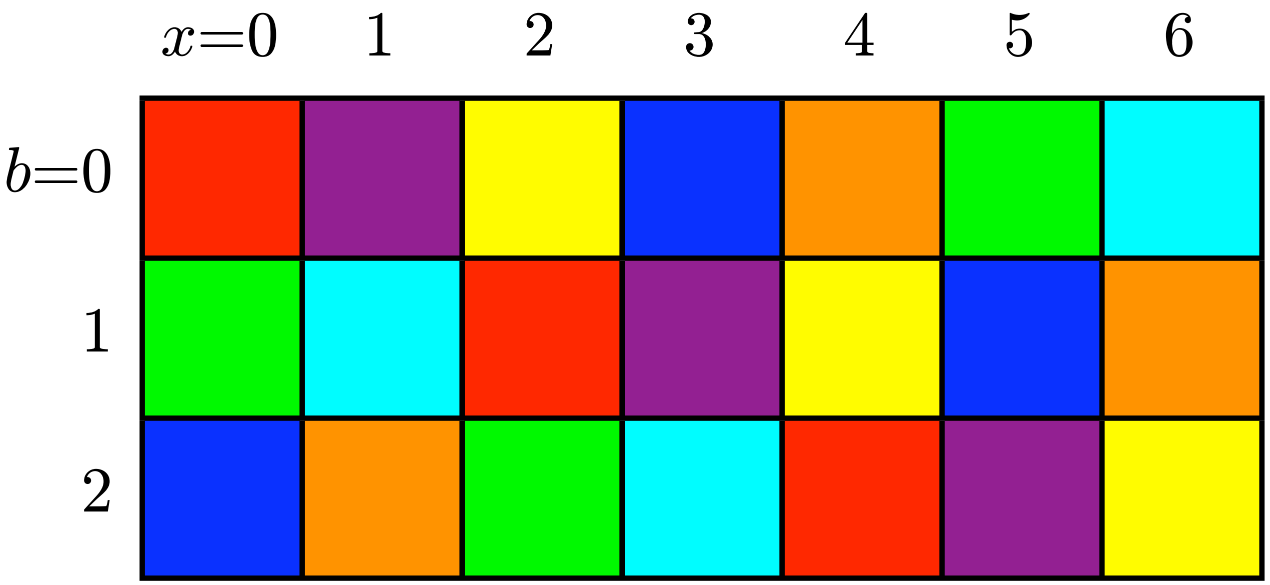
Satisfying: $f(0,x)$ injective

$$f(b+1, x+s) = f(b,x)$$

Find: s (the *hidden shift*)

$M = 2$ (hardest), … , N (easiest)

Example. $N = 7, M = 3, s = 2$



Classical complexity

Claim. To determine s, a classical, randomized algorithm must make exponentially many queries (in $\log N$) to f.

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Proof idea:

- Since the function values are arbitrary, they are not informative until we find two inputs that give the same output.
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Note: This holds independent of how big M is.

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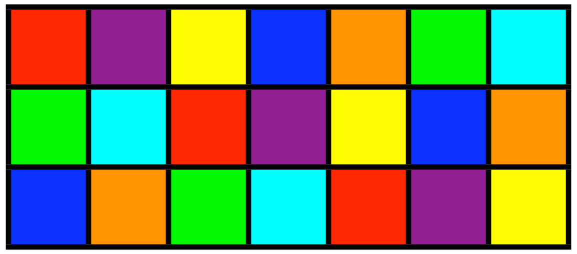

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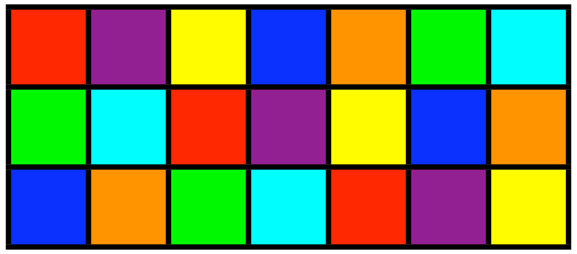
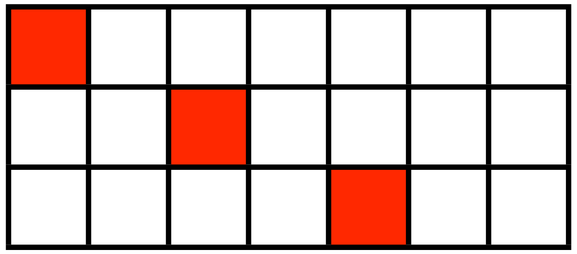


Measure function value: obtain (with equal probability)



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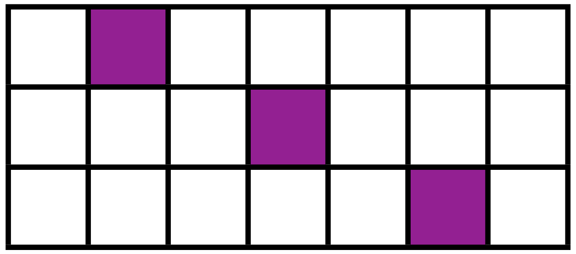

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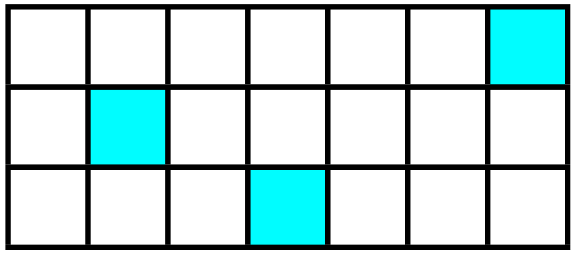

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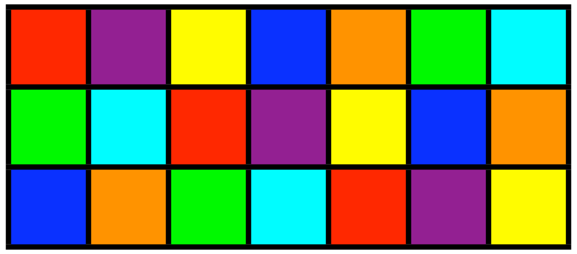

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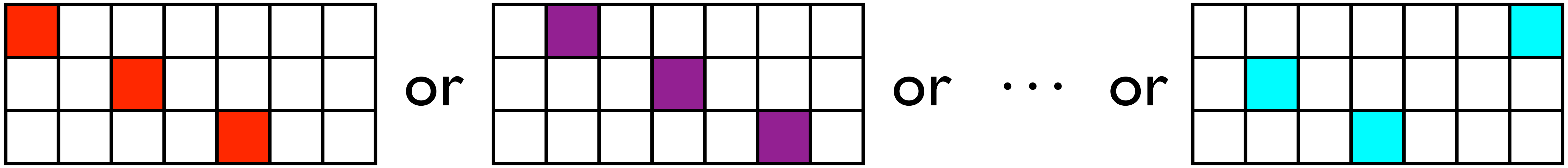

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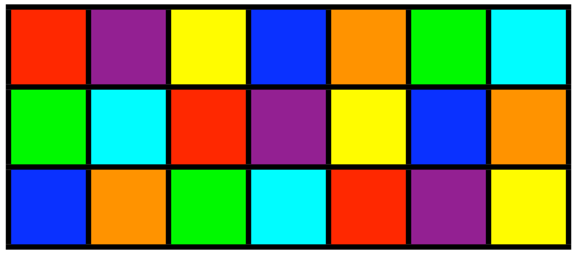

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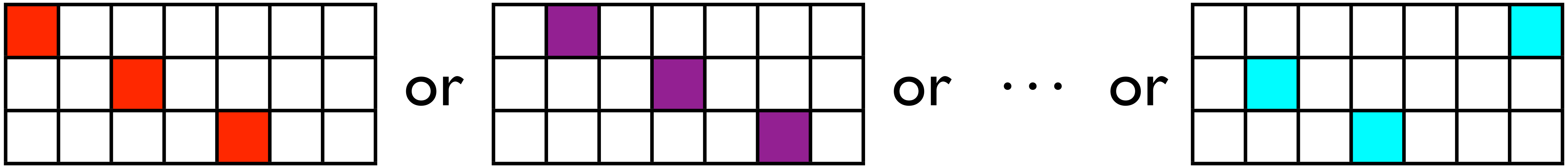

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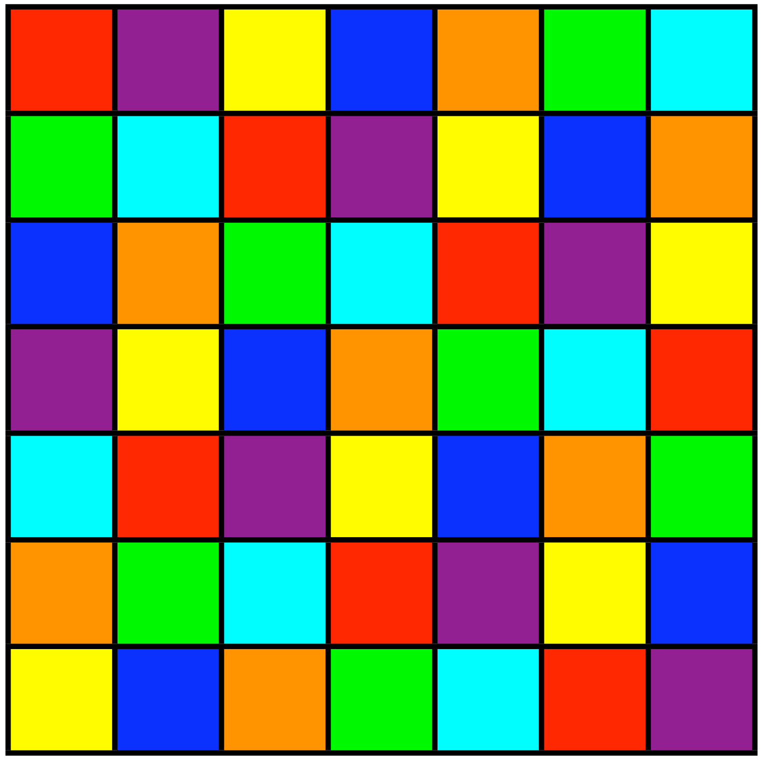
Main question: *Can we do it in* $\mathrm{poly}(\log N)$ *time?*

$M=N$: An abelian hidden subgroup problem

Easiest hidden shift problem:



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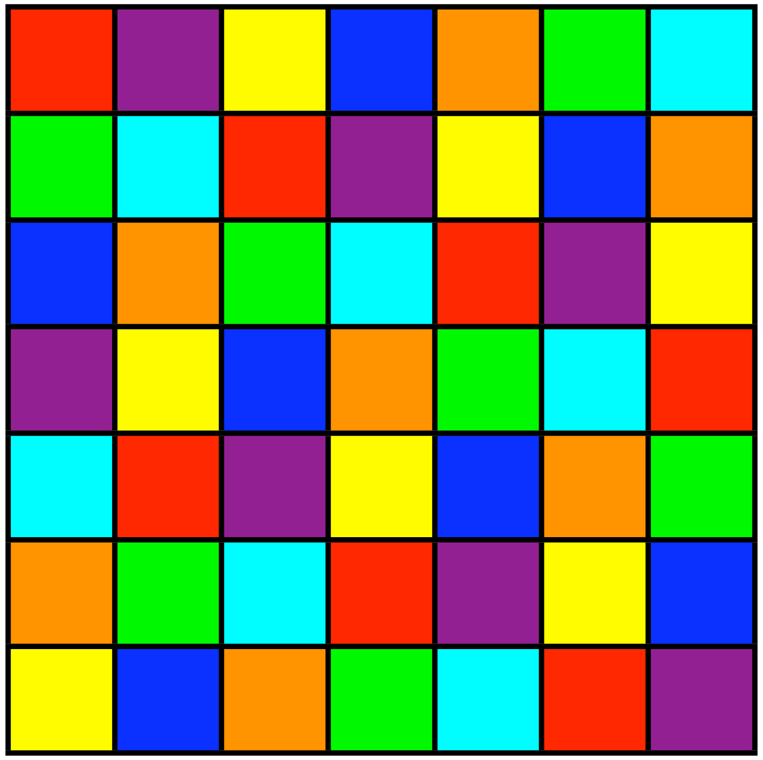

This is an instance of the *hidden subgroup problem* in the abelian group $G = \mathbb{Z}_N \times \mathbb{Z}_N$. Shor's algorithm ("Fourier transform and measure") finds s efficiently.

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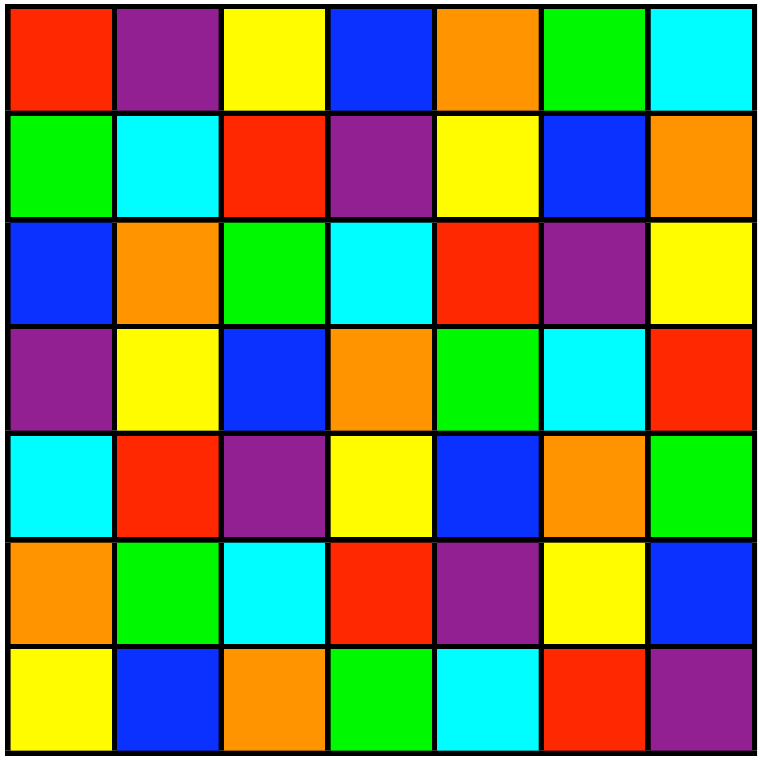

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The same approach works for any $M \geq N/\mathrm{poly}(\log N)$, but not smaller!

$M{=}2$: The dihedral hidden subgroup problem

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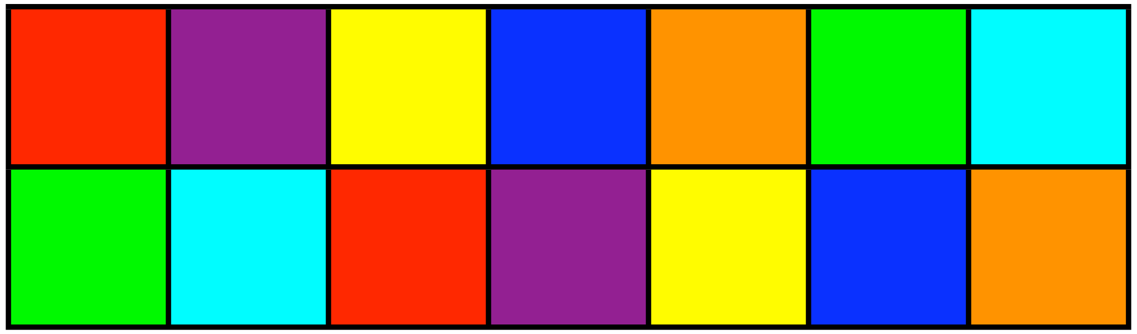

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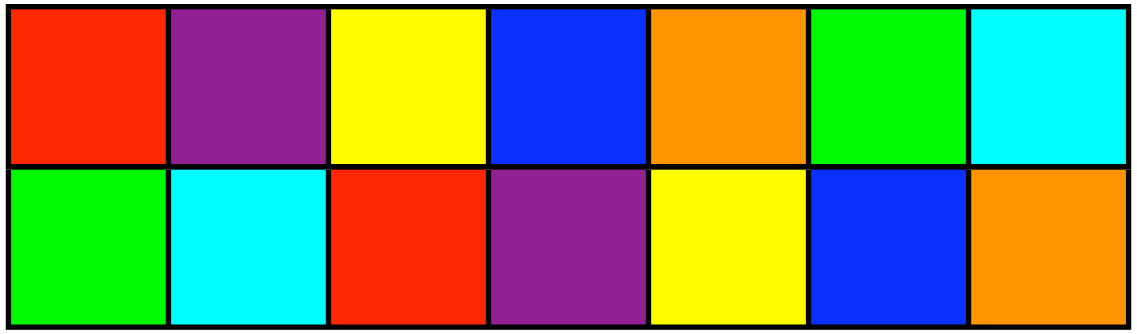

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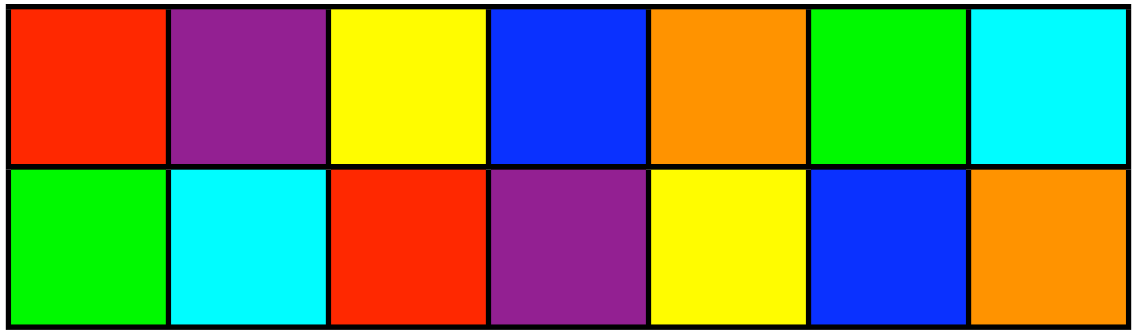

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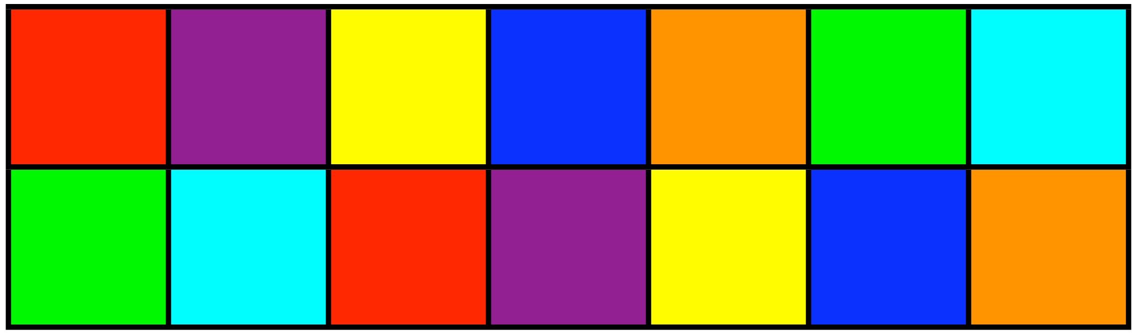

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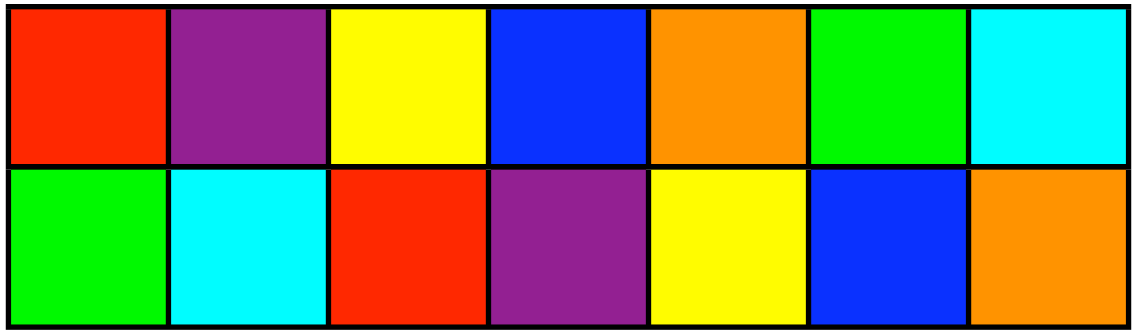

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Regev's reduction also works for larger M. Is this any easier?

Main result

Theorem. Let $M = N^\epsilon$ for any fixed $\epsilon > 0$. Then there is an efficient (i.e., run time $\mathrm{poly}(\log N)$) quantum algorithm for the generalized hidden shift problem, using entangled measurements on $k = \max\{3, \log \frac{1}{\epsilon}\}$ registers.

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Tools:

- "Pretty good measurement" on hidden shift states, à la Bacon, Childs, van Dam 2005.
- Integer programming in constant dimensions (Lenstra 1983).

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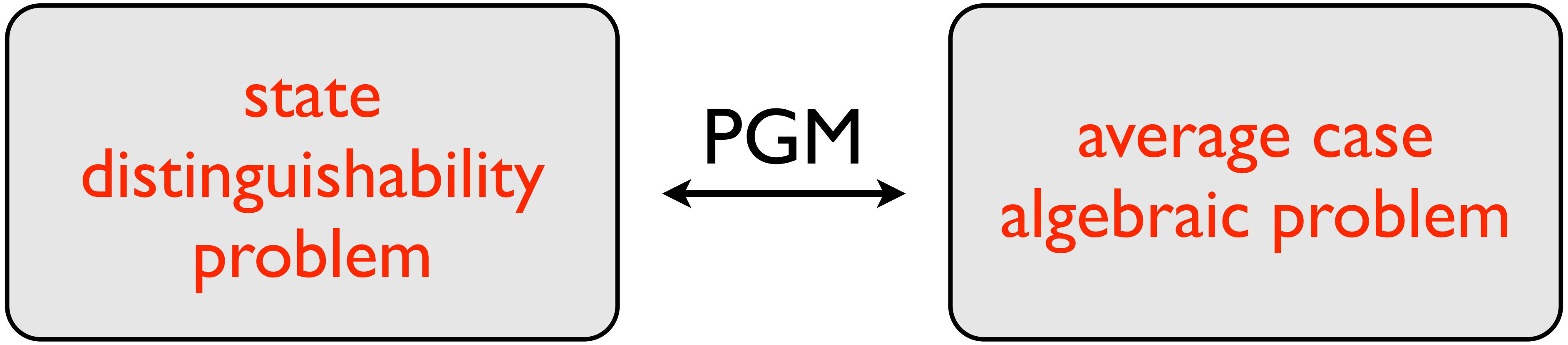

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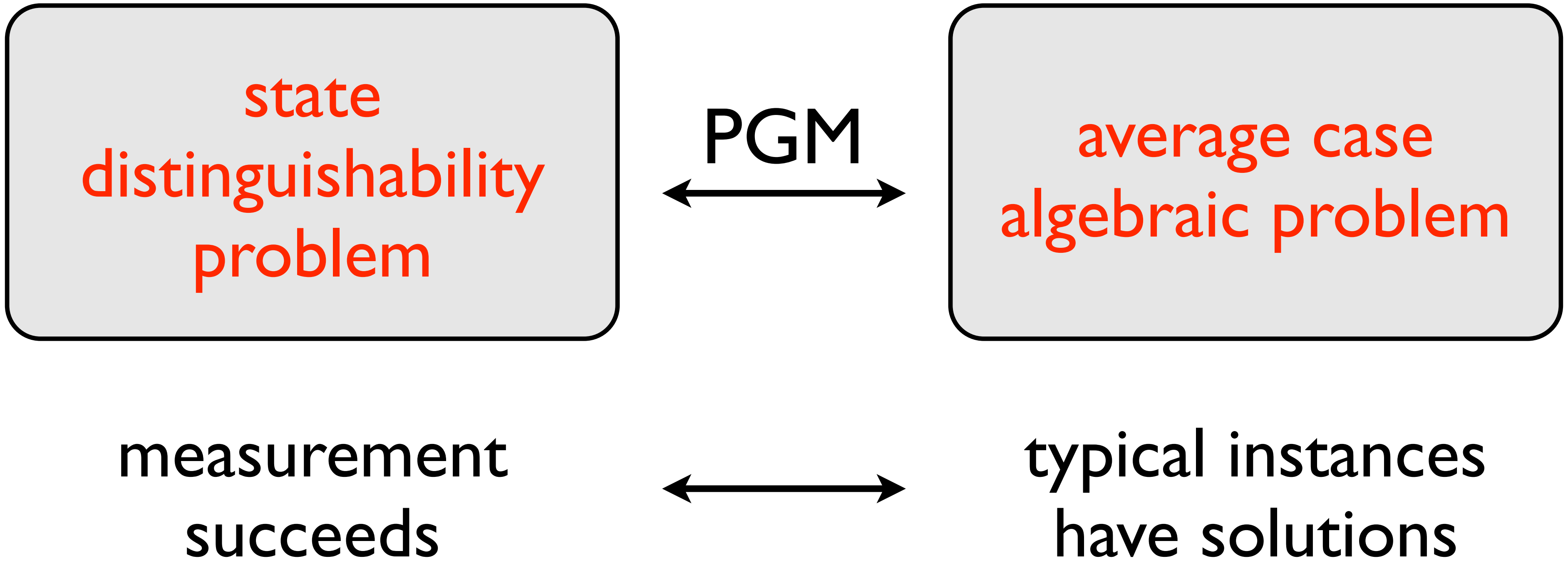

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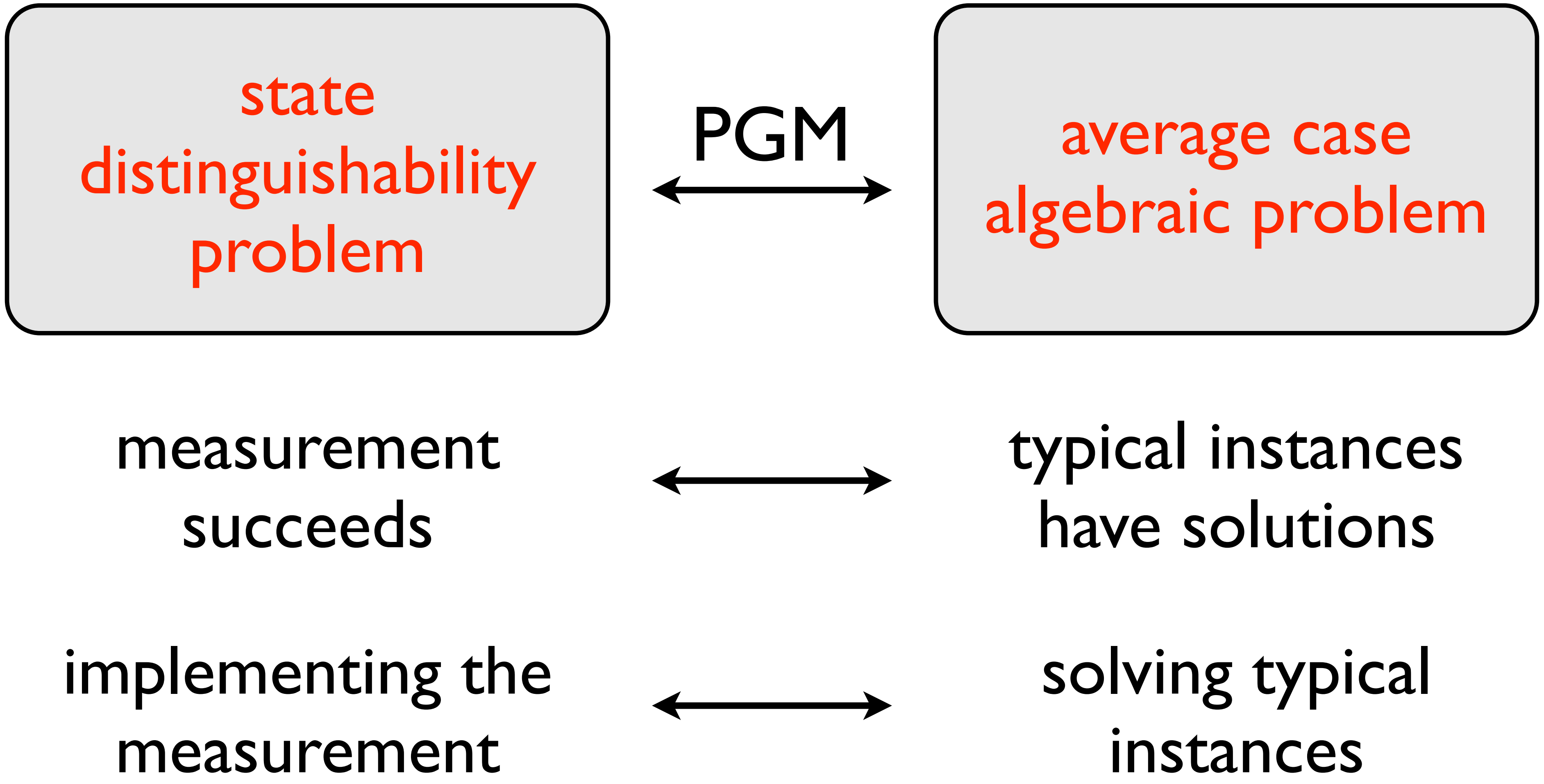

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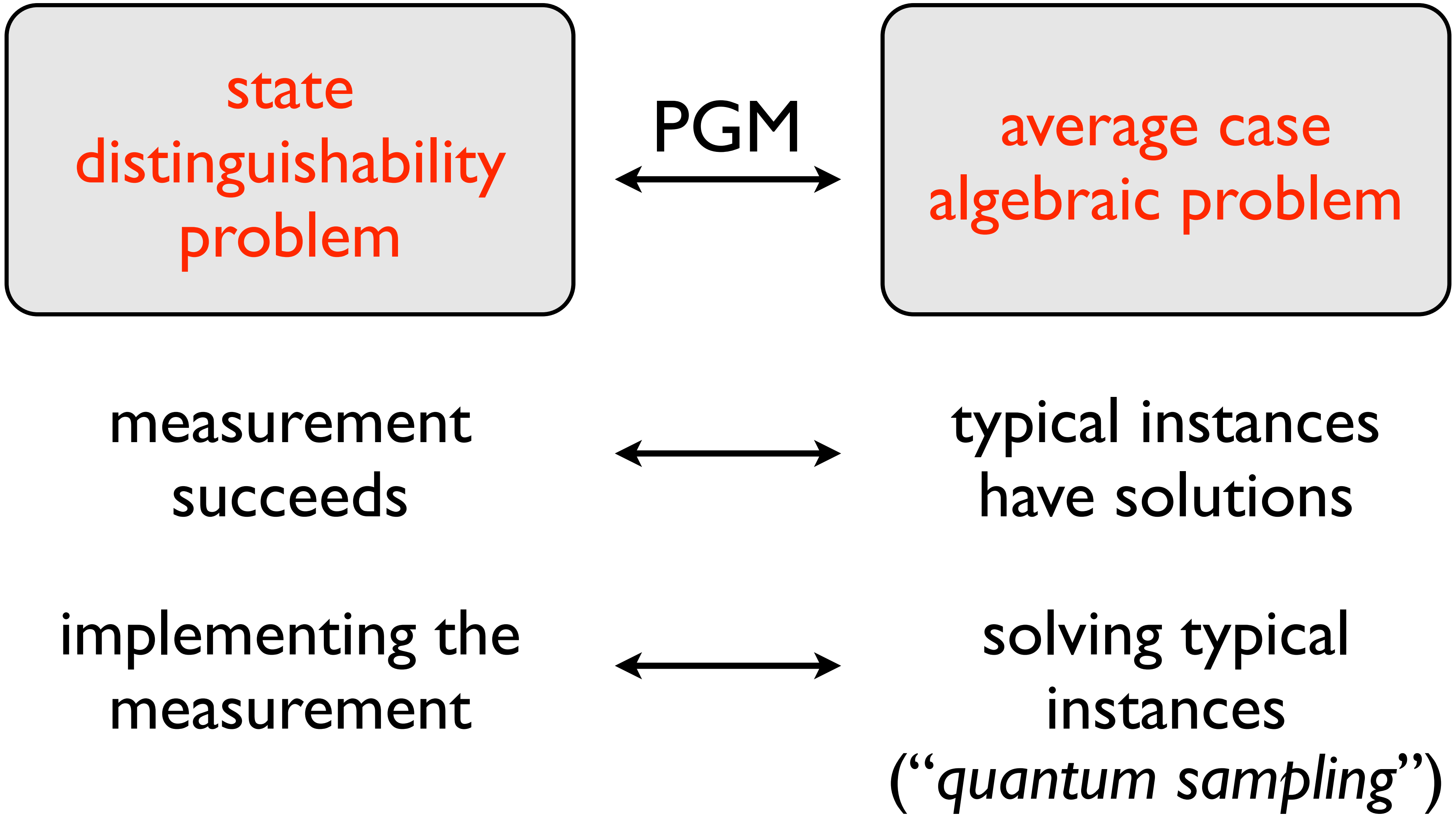

state distinguishability problem ⟷ (PGM) ⟷ average case algebraic problem

measurement succeeds ⟷ typical instances have solutions

implementing the measurement ⟷ solving typical instances (“*quantum sampling*”)

The algebraic problem

Given: random $x \in \mathbb{Z}_N^k$

random $w \in \mathbb{Z}_N$

Find: $b \in \{0, 1, \ldots, M-1\}^k$

such that $b \cdot x = w \bmod N$

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Key observation: This is a k-dimensional integer program.

- Solutions of $b \cdot x = w$ over $\mathbb{Z}$ form a shifted integer lattice
- "$\mathrm{mod}\ N$" can be enforced by adding a component
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Lenstra 1983: $2^{O(k^3)}$ time algorithm for integer programming in k dimensions (using LLL lattice basis reduction)

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$\square$

Questions

- Is the quantum solvability of the generalized hidden shift problem with $M = \Omega(N^\epsilon)$ useful for any problems going beyond factoring/discrete log?

- Can we solve the problem efficiently for smaller M?
  Can we at least interpolate with Kuperberg's algorithm?

- What if we replace $\mathbb{Z}_N$ by a nonabelian group?
  (Then even $M=2$ is not a hidden subgroup problem.)
  Can we solve this even for very large M?