Chu Spaces: Automata with quantum aspects

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Abstract

Chu spaces are a recently developed model of concurrent computation extending automata theory to express branching time and true concurrency. They exhibit in a primitive form the quantum mechanical phenomena of complementarity and uncertainty. The complementarity arises as the duality of information and time, automata and schedules, and states and events. Uncertainty arises when we define a measurement to be a morphism and notice that increasing structure in the observed object reduces clarity of observation. For a Chu space this uncertainty can be calculated numerically in an attractively simple way directly from its form factor to yield the usual Heisenberg uncertainty relation. Chu spaces correspond to wavefunctions as vectors of Hilbert space, whose inner product operation is realized for Chu spaces as right residuation and whose quantum logic becomes Girard's linear logic.

1 Introduction

1.1 Prospects for Chu Spaces

The automaton model of this paper, Chu spaces, is an outgrowth of automata theory research done in the 1980's, primarily in Europe, the US having more or less settled on the brand of automata theory arrived at by 1970. Our own *raison d'être* for Chu spaces is as a conceptual foundation for parallel programming and computer architecture. There is an intensive ongoing search for an attractive framework that can applied reliably to the modeling of computational behavior in the same general way that vector spaces are heavily used today as a basis for computational geometry and computer graphics. Chu spaces are our current favorite contender for this role.

This paper digresses from our main goal to pursue possible connections with quantum mechanics. We see a reasonable possibility that such connections could prove useful in both computer science and physics.

The full picture of Chu spaces that has emerged for us during the past two years since we began using them is beyond the scope (or at least available space) of this paper. We therefore refer the reader to other recent work [Pra94, Gup94, GP93, Pra93b], in that order. These papers are all available either as cited, or by anonymous ftp (start with /pub/ABSTRACTS from boole.stanford.edu), or via World-Wide Web (WWW) via "mosaic http://boole.stanford.edu".

In computer science Chu spaces as an automata-theoretic abstraction of quantum mechanics could well serve as a bridge between the familiar world of automata and the conceptually remote world of quantum mechanics. Such a bridge may ease the transitions of computer architecture to quantum devices, and of algorithms to quantum computing, by indicating how to relate concepts of quantum mechanics to a suitable blend of classical mechanics and automata theory, thereby capitalizing on familiar automata-theoretic intuitions. We continue this theme in the subsection below on applications to quantum computing.

For physics, Chu spaces offer an even more abstract quantum mechanics than presently available. Quantum mechanics is already very abstract: although it deals ostensibly with momentum p and position q, these may be interpreted as any conjugate pair of attributes such as angular momentum respectively along any two orthogonal axes in 3-space. General quantum reasoning is more fundamental than particular attributes such as space, mass, charge, gravity, relativity, etc. Pure quantum mechanics in its own right exists independently of quantum spin, quantum geometry, quantum field theory, quantum gravity, quantum white rabbits, etc. In this respect it is like group the-

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ory: it can be studied as a pure subject in its own right, albeit at some pedagogical cost.

What QM has *not* let go of is the field of complex numbers, which is taken to be an immutable and distinguishing feature of all quantum phenomena. Chu spaces show how to abstract away even that last vestige of quantum identity, not just to a group of complex numbers or even a monoid, but to a pure set!

This extreme level of abstraction is ordinarily confined to set theory and number theory. A set is a relational structure with the empty language. Without language such a structure can signify nothing beyond cardinality. Not even quantum logic [BvN36] abstracts that far, retaining the binary relation of orthogonality on an "orthoframe" as the primitive dictionary entry for its Kripke semantics, and the logical connectives of conjunction (interpretable semantically as for classical Boolean logic as intersection of closed subsets of the orthoframe) and negation (interpretable via orthogonality as "not mistakable for") as the primitive lexical items for its finite equational axiomatization.

Chu spaces replace language by the complementarity of quantum mechanics. How powerful is complementarity? We shall show that all of first-order mathematical language, including its meaning, is recovered, along with (quite literally) exponentially much more. Separable Hilbert space is recovered, but now in a small corner of a much larger universe.

Quantum mechanics has since its inception progressed along the two not altogether independent axes of *simplicity* and *generality.* A fundamental step in this development was Schrödinger's discovery of the duality between his wave mechanics and Heisenberg-Born-Jordan matrix mechanics, thereby putting Bohr's intuitions about complementarity on what later proved, thanks to von Neumann, to be a completely rigorous mathematical basis.

It seems to us that Chu spaces have the potential to continue this progress, not merely a long way but "all" the way. Our thesis is that the category **Set** is the ultimate abstraction of body, and that $\mathbf{Set}^{\mathrm{op}}$, equivalent to the category of complete atomic Boolean algebras (i.e. power sets), which we shall advocate thinking of as *antisets*, is dually the ultimate abstraction of mind. A Chu space over a set K is simply a K-valued binary relation from a set to an antiset. We submit as circumstantial evidence for this thesis the above-mentioned universality of Chu spaces. The thesis can be contradicted by improving on **Set** (and hence $\mathbf{Set}^{\mathrm{op}}$) in a suitable way. We are not presently aware of a more appropriate basis for the Chu construction than **Set**, and until one appears we shall stick to our thesis.

A core simplification that Chu spaces would achieve for quantum mechanics is the reduction of wave-particle duality to matrix transposition. When the Pontrjagin duality of locally compact Abelian groups (the category **AbGrp**) is taken as the mathematical essence of wave-particle duality, the realization of Abelian groups as Chu spaces realizes this duality simply and very surprisingly as matrix transposition.

1.2 Quantum Logic is Incomplete

We argue here that quantum logic lacks the capabilities that give Chu spaces their potential for quantum mechanics. Whereas Chu spaces single out complementarity as the essence of quantum mechanics, quantum logic abstracts away from complementarity in favor of projectivity. In essentially the same sense that Boolean logic is the logic of sets and intuitionistic logic that of partially ordered sets, quantum logic is the logic of abstract projective geometry.[1]

If we define mathematics to be the gamut from pure sets to pure logic, then Boolean logic, intuitionistic logic, and quantum logic all sit at the logic end of this gamut. Chu spaces in contrast run the full gamut, and hence are neither sets nor logics but *general mathematical structures.* Moreover they are exponentially more general than the relational structures constituting the domain of discourse of all of first-order-definable mathematics.

Quantum logic has thrown out the baby with the bathwater. Although projective geometry is an *accompanying* characteristic of standard quantum mechanics, it is far from being its *defining* characteristic, which we maintain is complementarity. When complementarity is made the basis for abstract quantum mechanics, as with Chu spaces, the complexity is stripped away as with quantum logic[2] but the essence of quantum mechanics is retained. *Projective geometry as the essence of quantum logic is far too simple to constitute the essence of quantum mechanics.*

Just as Boolean algebra is the natural logic of sets and Heyting or intuitionistic logic that of ordered sets, linear logic is the natural logic of Chu spaces. As we

[1] It can equivalently be understood as the logic of commuting operators, based on an arbitrary reflexive symmetric relation interpreted as commutability, but from its very beginning it has customarily been described from the dual perspective of projectivity, perhaps because the associated W^*-algebras of operators would have reduced the accessibility of a framework whose sheer novelty at the time already presented enough of a conceptual challenge.

[2] In fact even more so, Chu spaces viewed equivalently as either matrices or Boolean propositions being conceptually simpler than ortholattices.

discussed at length at the previous meeting [Pra93a], linear logic resembles quantum logic in some respects while improving on it in others. The resemblances are in details such as rejection of distributivity of conjunction over disjunction and acceptance of double negation, differentiating both logics from Boolean logic (which accepts the former) and intuitionistic logic (which rejects the latter).

A key difference is in the foundation of entailment $A \vdash B$. In all of Boolean, intuitionistic, and quantum logic, entailment is a truth value: A either does or does not entail B. In linear logic interpreted for Chu spaces, entailment is a set of alternative *proofs* of B from A, understood either as reasons why fact B should follow from fact A, ways of getting from point (of view) A to point B, or ways of transforming object A into object B. These proofs compose to make the semantic basis not a poset as with nonconstructive logic but rather a category. We argued there that while nonconstructive logic was all very well for armchair philosophers, anyone planning to actually *use* logic in the field is not adequately served by nonconstructive logic and should be reasoning constructively. Knowing that you can get from A to B is practically useless if you do not know even one method of doing so!

The universality of Chu spaces translates this into the concrete result that constructive linear logic *is* mathematics, in the sense that its proofs are the constructions of mathematics, namely its homomorphisms or structure-preserving functions. This is to be contrasted with first-order logic, which is a nonconstructive *symbolic* logic of mathematics.

1.3 Relevance to Cosmology

Our proposed role for Chu spaces as the ultimate abstraction of quantum mechanics is relatively clear, if not yet as convincing as we would like. In a rather more speculative vein, we may consider the significance of such an abstraction for cosmology. Now a popular premise of cosmologists of all stripes is that the space of competing conceptual frameworks must be very large, raising all sort of knotty questions such as, why this universe, could the universe have supported intelligent life if the periodic table had been organized differently, would TV have been a passing fad had there never been color, etc.

The universality of Chu spaces undermines the premise of this question, by raising the possibility that *Chu spaces have no competition,* in that any conceptual framework we have a chance of understanding can be found embedded in Chu spaces. This simplifies the procedure for manufacturing a universe by permitting the step of choosing a framework to be omitted.

Attention then focuses on quantity: how big to make the universe? Here Chu spaces have nothing to suggest. Nothing intrinsic in either cosmology or the present theory of Chu spaces suggests either Eddington's original number $2 \times 136 \times 2^{256}$, for the number of protons and electrons in the universe, nor its modern counterpart $2^{127} + 136$ (squared?), the basis for the "combinatorial hierarchy" popular with the neo-Eddingtonians (who I note have their own session at this conference). *So far as we know,* the universe would have worked just about as well at a quarter or a trillionth of its present size. On the other hand I rather doubt one would be well served by being teleported to a Chu space with only 2^{40} elements—not enough room for a conventional sun, a nuclear power station would be a must for maintaining temperature and atmosphere, and there would probably be a disconcerting lack of inertia.

We therefore ask, *how does the quality of a universe depend on its size?* Based on the existence of Chu as a universal framework, we gingerly advance the thesis that there are only two integers of any relevance at the exact moment of the Big Bang, the number of values permitted as entries of a Chu space (less than 100 surely suffices, and 2 is not inconceivable), and the integer size of the universe, which presumably differs from Eddington's number in not too many bits.

2 Recent Developments in Automata Theory

The classical 1970's conception of an automaton was as a device for accepting a formal language defined as a set of strings, possibly infinite in the case of so-called ω-automata. This conception made two automata equivalent when they accepted the same language. As models of behavior, each string of the accepted language was considered as one of the alternative or *possible* behaviors of that automaton, and the symbols in that string all occurred during that behavior, in the order of occurrence in that string.

The new automata theory raised two objections to this conception. The first was raised by Robin Milner in his book on CCS, a Calculus of Communicating Systems [Mil80]. The standard model appears to condense all choices about behavior into a single selection of a string from a language made at the start of the behavior. Real behavior however makes informed decisions on the fly as information comes to hand. Milner proposed a logic that took deferred branching into account by abandoning the equation $a(b+c) = ab + ac$, along with a model, synchronization trees, to serve as counterexamples for this equation.

The second objection, raised sporadically by various people over a long period [Pet62, Gre75, Maz77, Gra81, NPW81, Pra82], was that the standard model assigned a well-defined order to every pair of events (symbol occurrences) in the same string. Besides contradicting relativity, this assumption also contradicts practical engineering issues at all scales, from "data skew" on parallel signal lines within a single chip to detecting when a husband and wife are simultaneously making withdrawals from the same account at remote automatic teller machines.

A succession of models addressed these two issues during the 1980's, initially separately and then later jointly. R. van Glabbeek's thesis [vG90] provides a comprehensive summary of the state of the art in 1990. This paper is based on a model that is, we feel, a particularly clean example of the state of that art. It has two main sources for its basic structure, the event spaces of Winskel [NPW81, Win88], and the *-autonomous categories of Barr [Bar79], originally done entirely independently of any possibility of its application to computer science. More specifically it makes use of those *-autonomous categories arising from a construction studied by Barr's student Chu and reported in an appendix to Barr's book. Chu spaces specialize this construction to the category **Set**, greatly increasing its accessibility while at the same time nicely matching these new requirements imposed on automata theory, and with the added bonus that the loss of generality in passing from $\mathbf{Chu}(V, k)$ to $\mathbf{Chu}(\mathbf{Set}, K)$ appears to be negligible in practice in comparison with what is lost in the passage from V (any symmetric closed category with pullbacks) to **Set** (a very constrained instance of such a V).

This model also picks up where we left off in [Pra93a], where we proposed linear logic as an extension of quantum logic that equipped it with a dynamics, a glaring omission from Birkhoff and von Neumann's original formulation [BvN36]. At the end of that paper we briefly hinted at "partial distributive lattices" as a potentially superior model to the state and event spaces we had presented as a model of our extension of quantum logic:

> In a separate paper we will describe a uniform generalization of state and event spaces to a single category PDL of partial distributive lattices. Informally a PDL is a distributive lattice where any given meet or join may or may not be defined. Maps of such preserve those meets and joins that exist.

The "will" was rather optimistic and should have been "hope to," given that at that time we had only a hazy concept of this notion, and no nice properties. Chu spaces turn out to be an ideal realization of that concept, as well as a natural limiting case of the progression of the various European automaton models of the 1980's.

Besides being simple to define, Chu spaces are also definitionally robust in having (at least) three strikingly different definitions, with each having a situation for which it is the most appropriate of the three. Moreover they create certain links between automata theory, model theory for first order logic, and (the theme of this paper) quantum mechanics.

The objects of this model can be understood as binary relations from an "antiset" to a set. Complementarity is then found in the duality of antisets and sets, while uncertainty derives from the "area" of each object, which can be understood as a sort of phase space.

This notion also clarifies the connection between Stone duality and Pontrjagin duality, which both rely on the same basic mechanism with the former based on the two truth values and the latter based on the unit circle. Pontrjagin duality accounts for the Fourier transform; Stone duality is its counterpart for logic, and the associated transform is the contravariant power set functor. "Uncertainty" is intrinsic to the Fourier transform; complete certainty transforms to complete uncertainty. The Heisenberg uncertainty principle $\Delta p.\Delta q \geq \hbar$ of quantum mechanics is a specialized instance of this phenomenon.

3 Definitions

3.1 Basic Notions

Vector spaces are defined by first fixing an arbitrary *field* k, which then determines the category of vector spaces over k and their operators (linear transformations). Chu spaces are similarly defined by first fixing merely an arbitrary *set* K. A *Chu space* $\mathcal{A} = (A, X, \models)$ *over* K consists of sets A and X and a function $\models: X \times A \to K$. The binary application $\models(x, a)$ may alternatively be written as $x \models a$ and pronounced "x satisfies a", or as $a \mathrel{=\!\!|} x$ and pronounced "a holds in x". The *dual* of $(A, X, \models)$, written $(A, X, \models)^\perp$, is simply its transpose $(X, A, \mathrel{=\!\!|})$, itself a Chu space.

The *rows* of $(A, X, \models)$ are the functions $\rho_x : A \to K$, one for each $x \in X$, defined by $\rho_x(a) = \models(x, a)$. This makes ρ itself a function $\rho : X \to K^A$ (K^A is synonymous with $A \to K$, the set of all functions from A to K). An *extensional* Chu space is one for which ρ is injective. A *normal* Chu space is one for which ρ is the identity function on a subset $X \subseteq K^A$; the data for a normal Chu space may be abbreviated to (A, X), the binary application $\models(x, a)$ then being understood as unary application of x to a. The dual notions to "row" and "extensional" are respectively *column* and T_0 (since topological spaces are T_0 just when their Chu realizations are T_0).

The smallest nontrivial choice of K is the set $2 = \{0, 1\}$. This choice is of use for propositional logic, Stone duality, and automata theory. A considerably larger set is $\mathbb{C}$, the complex numbers, used in signal and image processing, Pontrjagin duality, and quantum mechanics. Chu spaces tie together these and other areas, including point set topology, Scott domain theory, vector spaces, and relational structures of the kind interpreting first order logic, using less machinery altogether than any one of them individually (due in effect to the nontrivial overhead of identifying any given subcategory of $\mathbf{Chu}_K$). In addition Chu spaces are very well organized, being furnished with the structure of a category that is bicomplete, symmetric monoidal closed, self-dual, concrete (A), and coconcrete (X), about as much fundamental structure as one could ever wish for in a category. This structure is preserved to varying degrees in each of these application areas; typically self-duality and the natural closed structure are lost (which quantum mechanics goes to considerable lengths to regain!), and often cocompleteness as well.

3.2 Behavioral Interpretation

The interpretation of Chu spaces relevant to this paper, both for automata theory and quantum mechanics, is as an automaton with state set X. Such an automaton may be viewed either *declaratively* or *imperatively.*

In the declarative view, each $a \in A$ is an atomic proposition or propositional variable, and $x \models a$ gives the truth value of that proposition in state x, as an element of K. Propositional Boolean logic obtains for $K = 2$, where a proposition is either true or false in any given state. A normal Chu space may then be understood as an (abstract) Boolean proposition $\varphi(A)$, with its rows as its satisfying assignments. This proposition asserts all and only the properties of $\mathcal{A}$, understood as all Boolean consequences of $\varphi(A)$ expressible with the variables of A. Each property is determined by those missing rows of $\mathcal{A}$ that are restored, whence $\mathcal{A}$ must have exactly $2^{2^A - X}$ properties. Quantum mechanics obtains for $K = \mathbb{C}$, where an atomic proposition such as "spin up" or "spin down" or "energy level 3" has a complex "truth value" at position x in space (Heisenberg) or space-time (Schrödinger). Hilbert spaces are a tiny fragment of $\mathbf{Chu}_C$; our thesis however is that the proper way to shrink $\mathbf{Chu}_C$ is not by taking either subobjects or quotients of $\mathbf{Chu}_C$ (the self-duality of Hilbert spaces means that the two approaches are the same) but by taking $K \ll \mathbb{C}$.

In the imperative view a becomes an *event.* The notion of event is a delicate one, sitting between the two closely related notions of *action* and *transition.* An event is the *performance* of an action. Thus whereas a given action may happen repeatedly, a given event can happen at most once. A *labeled* Chu space associates with each event the action of which it is an occurrence; we shall not say any more about labeled Chu spaces. See [Pra94, Gup94] for a considerably more detailed treatment of automata and schedules, including the use of partial distributive lattices or two-toned Hasse diagrams to depict Chu space automata and schedules graphically. This topic is of considerable interest to us and we wish we had more space to go into it here.

A *transition* is a relationship between two states. For $K = 2$ we say that there is a transition $x \to y$ from x to y just when for all a, if $x \models a$ then $y \models a$. Thus the notion of transition expresses exactly the idea that *what is done is done,* there is no taking back an event. Transition is a preorder (transitive reflexive relation) on states, and is a partial order (antisymmetric) if and only if the Chu space is extensional; for a normal Chu space it is the inclusion order on states as sets of events. The *effect* of transition $x \to y$ is the set of events that happened during that transition, namely $y\models - x\models$. An *atomic* transition is one with a singleton effect: exactly one event happens.

For $K = 2$ there is an algebraic expression for the transition relation in terms of ordinary binary relation composition of Chu spaces. Writing $\to$ for the transition relation on X and $R\dagger$ for the complement of the converse of any binary relation R (that is, uRv just when not $vR \dagger u$), the reader may verify that $\to\; = \;(\models; \models \dagger)\dagger$. The operation $R\dagger$ plays the exactly analogous role for binary relations that conjugate-of-transpose does for vectors and matrices over $\mathbb{C}$ in such expressions as $\langle\psi|\psi\rangle$ as a between-states relationship computed by summing over the atomic propositions or attribute-outcomes of ψ.

The expression $(R; S\dagger)\dagger$ plays a fundamental role in the calculus of binary relations. It is called the *right residual of* R by S, standardly notated $S\backslash R$. It has the property $S;(S\backslash R) \subseteq R$, and is the greatest relation (under inclusion) with that property. If we think of ; as a (noncommutative) conjunction and $\subseteq$ as logical entailment $\models$, this property amounts to modus ponens, and makes $S\backslash R$ the corresponding intuitionistic implication, S implies R.

The above expression for $\rightarrow$ then becomes $\models \backslash \models$, making it the right residual of $\models$ by itself. Elsewhere [Pra90] we have shown that the equation $(R\backslash R)^* = R\backslash R$, namely that $R\backslash R$ is its own reflexive transitive closure (the meaning of * here), amounts to an induction axiom leading to a complete and finite equational axiomatization for the theory of regular expressions expanded with right residuation and its dual left residuation. This creates a remarkable connection between the equational logic of the iterative behavior of finite state automata and the form $\langle\psi|\psi\rangle$ basic to the computation of probabilities or correlations between states. This connection urgently needs to be developed further.

It should now be clear that *even though an event can only happen once, it may nevertheless be associated with many transitions.* We think of the event as a physically real thing and the many transitions it participates in as mental abstractions in our *interpretation* of what is going on, distinguished according to the states we are contemplating at the time this event becomes relevant.

The QM Interpretation Problem. Lifted to $K = \mathbb{C}$, this interpretation of events in terms of transitions suddenly becomes a much more intricate one. We claim that *the interpretation problem for quantum mechanics resides in the difficult notion of state transition.* There is nothing wrong with Schrödinger's model by itself, which we can understand simply as a Chu space. What is impossibly complicated is the notion of transition as a concept in its own right. We understand intuitively what we mean by transition, and for Chu spaces over 2 this intuition is rendered concrete in a reasonably coherent way when we associate each state transition $x \rightarrow y$ with a *set* of events. General state transitions in Hilbert space however are a conceptual abomination: they entail arbitrarily much "phase entanglement."

The interpretation problem for quantum mechanics is the problem of relating the obscure notion of transition as understood quantum mechanically to our perceived notions of transition, which somehow we do not find obscure even though the logic of quantum mechanics tells us that we should. The basis for perceived transition is perceived state, which is different from the quantum notion of state as intensionally a point x in space (or x, t in space-time) and extensionally the value $\psi(x)$ of the universal wavefunction at that point.

While this should be a reasonably clear statement of the interpretation problem, we regrettably have little to add to the many proposed solutions besides the following. Our own position is on the pessimistic end of the spectrum of interpretations, namely that we have evolved to perceive only computationally tractable approximations to state and transition, and that the rest, von Neumann notwithstanding, constitutes hidden variables.[3] Such an approximation is more or less nicely understood as a mixed state in the sense of a distribution over Hilbert space, but the domain of mixed states as "blurry" points of Hilbert space makes this less rather than more. Not only is the appealing algebraic structure of the domain of pure states as the points of Hilbert space lacking, but the blurring, whatever it is exactly, is apparently not itself a solution to the Schrödinger wave equation, and may be many other bad things besides. This all leads to seemingly paradoxical faster-than-light correlations between spins of formerly associated particles, violating the Bell inequalities and confusing even such luminaries as Einstein.

A more optimistic resolution would be to show that quantum mechanics can be as well or better understood with $K = 4$ or 3 or even 2, and that the surprising interpretational paradoxes of quantum mechanics are merely approximations to ordinary (but still surprising) chaos but in the discrete setting of say $\mathbf{Chu}_2$. Current experimental research into the complexity of Boolean satisfiability (NP) hints at severe chaos for satisfiability of conjunctive normal form formulas with certain critical numbers of variables per clause. This would leave unchanged our hypothesis that we only perceive the computationally tractable approximations to states, while however replacing the exotic notion of phase entanglement complexity by the more conventional combinatorial complexity of *ordinary chaos* of dynamical systems but with discrete Chu spaces replacing the continuous spaces of classical mechanics. It *is* hard to believe that nature drew up the complex numbers in their entirety before embarking on the Big Bang. Better that they evolved later, and better still that our theory of evolution admits of a simplification in which complex numbers are dropped

[3] Von Neumann's "no-hidden-variables theorem" excludes only variables postulated to account for the "missing information" associated with Heisenberg uncertainty, and imposes no other limitation on how uncertain a mixed state can get.

other than for the purposes of elegant approximations to the real truth.

3.3 Chu Transforms

A *Chu transform* $(f,g) : (A,X,\models) \to (B,Y,\models')$ consists of functions $f : A \to B$ and $g : Y \to X$ such that for all $a \in A$ and $y \in Y$, $f(a)\mathrel{=\!\!|}'y = a\mathrel{=\!\!|}g(y)$, the *adjointness condition*. If we write the first expression as $f;\mathrel{=\!\!|}'$ and the second as $\mathrel{=\!\!|};g^{\smile}$, in each case thinking of the semicolon as a form of composition, and thinking of $g^{\smile}$ as a sort of "antifunction," we may express the adjointness condition as the commutativity of

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mathrel{=\!\!|}"'] & B \arrow[d, "\mathrel{=\!\!|}'"] \\ X \arrow[r, "g^{\smile}"] & Y \end{tikzcd}$$

The case $K = 2$ makes this a diagram in the category **Rel** of sets and their binary relations, with $g^{\smile}$ being the binary relation obtained as the converse of g viewed as a binary relation. For $K = \mathbb{C}$ we may take semicolon to be matrix multiplication, f to be an $A \times B$ matrix over $\mathbb{C}$ having one 1 per row and the rest 0, and dually $g^{\smile}$ to have one 1 per column. Then $f;\mathrel{=\!\!|}'$ becomes the product of an $A \times B$ matrix with a $B \times Y$ matrix while $\mathrel{=\!\!|};g^{\smile}$ is that of an $A \times X$ matrix with an $X \times A$ matrix, both products yielding an $A \times Y$ complex matrix.

It should be clear from the definition of Chu transform that duality of Chu spaces is a true categorical duality, in the sense that it sends Chu transforms (f,g) from $\mathcal{A}$ to $\mathcal{B}$ to their dual transforms (g,f) from $\mathcal{B}^{\perp}$ to $\mathcal{A}^{\perp}$. This makes the category $\mathbf{Chu}_K$ of Chu spaces and their transforms a self-dual category.

It is tempting to interpret an $X \times A$ Chu space over $\mathbb{C}$ as the usual matrix representation of an operator from $\mathbb{C}^X$ to $\mathbb{C}^A$. However no basis has been singled out here, and $\models$ is not an operator but rather a form of inner product for a *generalized vector space* V over K, with point set A and dual point set X (the functionals from H to K). As observed by Lafont and Streicher [LS91], such a Chu space, or *game* as they call it, is an ordinary vector space just when K is the underlying set of a field k, A is the underlying set of a vector space V over k, and X consists of the functionals on V, meaning the operators from V to k as the one-dimensional vector space over k. The Chu transforms between two vector spaces U, V realized in this way as Chu spaces then have as their f component exactly the operators $f : U \to V$. This is the realization of the category $\mathbf{Vct}_k$ of vector spaces over k as a full, faithful, and concrete subcategory of $\mathbf{Chu}_K$.

For finite-dimensional vector spaces X is itself a vector space with A as its functionals to k, namely the dual space V^*. Quantum mechanics covets this duality but sets no intrinsic bound on the number of energy levels of an electron (a discrete set) or the number of positions of a particle (a continuum). This motivates the notion of an *inner product space*, namely a vector space over $\mathbb{C}$ having a self-adjoint realization as a Chu space $(A,A,\models)$. By self-adjoint here we mean $a \models b = (b \models a)^*$ (complex conjugate) and $a \models a > 0$ for a not the origin, taken to be the squared *length* $||a||^2$ of a. This furnishes V with a metric $d(a,b) = ||a-b||$ as usual making V a metric space. The Chu transforms of such are the unitary operators, that is, the isometric (length-preserving) linear transformations.

A *Hilbert space* H is an inner product space which is *complete* in the sense that all Cauchy sequences converge, that is, when H contains the limits of all sequences in H the diameter of whose suffixes ultimately vanishes. There is only one countably dimensioned Hilbert space up to isomorphism, and its points are equally nicely represented either as the set of all square-summable complex sequences, each being a row of the Chu space restricted to those columns indexed by suitable basis vectors $h_0, h_1, h_2, \ldots$ of H, or the set of all square-integrable complex functions over H, each being an unrestricted row of the whole Chu space. The Chu transforms of Hilbert spaces realized as above as Chu spaces are automatically continuous in their metric, as an obvious consequence of being unitary.

A *rigged* Hilbert space is obtained from a Hilbert space as a subset of its rows, namely those rows which when restricted to basis vectors (for any choice of basis) remain square summable when the square of h_i is scaled by i^n for any integer $n > 0$ remaining fixed over the summation. The missing rows remain as columns, which are then to be understood as points infinitely far from the origin and constituting a *completion at infinity* of the original Hilbert space. (This is analogous to compactifying the real line by adding the points ∞ and $-\infty$, bearing in mind that the above construction yields no new such points when H is finite-dimensional to begin with.) Rigged Hilbert spaces have the pleasant property that the set of eigenvectors of *every* operator span the space, permitting any operator one might draw out of a hat to be meaningfully understood as a real-valued physical variable.

4 Universality of Chu Spaces

We say that a concrete category D (such as groups, Hilbert spaces, etc.) *realizes* a concrete category C when there exists a functor $F : C \to D$ that is full and faithful and which commutes with the respective underlying-set functors of C and D. The category $\mathbf{Str}_\kappa$ of κ-ary relational structures and their homomorphisms where κ is any ordinal is a universal category for mathematics to the extent that it realizes all categories definable by first order logic in a language with relational symbols of total arity κ and whose homomorphisms are assumed to as usual for any first-order relational structure. Groups, lattices, and Boolean algebras appear at $\kappa = 3$, monoids, rings, fields, and categories at $\kappa = 4$, etc.

The significance of fullness and faithfulness here is that they ensure that the representing object $F(c)$ transforms in the same way that c does. Concreteness means that c and its representation $F(c)$ have the same underlying set, an essential missing detail of previous such universality results [HL69]. For example groups with carrier (underlying set) G are realized as ternary relational structures on G (whence concreteness), the ternary relation being $xy = z$; monoids are quaternary because the unit must be given explicitly, in groups it can be read off from the multiplication and the homomorphisms preserve it automatically.

Elsewhere [Pra93b, p.153-4] we have proved that the self-dual category $\mathbf{Chu}_{2^\kappa}$ of Chu spaces over the power set $\kappa = \{0, 1, \ldots, \kappa - 1\}$ realizes $\mathbf{Str}_\kappa$. This makes the hierarchy of categories of Chu spaces at least as universal as first-order definable mathematics. We have recently shortened and clarified that argument, and have squeezed the full proof onto a single page, available electronically as /pub/uni.tex.Z on boole.stanford.edu.

5 Pontrjagin Duality

The representation of the preceding section is somewhat arbitrary. While it does the claimed job, one would hope that it preserved other worthwhile structure when present. Here we describe one instance of a quite different representation that is a better fit to the self-duality of Chu spaces.

We know of no reasonable duality theorem for general groups themselves, where instead one resorts to Hopf algebras. But there is a beautiful duality theory for locally compact Abelian groups, namely Pontrjagin duality; indeed this category is self-dual. This particular self-duality gets to the heart of standard quantum mechanics as based on complex numbers, being the algebraic basis for both the discrete Fourier transform (finite or infinite) and the continuous. Its exact counterpart for logic is Stone duality, where $K = 2$, which we do not treat here. The following requires no knowledge of groups beyond the definitions of group and group homomorphism.

Given any two abelian groups G and H, the set H^G of all group homomorphisms $f : G \to H$ can be made a group by pointwise combination, i.e. $(f \cdot g)(x) = f(x) \cdot g(x)$. Let T denote the "circle" group of complex numbers on the unit circle under complex multiplication. Define the *dual* of a group G (not yet a true dual) to be the group T^G. This operation on groups, colloquially called "homming into" T, is a functor $D : \mathbf{AbGrp}^{\mathrm{op}} \to \mathbf{AbGrp}$, that is, a *contravariant* functor on the category $\mathbf{AbGrp}$ of all Abelian groups. The significance of being a functor is that it maps not just groups but group homomorphisms (and preserves their composition). The significance of being contravariant is that it reverses the homomorphisms. Thinking of the homomorphisms of $\mathbf{AbGrp}$ as its highways and byways, namely how groups get around in $\mathbf{AbGrp}$, we see that duality has the feel of "looking in the mirror," in that it reverses the direction of the highways.

Now show that the set $G^{\mathbf{Z}}$ from the group $\mathbf{Z}$ of integers to an arbitrary abelian group G is isomorphic to G when the homomorphisms are made a group. (Hint: where can f send 1? Where must everything else go?) Now take G to be the "circle" group T of complex numbers on the unit circle under complex multiplication to yield $T^{\mathbf{Z}} \cong T$. Thus the circle group is the dual of the integers.

Now duality ought to be a symmetric relationship, for which we require $T^T \cong \mathbf{Z}$. To see that this is indeed the case, observe first that any group homomorphism $f : T \to T$ must be a "speedup of travel round the circle," definable formally by $f(e^{i\theta}) = e^{is\theta}$ for some real s constituting the speedup. We then have $e^{2\pi is} = f(e^{2\pi i}) = f(1) = 1$, whence $2\pi s$ is an integer multiple of 2π, i.e. s is an integer. Hence $T^T \cong \mathbf{Z}$, completing the argument that the group of integers and the circle group are Pontrjagin duals of each other.

Now this duality underpins that between Heisenberg's matrix mechanics and Schrödinger's wave mechanics for the periodic case, as arising in situations of resonance. Heisenberg's discrete particle-oriented matrices are indexed by the integers, while Schrödinger's waves oscillate smoothly by going around the unit circle. The aperiodic case corresponds to the Pontrjagin duality of the additive group of reals with themselves, a self-duality. Another self-duality is that of the cyclic group of any order (finite by definition), which arises in connection with particle spin, which can have only finitely many values, namely $2s+1$ for a particle of spin s: $\frac{1}{2}$ for electrons, 1 for photons, etc.

Now it turns out that the subcategory of **AbGrp** consisting of the *locally compact* Abelian groups (compact means essentially that its dual can be completely described by a (in general infinite) set of finite statements) is *self-dual* with respect to the duality of homming into T, making homming into T a true dual (which we did not have until now). Such a category can be realized in a category of Chu spaces along quite different lines to our previous representation, in such as way as to *preserve the duality.* Since duality in Chu spaces is always matrix transposition, this means that in this realization, the duality of interest is realized simply as transposition!

The representation is as follows. Take the realizing category to be $\mathbf{Chu}_T$ where T is simply the *set* of complex numbers on the unit circle. Realize the group G as the normal Chu space (G, T^G) where G is taken to be the underlying set of G and T^G is the set of group homomorphisms from G to T. A Chu transform between two such representations is then exactly a group homomorphism between the corresponding groups. Hence Chu duality is **AbGrp** duality, that is, this realization preserves the self-dual structure of **AbGrp**, in the sense of mapping G and its dual to a Chu space and its Chu dual, i.e. its transpose, obvious from the realization as (G, T^G). What is easy about this is the duality as transposition, what is interesting is that this duality is also a representable duality in Chu (reflecting that in **AbGrp**), with (T, Z) as the dualizing object in Chu realizing **AbGrp**'s T (since $T^T \cong Z$).

6 Uncertainty

We give a naive argument that the joint uncertainty of a Chu space and its dual satisfies $\Delta x \Delta a \geq \hbar$ for a plausible interpretation of these three symbolic quantities.

The essential idea is to measure the visibility or otherwise of a normal Chu space $\mathcal{A} = (A, X)$ by the number of Chu transforms from it to the Chu space $(K, 1)$ (when $K = 2$, the two-element Boolean algebra), which we may regard as the possible states it can be observed in. If $\mathcal{A}$ is a set, corresponding to X being K^A, we think of the points of A as being independent. This is the situation with the set of pixels on a computer screen, which can display all possible black-and-white images as its messages to the user. If however A has some structure, e.g. a linear ordering imposed on those pixels, the variety of possible messages can drop sharply; we then think of this additional structure as creating a sort of veil that defocuses the screen, making it less distinct. This gives a primitive model of the intuitively plausible idea that while one can see straight through an idiot, deeper thinkers are harder to understand.

But in a Chu space, X indexes the possible messages to $(K, 1)$, whence there can be at most $|X|$ (exactly $|X|$ just when $\mathcal{A}$ is extensional). The reciprocal $1/|X|$ of the size of the message space then gives the intrinsic uncertainty in a message from $\mathcal{A}$, which we call Δx. Similarly $1/|A|$ bounds from below is the intrinsic uncertainty in a message from the dual $\mathcal{A}^\perp$, which we call Δa.

The amount of information in a Chu space is bounded above by the number of bits in it (assuming $K = 2$), which is just $|X| \times |A|$. The reciprocal of that quantity then measures the intrinsic uncertainty of that Chu space as a whole, ignoring its form factor, which we shall call $\hbar$. Relative to that uncertainty we then have $\Delta x \Delta a \geq \hbar$.

If the universe were a single Chu space, which fits well with the Schrödinger picture in which it is a single wavefunction, both $\hbar$ and the uncertainty principle would arise in this way. In this case we are talking about uncertainty of x and a for the whole universe, which may or may not be a square Chu space, but the same idea obtains for neighborhoods of the universe, some of which will be highly rectangular, i.e. have a relatively precise position in space-time or have a precise value for momentum-energy. See [Pra94, §5] for a more detailed discussion of these ideas.

7 Conclusion

Chu spaces are the latest in a series of models addressing foundational concerns about the nature of concurrent behavior in the context of specification and implementation of concurrent hardware and software, as well as other systems found in e.g. telecommunications, manufacturing, transportation, and the service sector. They are particularly attractive with regard to the "total package" of requirements for such a model, namely simplicity, generality, and structure.

An unexpected spinoff of meeting all three of these needs simultaneously has been that they exhibit the basic phenomena of quantum mechanical behavior about as well as could be expected of a model based on truth values (the case $K = 2$) rather than complex numbers. When K is the set of complex numbers, the universality theorem for Chu spaces suggests that they may realize the objects not only of quantum mechanics but also of quantum electrodynamics. A yet more speculative possibility is that they may also realize facets of physics not fully accounted for by quantum mechanics and electrodynamics alone, in particular particle physics and quantum gravity.

Whether all this requires K to be infinite is a very interesting question. It may well be that $K = 4$ is already generous, and that the seeming dependence of quantum phenomena on as large a set as the complex numbers is only a consequence of finding numbers like 2^{256} best accommodated by approximating them with infinity. No combinatorialist should have any qualms with this approximation; the combinatorics of propositions about sets is already quite hard enough with sets of size 50.

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