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Grammar Theory Based on Quantum Logic

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Motivated by Ying' work on automata theory based on quantum logic (Ying, M. S. (2000). International Journal of Therotical Physics, 39(4): 985–996; 39(11): 2545–2557) and inspired by the close relationship between the automata theory and the theory of formal grammars, we have established a basic framework of grammar theory on quantum logic and shown that the set of l-valued quantum regular languages generated by l-valued quantum regular grammars coincides with the set of l-valued quantum languages recognized by l-valued quantum automata.

KEY WORDS: quantum logic; quantum automata; quantum grammar.

1. PRELIMINARIES

To provide a new model of quantum computation, Ying used the semantically analysis approach to study the automata theory based on quantum logic. Ying presented a basic framework of automata theory on quantum logic (Ying, 2000a,b). In particular, Ying introduced the orthomodular lattice-valued quantum predicate of recognizability and established some of its fundamental properties. The most interesting result obtained is the Proposition 2 in Ying (2000b) that says that the language recognized by the product of automata is the intersection of the languages recognized by the factors iff the truth-value lattice of the underlying logic is distributive. But an orthomodular lattice possessing distributivity is a Boolean algebra! This negative result may help us to clarify the boundary between classical computation and quantum computation. Lu and Zheng (2002) defined and studied three different types of lattice-valued finite state quantum automata (LQA) and four different kinds of LQA operation, discussed their advantages, disadvantages, and various properties. The most interesting results (Lu and Zheng, 2002) obtained are the Theorem 3.14, Theorem 3.15, and Theorem 3.16 that say that the validity of many properties of the lattice *LAT* (l, Σ, Θ), such as whether it is complete,

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distributive, or modular, depends on the corresponding properties of the original lattice.

With the close relationship between automaton theory and the theory of formal grammars in our minds, we have to consider whether or not we can establish a grammar theory based on quantum logic corresponding to the automaton theory based on quantum logic established by Ying (2000a,b). If we can do so, could we obtain the relation between quantum automa and quantum grammars corresponding to the classical one?

First let's review the classical automata theory and formal grammar theory.

1.1. Classical Automaton Theory

Definition 1.1. A finite state automaton is a quintuple M (Howie, 1991), where

$$M = (Q, A, \varphi, i, T)$$

Q is a finite nonempty set, called the states of M;
A is a finite nonempty set, called the set of inputs or the alphabet of M;
$i \in Q$ is the initial state of M;
T is a nonempty subset of Q and the elements of T are called the terminal states of M; φ is a mapping from $Q \times A$ to Q, called the state trnsition function of M. It is natural to expand φ to be a mapping from $Q \times A^*$ to Q in a recursive way by stipulating that

$$\varphi(q, 1) = q(q \in Q)\,(1 \text{ stands for the empty word})$$

$$\varphi(q, wa) = \varphi(\varphi(q, w), a)(q \in Q, w \in A^*, a \in A)$$

An element w of A^* is said to recognized by M if $\varphi(i, w) \in T$. The language $L(M)$ recognized by M is the set of all elements w in A^* that are recognized by M, that is to say

$$L(M) = \{w \in A^* | \varphi(i, w) \in T\}$$

Let $q(a_1a_2 \ldots a_n) = q'$, then the states q, q' in the automaton are connected by a path

$$q \xrightarrow{a_1} q_2 \xrightarrow{a_2} q_3 \rightarrow \cdots \rightarrow q_n \xrightarrow{a_n} q'$$

The word $a_1a_2 \ldots a_n$ is called the label of the path. A path will be called successful if it begins with the initial i and ends with a terminal state t in T. Thus $w \in L(M)$ if and only if there exists a successful path with label w.

1.2. The Classical Formal Grammar Theory

Definition 1.2. A formal grammar (phrase structure grammar) is a quadruple (Howie, 1991)

$$\Gamma = (V, A, \pi, \sigma);$$

V is a finite set of symbols called the vocabulary of Γ;
A is a nonempty subset of V called the terminal alphabet of Γ;
π is a finite subset of $(V\backslash A)^{+} \times V^{*}$. The elements (u, v) of π are called the productions of Γ, and we write $u \to v$ whenever $(u, v) \in \pi$;
$\sigma \in V\backslash A$ is an initial symbol.

Formally, for w, w' in V^* we write $w \Rightarrow w'$ if there exist x, y in V^* and a production $u \to v$ in π such that $w = xuy$, $w' = xvy$. We say that w' derives from w. We write $w \overset{*}{\Rightarrow} z$ if either $w = z$ or there exist $w_1, w_2, \ldots, w_n$ (with $n \geq 2$) in V^* such that $w = w_1$, $z = w_n$ and $w_i \Rightarrow w_{i+1}(i = 1, 2, \ldots, n-1)$. We refer to this chain of transformations as a derivation in Γ, and say that z derives from w.

The language $L(\Gamma)$ generated by Γ is the set of words in the terminal alphabet A that can be derived in this way from σ, i.e., $L(\Gamma) = \{w \in A^* : \sigma \overset{*}{\Rightarrow} w\}$.

The simplest type of grammar is a regular grammar, where every production in π is either of the form

$$\alpha \to x\beta(x \in A^{+}, \alpha, \beta \in V\backslash A)$$

or of the form

$$\alpha \to y(\alpha \in V\backslash A, y \in A^{*})$$

A grammar is hyper-regular if all productions have the form

$$\alpha \to a\beta(\alpha, \beta \in V\backslash A, a \in A)$$

or the form

$$\alpha \to 1(\alpha \in V\backslash A)$$

A language is regular (hyper-regular) if it can be generated by a regular (hyper-regular) grammar.

2. GRAMMAR THEORY BASED ON QUANTUM LOGIC

2.1. Automaton Theory Based on Quantum Logic

Ying (2000a,b) presented a basic framework for automaton theory based on quantum logic. We repeat the concept of an l-valued quantum automaton over $\sum$ defined by Ying (2000a,b) in a slightly different notation and definition.

Definition 2.1. Let $l = (L, \leq, \wedge, \vee, \perp, 0, 1)$ be an orthomodular lattice and A be a finite alphabet (Ying, 2000b). Then an l-valued quantum automaton over A is a quintuple M, where

$$M = (Q, A, \varphi, i, T);$$

Q is a finite set of states;
A is a finite alphabet;
$i \in Q$ is the initial state;
$T \subseteq Q$ is the set of terminal states;

φ is an l-valued subset of $Q \times A \times Q$, i.e., a mapping from $Q \times A \times Q$ into L and is called the l-valued quantum transition relation of M. Intuitively, for and $p, q \in Q$ and $\sigma \in A$, $\varphi(p, \sigma, q)$ indicates the truth-value of the proposition that input σ causes state p to become q.

An l-valued quantum automaton over A determines an l-valued (unary) predicate rec_M on $A^* \cup_{k=0}^{\infty} A^k$, and it is defined as follows: for all $k \geq 0$, $w = \sigma_1, \ldots, \sigma_k \in A$,

$$\begin{aligned} \mathrm{rec}_M(w) &= \mathrm{rec}_M(\sigma_1 \ldots \sigma_k) \\ &\stackrel{\text{def}}{=} (\exists q_0 = i, q_1, \ldots, q_{k-1} \in Q, q_k \in T)\mathrm{path}_M(q_0\sigma_1 q_1 \ldots q_{k-1}\sigma_k q_k) \end{aligned}$$

where

$$\mathrm{path}_M(q_0\sigma_1 q_1 \ldots q_{k-1}\sigma_k q_k) \stackrel{def}{=} \wedge_{j=0}^{k}[(q_j, \sigma_{j+1}, q_{j+1}) \in \varphi]$$
$$q_{k+1} = q_k, \sigma_{k+1} = 1 \quad \text{and} \quad \varphi(q_k, \sigma_{k+1}, q_{k+1}) = \varphi(q_k, 1, q_k) \in L.$$

Intuitively, $\mathrm{rec}_M(w)$ stands for the proposition that the word w is recognized by the quantum automaton M and its truth-value is

$$\mathrm{value}(\mathrm{rec}_M(w)) = \mathrm{value}(\mathrm{rec}_M(\sigma_1 \ldots \sigma_k)) \stackrel{\text{def}}{=} \vee_{q_0=i, q_1, \ldots, q_{k-1} \in Q, q_k \in T} \wedge_{j=0}^{k}$$
$$\varphi(q_j, \sigma_{j+1}, q_{j+1}) l_M(w) \stackrel{\text{def}}{=} \mathrm{value}(\mathrm{rec}_M(w)).$$

We call an l-valued subset of A^* an l-valued quantum language over A. Thus, the l-valued quantum language over A generated by M is $L(M)$, where

$$L(M) = \{(w, l_M(w)) | w \in A^*)\}.$$

2.2. Grammar Theory Based on Quantum Logic

Definition 2.2. Let $l = (L, \leq, \wedge, \vee, \perp, 0, 1)$ be an orthomodular lattice. Then an l-valued quantum grammar is a quadruple G, where

$$G = (V, T, I, P)$$

V is a finite alphabet of variables;
T is a finite alphabet of terminals;
$I \in V$ is an initial variable;
P is a finite set of productions $\alpha \to \beta, \alpha \in V^+, \beta \in (V \cup T)^*$.

Every production $\alpha \to \beta \in P$ has a value in L, i.e., there exists a mapping l from P into L such that $l(\alpha \to \beta) \in L$ for any $\alpha \to \beta \in P$.

If w' derives from w, i.e., $w \Rightarrow w'$ by the production $\alpha \to \beta \in P$, we define $l(w \Rightarrow w') = l(\alpha \to \beta) \in L$. If $w \overset{*}{\Rightarrow} z$, i.e., either $w = z$, then $l(w \overset{*}{\Rightarrow} z) \in L$ or there exist $w_1, w_2, \ldots, w_n$ (with $n \geq 2$) in V^* such that $w = w_1, z = w_n$ and $w_i \Rightarrow w_{i+1}(i = 1, 2, \ldots, n-1)$ with the corresponding production $\alpha_i \to \beta_i$ for every $w_i \Rightarrow w_{i+1}(i = 1, 2, \ldots, n-1)$, then

$$l(w \overset{*}{\Rightarrow} z) = \wedge_{i=1}^{n-1} l(w_i \Rightarrow w_{i+1}) = \wedge_{i=1}^{n-1} l(\alpha_i \to \beta_i).$$

An l-valued quantum grammar G is regular if it has only productions of the form

$$\alpha_1 \to \beta\alpha_2(\beta \in T^+, \alpha_1, \alpha_2 \in V) \quad \text{with} \quad l(\alpha_1 \to \beta_1\alpha_2) \in L$$

or of the form

$$\alpha_1 \to \beta(\beta \in T^*, \alpha_1, \in V) \quad \text{with} \quad l(\alpha_1 \to \beta) \in L.$$

An l-valued quantum grammar G is hyper-regular if it has only productions of the form

$$\alpha_1 \to \beta\alpha_2(\beta \in T, \alpha_1, \alpha_2 \in V) \quad \text{with} \quad l(\alpha_1 \to \beta_1\alpha_2) \in L$$

or of the form

$$\alpha \to 1(\alpha \in V) \quad \text{with} \quad l(\alpha_1 \to 1) \in L.$$

An l-valued quantum grammar determines an l-valued (unary) predicate rec_G on $T^* = \cup_{k=0}^{\infty} T^k$, and it is defined as follows: for all $k \geq 0$, $w \in T^*$,

$$\text{rec}_G(w) \overset{\text{def}}{=} (\exists w_1 = I, w_2, \ldots, w_{k-1} \in (V \cup T)^*, w_k = w)\text{deri}_G$$
$$\times (w_1 \Rightarrow w_2, \ldots, w_{k-1} \Rightarrow w_k),$$

where the derivation degree of w is defined as follows:

$$\text{deri}_G(w_1 \Rightarrow w_2, \ldots, w_{k-1} \Rightarrow w_k) \overset{\text{def}}{=} \wedge_{i=1}^{k-1} l(w_i \Rightarrow w_{i+1}),$$

$$\text{value}(\text{rec}_G(w)) \overset{\text{def}}{=} \vee_{w_1 = I, w_2, \ldots, w_{k-1} \in (V \cup T)^*, w_k = w} \wedge_{i=1}^{k-1} l(w_i \Rightarrow w_{i+1}),$$

$$l_G(w) \overset{\text{def}}{=} \text{value}(\text{rec}_G(w)).$$

The language $L(G)$ generated by G is

$$L(G) = \{(w, l_G(w))|w \in T^*\}.$$

An l-valued quantum (hyper-) regular language is one generated by some l-valued quantum (hyper-) regular grammar.

Let $l = (L, \leq, \wedge, \vee, \perp, 0, 1)$ be an orthomodular lattice. Two l-valued quantum grammar $G_1 = (V_1, T, I_1, P_1)$ and $G_2 = (V_2, T, I_2, P_2)$ are equivalent, i.e., $L(G_1) = L(G_2)$, if $l_{G_1}(w) = l_{G_2}(w)$ for all $w \in T^*$.

First we can obtain the important relation between l-valued quantum regular languages and quantum hyper-regular languages as same as that in class one.

Lemma 2.1. *Every l-valued quantum regular language is an l-valued quantum hyper-regular language.*

Proof: Let $l = (L, \leq, \wedge, \vee, \perp, 0, 1)$ be an orthomodular lattice and $G = (V, T, I, P)$ is an l-valued quantum regular grammar. Then the language $L(G)$ generated by G is

$$L(G) = \{(w, l_G(w))|w \in T^*\}$$

We can define an l-valued hyper-regular grammar $G_1 = (V_1, T, I, P_1)$ such that $L(G) = L(G_1)$.

For each production

$$\alpha \to a_1 a_2 \ldots a_m \beta \in P \quad \text{with} \quad l_G(\alpha \to a_1 a_2 \ldots a_m \beta) \in L \tag{1}$$

When $m = 1$, we have $a_1 \in T, \alpha, \beta \in V \subseteq V_1$, and $\alpha \to a_1\beta \in P_1$ with $l_{G_1}(\alpha \to a_1\beta) = l_G(\alpha \to a_1\beta) \in L$, as required; when $m \geq 2, a_1, a_2, \ldots, a_m \in T, \alpha, \beta \in V \subseteq V_1$, and we can define non-terminal symbols $\zeta_1, \zeta_2, \ldots, \zeta_{m-1}$ in V_1 and within P_1 mimic the production (1) by means of productions

$$\begin{aligned} &\alpha \to a_1\zeta_1, \zeta_1 \to a_2\zeta_2, \ldots, \zeta_{m-1} \to a_m\beta \in P_1 \quad \text{with} \quad l_{G_1}(\alpha \to a_1\zeta_1) \\ &\quad = l_{G_1}(\zeta_1 \to a_2\zeta_2) = \ldots = l_{G_1}(\zeta_{m-1} \to a_m\beta) = l_G(\alpha \to a_1a_2\ldots a_m\beta). \end{aligned} \tag{2}$$

Then

$$\begin{aligned} &l_{G_1}(\alpha \to a_1a_2\ldots a_m\beta) \\ &\quad = l_{G_1}(\alpha \to a_1\zeta_1) \wedge l_{G_1}(\zeta_1 \to a_2\zeta_2) \wedge \ldots \wedge l_{G_1}(\zeta_{m-1} \to a_m\beta) \\ &\quad = l_G(\alpha \to a_1a_2\ldots a_m\beta). \end{aligned}$$

For each production

$$\alpha \to b_1b_2\ldots b_n \in P \quad \text{with} \quad l_G(\alpha \to b_1b_2\ldots b_n) \in L. \tag{3}$$

When $n = 0, \alpha \in V \subseteq V_1$, and we have $\alpha \to 1 \in P_1$ with $l_{G_1}(\alpha \to 1) = l_G(\alpha \to 1) \in L$, as required; when $n \geq 1, b_1, b_2, \ldots, b_n \in T, \alpha \in V \subseteq V_1$, and we can define nonterminal symbols $\eta_1, \eta_2, \ldots, \eta_n$ in V_1 and within P_1 mimic the production (3) by means of productions

$$\begin{aligned}\alpha \to b_1\eta_1, \eta_1 \to b_2\eta_2, \ldots, \eta_{n-1} \to b_n\eta_n, \eta_n \to 1 \in P_1 \quad \text{with} \quad & l_{G_1}(\alpha \to b_1\eta_1) \\ = l_{G_1}(\eta_1 \to b_2\eta_2) = \ldots = l_{G_1}(\eta_{n-1} \to b_n\eta_n) = l_{G_1}(\eta_n \to 1) & \\ = l_G(\alpha \to b_1 b_2 \ldots b_n) & \end{aligned} \tag{4}$$

Then

$$\begin{aligned} & l_{G_1}(\alpha \to b_1 b_2 \ldots b_n) \\ & = l_{G_1}(\alpha \to b_1\eta_1) \wedge l_{G_1}(\eta_1 \to b_2\eta_2) \wedge \ldots \wedge l_{G_1}(\eta_{n-1} \to b_n\eta_n) \wedge l_{G_1}(\eta_n \to 1) \\ & = l_G(\alpha \to b_1 b_2 \ldots b_n) \end{aligned}$$

Certainly G_1 is an l-valued quantum hyper-regular grammar. Moreover, it is clear that the productions (2) give that $I \overset{*}{\Rightarrow} a_1 a_2 \ldots a_m\beta$ in G_1 with $l_{G_1}(I \overset{*}{\Rightarrow} a_1 a_2 \ldots a_m\beta) = l_G(I \overset{*}{\Rightarrow} a_1 a_2 \ldots a_m\beta) \in L$. We similarly obtain that $I \overset{*}{\Rightarrow} b_1 b_2 \ldots b_n$ in G_1 from the productions (4) with $l_{G_1}(I \overset{*}{\Rightarrow} b_1 b_2 \ldots b_n) = l_G(I \overset{*}{\Rightarrow} b_1 b_2 \ldots b_n) \in L$. Thus very derivation $I \overset{*}{\Rightarrow} w$ in G with $l_G(I \overset{*}{\Rightarrow} w) \in L$ can be simulated by a longer derivation $I \overset{*}{\Rightarrow} w$ in G_1 with $l_{G_1}(I \overset{*}{\Rightarrow} w) \in L$ and $l_{G_1}(I \overset{*}{\Rightarrow} w) = I_G(I \overset{*}{\Rightarrow} w)$. Then for all $w \in T^*$, we have $l_{G_1}(w) = l_G(w)$, Thus we conclude that $L(G) \subseteq L(G_1)$.

To prove the reverse inclusion, suppose that for some $w \in T^*$, there is a derivation $I \overset{*}{\Rightarrow} w$ in G_1 with $l_{G_1}(I \overset{*}{\Rightarrow} w) \in L$. Then certainly there is a derivation

$$I \overset{*}{\Rightarrow} w \text{ in } G_2 \quad \text{with} \quad L_{G_2}(I \overset{*}{\Rightarrow} w) = l_{G_1}(I \overset{*}{\Rightarrow} w) \in L \tag{5}$$

where $G_2 = (V_1, T, I, P \cup P_1)$ has all the productions in G together with all the productions in G_1.

We shall show that there is a derivation of w in G, by induction on the number of symbols from $V_1 \backslash V$ appearing in the derivation (5). If no such symbols appear then (5) is already a derivation in G. Otherwise the first appearance of a symbol of $V_1 \backslash V$ is based either on a production

$$\alpha \to \alpha_1\zeta_1 \quad \text{with} \quad l_{G_2}(\alpha \to \alpha_1\zeta_1) = l_{G_1}(\alpha \to \alpha_1\zeta_1) \in L$$

where $\alpha \to a_1 a_2 \ldots a_m\beta$ is a production in G, or on a production

$$\alpha \to b_1\eta_1 \quad \text{with} \quad l_{G_2}(\alpha \to b_1\eta_1) = l_{G_1}(\alpha \to b_1\eta_1) \in L$$

where $\alpha \to b_1 b_2 \ldots b_n$ is a production in G. Consider the first of these cases. Since the final word w in the derivation (5) has no nonterminal symbols, and since the grammar G_2 has no productions of the type $\zeta_i \to y (y \in T^*)$ with $l_{G_2}(\zeta_i \to y) \in L$

for any of the symbols of $V_1 \backslash V$, the only way in which ζ_1, once introduced, can subsequently disappear must involve changes from ζ_1 to $a_2\zeta_2, \ldots, \zeta_{m-1}$ to $a_m\beta$. But then the sequence of transitions

$$\alpha \to \alpha_1\zeta_1, \zeta_1 \to a_2\zeta_2, \ldots, \zeta_{m-1} \to a_m\beta \quad \text{with}$$
$$l_{G_2}(\alpha \to \alpha_1\zeta_1) = l_{G_1}(\alpha \to \alpha_1\zeta_1), l_{G_2}(\zeta_1 \to \alpha_2\zeta_2) = l_{G_1}(\zeta_1 \to \alpha_2\zeta_2)$$
$$, \ldots, l_{G_2}(\zeta_{m-1} \to \alpha_m\beta) = l_{G_1}(\zeta_{m-1} \to \alpha_m\beta) \in L$$

can be replaced by a single transition

$$\alpha \to a_1a_2 \cdots a_m\beta$$

with

$$\begin{aligned}
&l_{G_2}(\alpha \to a_1a_2 \ldots a_m\beta)\\
&= l_{G_2}(\alpha \to a_1\zeta_1) \wedge l_{G_2}(\zeta_1 \to a_2\zeta_2) \wedge \cdots \wedge l_{G_2}(\zeta_{m-1} \to a_m\beta)\\
&= l_{G_1}(\alpha \to a_1\zeta_1) \wedge l_{G_1}(\zeta_1 \to a_2\zeta_2) \wedge \cdots \wedge l_{G_1}(\zeta_{m-1} \to a_m\beta)\\
&= l_G(\alpha \to a_1a_2 \ldots a_m\beta)
\end{aligned}$$

in $G2$. Thus the number of symbols from $V_1 \backslash V$ has been reduced.

Equally, in the second case the derivation must involve subsequent changes from η_1 to $b_2\eta_2, \ldots, \eta_{n-1}$ to $b_n\eta_n$, and these n transitions can be replaced by a single transition in G_2 from α to $b_1b_2 \cdots b_n$ with

$$\begin{aligned}
&l_{G_2}(\alpha \to b_1b_2 \ldots b_n)\\
&= l_{G_2}(\alpha \to b_1\eta_1) \wedge l_{G_2}(\eta_1 \to b_2\eta_2) \wedge \cdots \wedge l_{G_2}(\eta_{n-1} \to b_n\eta_n) \wedge l_{G_2}(\eta_n \to 1)\\
&= l_{G_1}(\alpha \to b_1\eta_1) \wedge l_{G_1}(\eta_1 \to b_2\eta_2) \wedge \cdots \wedge l_{G_1}(\eta_{n-1} \to b_n\eta_n) \wedge l_{G_1}(\eta_n \to 1)\\
&= l_G(\alpha \to b_1b_2 \ldots b_n).
\end{aligned}$$

In both cases the derivation (5) is replaced by one with fewer occurrences of symbols from $V_1 \backslash V$ with $l_{G_2}(w) = l_{G_1}(w)$. By induction it now follows that $L(G_2) \subseteq L(G)$. Hence certainly $L(G_1) \subseteq L(G)$. □

In **Theorem 2** and **Theorem 3** below, we shall prove that the set of l-valued quantum regular languages coincides with the set of l-valued quantum languages, as same as that in class one.

Theorem 2.2. *Let $l = (L, \leq, \wedge, \vee, \perp, 0, 1)$ be an orthomodular lattice and M and l-valued quantum automaton over a finite alphabet A, then there exists an l-valued quantum regular grammar G such that $L(G) = L(M)$.*

Proof: Let M is an l-valued quantum automaton over A, where

$$M = (Q, A, \varphi, i, T)$$

The l-valued quantum language $L(M)$ over A generated by M is

$$L(M) = \{(w, l_M(w)))\backslash w \in A^*\}$$

We define an l-valued quantum regular grammar G as follows: $G = (Q, A, I(= i), P)$ where P consists of the productions

$$p \to aq(p, q \in Q, a \in A) \quad \text{with} \quad l_G(p \to aq) = \varphi(p, a, q) \in L$$

and

$$t \to 1(t \in T) \quad \text{with} \quad l_G(t \to 1) = \varphi(t, 1, t) \in L.$$

We show that $L(G) = L(M)$.

Suppose first that $(w, l_G(w)) = (a_1a_2 \ldots a_n, l_G(a_1a_2 \ldots a_n)) \in L(G)$. Because

$$l_G(w) = \vee_{w_1=1, w_2, \ldots, w_{k-1} \in (V \cup T)^*, w_k = w} \wedge_{i=1}^{k-1} l(w_i \Rightarrow w_{i+1})$$

there is a derivation

$$I \overset{*}{\Rightarrow} a_1a_2 \ldots a_n \quad \text{with} \quad l_G(I \overset{*}{\Rightarrow} a_1a_2 \ldots a_n) = l_G(w)$$

We see that this derivation must be of the form

$$I \Rightarrow a_1q_1 \Rightarrow a_1a_2q_2 \Rightarrow \cdots \Rightarrow a_1a_2 \ldots a_nq_n \Rightarrow a_1a_2 \ldots a_n$$

where $I \to a_1q_1, q_1 \to a_2q_2, \ldots, q_{n-1} \to a_nq_n, q_n \to 1$ are productions in G with $l_G(I \to a_1q_1), l_G(q_1 \to a_2q_2), \ldots, l_G(q_{n-1} \to a_nq_n), l_G(q_n \to 1) \in L$, and $q_n \in T$. Because

$$\begin{aligned}
&l_G(I \to a_1q_1) = \varphi(I, a_1, q_1), \\
&l_G(q_1 \to a_2q_2) = \varphi(q_1, a_2, q_2), \\
&\ldots, \\
&l_G(q_{n-1} \to a_nq_n) = \varphi(q_{n-1}, a_n, q_n), \\
&l_G(q_n \to 1) = \varphi(q_n, 1, q_n), \\
&\text{and } q_n \in T,
\end{aligned}$$

we have $(I \to a_1q_1), (q_1 \to a_2q_2), \ldots, q_{n-1}, a_n, q_n), (q_n, 1, q_n) \in \varphi$ and $q_n \in T$. Thus, there exists a successful path

$$I \xrightarrow{a_1} q_1 \xrightarrow{a_2} Q_2 \to \cdots \to q_{n-1} \xrightarrow{a_n} q_n$$

in M such that

$$\begin{aligned}
&\text{path}_M(Ia_1q_1a_2q_2\dots q_{n-1}a_nq_n)\\
&=\varphi(I,a_1,q_1)\wedge\varphi(q_1,a_2,q_2)\wedge\cdots\wedge\varphi(q_{n-1},a_n,q_n)\wedge\varphi(q_n,1,q_n)\\
&=l_G(I\to a_1q_1)\wedge l_G(q_1\to a_2q_2)\wedge\cdots\wedge l_G(q_{n-1}\to a_nq_n)\wedge l_G(q_n\to 1)\\
&=l_G(I\overset{*}{\Rightarrow}a_1a_2\dots a_n)=l_G(w)\in L.
\end{aligned}$$

Because for every derivation deri^k in G, there exists a successful path path^k with label w such that $l_M(\text{path}^k)=l_G(\text{deri}^k)$, it follows that

$$\begin{aligned}
&l_M(w)=\vee_{\text{path}^k}\wedge l_M(\text{path}^k)=\vee_{q_0=1,q_1,\dots,q_{k-1}\in Q,q_n\in T}\wedge_{i=0}^{n}\varphi(q_i,a_{i+1},q_{i+1})\\
&=\vee_{I\to a_1q_1,q_1\to a_2q_2,\dots,q_{n-1}\to a_nq_n,q_n\to 1}(l_G(I\to a_1q_1)\wedge l_G(q_1\to a_2q_2)\wedge\\
&\cdots\wedge l_G(q_{n-1}\to a_nq_n)\wedge l_G(q_n\to 1))\\
&=\vee_{\text{deri}^k}\wedge l_G(\text{deri}^k)=\vee_{I\overset{*}{\Rightarrow}w}l_G(I\overset{*}{\Rightarrow}w)=l_G(w).
\end{aligned}$$

Thus $L(G)\subseteq L(M)$.

Conversely, suppose that

$$(w,l_G(w))=(a_1a_2\dots a_n,l_G(a_1a_2\dots a_n))\in L(M)$$

so that there is a successful path

$$I\overset{a_1}{\to}q_1\overset{a_2}{\to}q_2\to\cdots\to q_{n-1}\overset{a_n}{\to}q_n$$

in M with

$$\begin{aligned}
l_M(w)&=\text{path}_M(Ia_1q_1a_2q_2\dots q_{n-1}a_nq_n)\\
&=\varphi(I,a_1,q_1)\wedge\varphi(q_1,a_2,q_2)\wedge\cdots\wedge\varphi(q_{n-1},a_n,q_n)\wedge\varphi(q_n,1,q_n)
\end{aligned}$$

and $\varphi(I,a_1,q_1),\varphi(q_1,a_2,q_2),\dots,\varphi(q_{n-1},a_n,q_n),\varphi(q_n,1,q_n)\in L$ and $q_n\in T$. Then there are productions

$I\to a_1q_1,q_1\to a_2q_2,\dots,q_{n-1}\to a_nq_n,q_n\to 1$ in G with $l(I\to a_1q_1)$, $l(q_1\to a_2q_2),\dots,l(q_{n-1}\to a_nq_n),l(q_n\to 1)\in L$ and

$$\begin{aligned}
&l(I\to a_1q_1)=\varphi(I,a_1,q_1),\\
&l(q_1\to a_2q_2)=\varphi(q_1,a_2,q_2),\\
&\dots,\\
&l(q_{n-1}\to a_nq_n)=\varphi(q_{n-1},a_n,q_n),\\
&l(q_n\to 1)=\varphi(q_n,1,q_n).
\end{aligned}$$

So there is a derivation

$$I\Rightarrow a_1q_1\Rightarrow a_1a_2q_2\Rightarrow\cdots\Rightarrow a_1a_2\dots a_nq_n\Rightarrow a_1a_2\dots a_n$$

in G, and

$$\begin{aligned}
&\mathrm{deri}_G(I \Rightarrow a_1q_1 \Rightarrow a_1a_2q_2 \Rightarrow \cdots \Rightarrow a_1a_2\ldots a_nq_n \Rightarrow a_1a_2\ldots a_n)\\
&= l(I \to a_1q_1) \wedge l(q_1 \to a_2q_2) \wedge \cdots \wedge l(q_{n-1} \to a_nq_n) \wedge l(q_n \to 1)\\
&= \varphi(I, a_1, q_1) \wedge \varphi(q_1, a_2, q_2) \wedge \cdots \wedge \varphi(q_{n-1}, a_n, q_n) \wedge \varphi(q_n, 1, q_n)\\
&= l_M(w)
\end{aligned}$$

Because for every successful path path^k in M there exists a derivation deri^k in G such that $l_G(\mathrm{deri}^k) = l_M(\mathrm{path}^k)$, it follows that

$$\begin{aligned}
l_G(w) &= \vee_{\mathrm{dert}^k} \wedge l_G(\mathrm{deri}^k) = \vee_{I \overset{*}{\Rightarrow} w} \mathrm{deri}_G(I \overset{*}{\Rightarrow} w)\\
&= \vee_{I\Rightarrow a_1q_1\Rightarrow a_1a_2q_2\Rightarrow\cdots\Rightarrow a_1a_2\ldots a_nq_n\Rightarrow a_1a_2\ldots a_n} \mathrm{deri}_G(I \Rightarrow a_1q_1 \Rightarrow a_1a_2q_2 \Rightarrow\\
&\quad \cdots \Rightarrow a_1a_2\ldots a_nq_n \Rightarrow a_1a_2\cdots a_n)\\
&= \vee_{I\to a_1q_1, q_1\to a_2q_2,\ldots,q_{n-1}\to a_nq_n, q_n\to l}(l_G(I \to a_1q_1) \wedge l_G(q_1 \to a_2q_2) \wedge\\
&\quad \cdots \wedge l_G(q_{n-1} \to a_nq_n) \wedge l_G(q_n \to 1))\\
&= \vee_{q_0=1,q_1,q_2,\ldots,q_{n-1}\in Q, q_n\in T} \wedge_{i=0}^{n} \varphi(q_i, a_i, q_{i+1})\\
&= \vee_{\mathrm{path}^k} \wedge l_M(\mathrm{path}^k) = l_M(w)
\end{aligned}$$

Thus $L(M) \subseteq L(G)$. □

We have shown that every l-valued quantum language is an l-valued quantum regular language. Next we shall prove that every l-valued quantum regular language is also an l-valued quantum language.

Theorem 2.3. *Let $l = (L, \leq, \wedge, \vee, \perp, 0, 1)$ be an orthomodular lattice and G an l-valued quantum regular grammar, then there exists an l-valued quantum automaton M such that $L(M) = L(G)$.*

Proof: Suppose that $L(G)$ is an l-valued quantum regular language, where $G = (V, T, I, P)$ is an l-valued quantum regular grammar. By Lemma 1 we can assume that the grammar G is an l-valued quantum hyper-regular grammar. i.e., that all productions are of the form

$$\alpha \to a\beta (\text{with } a \in T, \alpha, \beta \in V) \quad \text{with} \quad l_G(\alpha \to a\beta) \in L$$

or

$$\alpha \to 1 (\text{with } \alpha \in V) \quad \text{with} \quad l_G(\alpha \to 1) \in L.$$

Let $M = (V, T, \varphi, I, T')$ be the l-valued quantum automaton, where

$$T' = \{\alpha \in V : \alpha \to 1 \in P \quad \text{with} \quad l(\alpha \to l) \in L\}$$

$(\alpha, a, \beta) \in \varphi$, whenever $\alpha \to a\beta \in P$ and $\alpha, 1, \alpha) \in \varphi$, whenever $\alpha \to 1 \in P$.

Suppose first that $(w, l_G(w)) = (a_1a_2 \dots a_n, l_G(a_1a_2 \dots a_n)) \in L(G)$. The derivation of w must be of the form

$$I \Rightarrow a_1\beta_1 \Rightarrow a_1a_2\beta_2 \Rightarrow \cdots \Rightarrow a_1a_2 \dots a_n\beta_n \Rightarrow a_1a_2 \dots a_n$$

where

$$I \to a_1\beta_1, \beta_1 \to a_2\beta_2, \dots, \beta_{n-1} \to a_n\beta_n, \beta_n \to 1 \in P$$

with

$$l_G(I \to a_1\beta_1), l_G(\beta_1 \to a_2\beta_2), \dots, l_G(\beta_{n-1} \to a_n\beta_n), l_G(\beta_n \to 1) \in L.$$

Thus $\beta_n \in T'$

$$I \xrightarrow{a_1} \beta_1 \xrightarrow{a_2} \beta_2 \xrightarrow{a_3} \cdots \xrightarrow{a_n} \beta_n$$

is a successful path in M with label w and

$$\begin{aligned}
&\varphi(I, a_1, \beta_1) = l_G(I \to a_1\beta_1),\\
&\varphi(\beta_1, a_2, \beta_2) = l_G(\beta_1 \to a_2\beta_2),\\
&\cdots,\\
&\varphi(\beta_{n-1}, a_n, \beta_n) = l_G(\beta_{n-1} \to a_n\beta_n),\\
&\varphi(\beta_n 1, \beta_n) = l_G(\beta_n \to 1).
\end{aligned}$$

Thus

$$\begin{aligned}
\text{path}_M(Ia_1\beta_1a_2\beta_2 \dots \beta_{n-1}a_n\beta_n) &= \varphi(I, a_1, \beta_1) \wedge \varphi(\beta_1, a_2, \beta_2) \wedge \dots\\
&\quad \times \wedge \varphi(\beta_{n-1}, a_n, \beta_n) \wedge \varphi(\beta_n, 1, \beta_n)\\
&= l_G(I \to a_1\beta_1) \wedge l_G(\beta_1 \to a_2\beta_2) \wedge \cdots\\
&\quad \times \wedge l_G(\beta_{n-1} \to a_n\beta_n) \wedge l_G(\beta_n \to 1)\\
&= l_G(w).
\end{aligned}$$

Because for every derivation deri^k in G, there exists a successful path path^k, such that $l_M(\text{path}^k) = l_G(\text{deri}^k)$, it follows that

$$\begin{aligned}
l_M(w) &= \vee_{\text{path}^k} \wedge l_M(\text{path}^k) = \vee_{\beta_1, \beta_2, \dots, \beta_{n-1} \in \mathrm{V}, \beta_n \in \mathrm{T}'}(\varphi(\mathrm{I}, \mathrm{a}_1, \beta_1)\\
&\quad \wedge\varphi(\beta_1, a_2, \beta_2) \wedge \cdots \wedge \varphi(\beta_{n-1}, a_n, \beta_n) \wedge \varphi(\beta_n, 1, \beta_n))\\
&= \vee_{I \to a_1\beta_1, \beta_1 \to a_2\beta_2, \dots, \beta_{n-1} \to a_n\beta_n, \beta_n \to I}(l_G(I \to a_1\beta_1) \wedge l_G(\beta_1 \to a_2\beta_2) \wedge\\
&\quad \cdots \wedge l_G(\beta_{n-1} \to a_n\beta_n) \wedge l_G(\beta_n \to 1))\\
&\quad \vee_{\text{deri}^k} \wedge l_G(\text{deri}^k) = \vee_{I \overset{*}{\Rightarrow} w} l_G(I \overset{*}{\Rightarrow} w) = l_G(w).
\end{aligned}$$

Thus $L(G) \subseteq L(M)$.

We have shown that $L(G) \subseteq L(M)$. To show the reverse inclusion suppose that $(w, l_M(w)) = (a_1a_2 \ldots a_n, l_M(a_1a_2 \ldots a_n)) \in L(M)$, so that there is a successful path

$$I \xrightarrow{a_1} \beta_1 \xrightarrow{a_2} \beta_2 \xrightarrow{a_3} \cdots \xrightarrow{a_n} \beta_n (\in T')$$

and

$$l_M(w) = \varphi(I, a_1, \beta_1) \wedge (\beta_1, a_2, \beta_2) \wedge \cdots \wedge \varphi(\beta_{n-1}, a_n, \beta_n) \wedge \varphi(\beta_n, 1, \beta_n)$$

and so $I \to a_1\beta_1, \beta_1 \to a_2\beta_2, \ldots, \beta_{n-1} \to a_n\beta_n, \beta_n \to 1 \in P$ and

$$\begin{aligned} &l(I \to a_1\beta_1) = \varphi(I, \alpha_1, \beta_1),\\ &l(\beta_1 \to a_2\beta_2) = \varphi(\beta_1, a_2, \beta_2),\\ &\cdots,\\ &l(\beta_{n-1} \to a_n\beta_n) = \varphi(\beta_{n-1}, a_n, \beta_n),\\ &l_G(\beta_n \to 1) = \varphi(\beta_n, 1, \beta_n). \end{aligned}$$

Hence there is a derivation

$$I \Rightarrow a_1\beta_1 \Rightarrow a_1a_2\beta_2 \Rightarrow \cdots \Rightarrow a_1a_2 \ldots a_n\beta_n \Rightarrow a_1a_2 \ldots a_n$$

in G such that

$$\begin{aligned} &\mathrm{deri}_G(I \Rightarrow a_1\beta_1 \Rightarrow a_1a_2\beta_2 \Rightarrow \cdots \Rightarrow a_1a_2 \ldots a_n\beta_n \Rightarrow a_1a_2 \ldots a_n)\\ &= l_G(I \to a_1\beta_1) \wedge l_G(\beta_1 \to a_2\beta_2) \wedge \cdots \wedge l_G(\beta_{n-1} \to a_n\beta_n) \wedge l_G(\beta_n \to 1)\\ &= \varphi(I, a_1, \beta_1) \wedge \varphi(\beta_1, a_2, \beta_2) \wedge \cdots \wedge \varphi(\beta_{n-1}, a_n, \beta_n) \wedge \varphi(\beta_n, 1, \beta_n)\\ &= l_M(w). \end{aligned}$$

Because for every successful path path^k in M there exists a derivation deri^k in G such that $l_G(\mathrm{deri}^k) = l_M(\mathrm{path}^k)$, we have

$$\begin{aligned} l_G(w) &= \vee_{\mathrm{deri}^k} \wedge l_G(\mathrm{deri}^k) = \vee_{I \overset{*}{\Rightarrow} w} \mathrm{deri}_G(I \overset{*}{\Rightarrow} w)\\ &= \vee_{I \Rightarrow a_1\beta_1 \Rightarrow a_1a_2\beta_2 \Rightarrow \cdots \Rightarrow a_1a_2 \ldots a_n\beta_n \Rightarrow a_1a_2 \ldots a_n} \mathrm{deri}_G(I \Rightarrow a_1\beta_1 \Rightarrow a_1a_2\beta_2 \Rightarrow\\ &\quad \cdots \Rightarrow a_1a_2 \ldots a_n\beta_n \Rightarrow a_1a_2 \ldots a_n)\\ &= \vee_{I \to a_1\beta_1, \beta_1 \to a_2\beta_2, \ldots, \beta_{n-1} \to a_n\beta_n, \beta_n \to I}(l_G(I \to a_1\beta_1) \wedge l_G(\beta_1 \to a_2\beta_2) \wedge\\ &\quad \cdots \wedge l_G(\beta_{n-1} \to a_n\beta_n) \wedge l_G(\beta_n \to 1))\\ &= \vee_{\beta_1, \beta_2, \ldots, \beta_{n-1} \in V, \beta_n \in T'}(\varphi(I, a_1, \beta_1) \wedge \varphi(\beta_1, a_2, \beta_2) \wedge \cdots \wedge \varphi(\beta_{n-1}, a_n, \beta_n)\\ &= \wedge \varphi(\beta_n, 1, \beta_n))\\ &= \vee_{\mathrm{path}^k} \wedge l_M(\mathrm{path}^k) = l_M(w). \end{aligned}$$

Thus $L(M) \subseteq L(G)$. □

Corollary 2.1. *The set of l-valued quantum regular languages coincides with the set of l-valued quantum languages.*

Proof: Straightforward by Theorem 2 and Theorem 3. □

3. CONCLUSION

In this paper, we outlined a framework of grammar theory based on quantum logic corresponding to the automata theory based on quantum logic established by Ying (2000a,b). We defined the l-valued quantum regular grammar and hyper-regular grammar. Then we proved that every l-valued quantum regular language is an l-valued quantum hyper-regular language. The most important results obtained in this paper is the Theorem 2 and Theorem 3 that say the set of l-valued quantum regular languages coincides with the set of l-valued quantum languages. The further study about the properties of the l-valued quantum grammars will be left to a later paper.

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