PROPERTIES AND CALCULATION OF PAIRED HAAR TRANSFORM

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Abstract

Different properties of recently introduced Paired Haar transform have been shown. Non-polynomial Haar expansion of incompletely specified Boolean functions has been presented. Based on the above properties and expansion some applications of Paired Haar spectrum have been proposed. Algorithm for the calculation of Haar Pair spectrum from disjoint cubes for systems of incompletely specified Boolean functions has also been developed.

1 Introduction

Spectral techniques have been applied widely to Boolean function classification, disjoint decomposition, parallel and serial linear decomposition, spectral translation synthesis (extraction of linear pre-and post-filters), multiplexer synthesis, prime implicant extraction, threshold logic synthesis, state assignment, testing and evaluation of logic complexity [7, 8, 10—13]. There are at least two transforms which are based on square—wave like functions that are well suitable for Boolean functions: Walsh and Haar transforms. The Walsh functions are global like the Fourier functions and consist of a set of irregular rectangular waveforms with only two amplitude values +1 and −1 [1, 3, 6—8]. Each but two basis functions in Haar transform consists of a square wave pulse located on an otherwise zero amplitude interval. Computation of the fast Haar transform (FHT) requires order N (N is a number of spectral coefficients) additions and subtractions, which makes it much faster than the fast Walsh transform (FWT) [1, 7—9]. Due to its low computing requirements, Haar transform has been used mainly for pattern recognition and image processing [1,14]. Such a transform is also well suited in communication technology for data coding, multiplexing and digital filtering [14]. The advantages of computational and memory requirements of the Haar transform make it of big interest to VLSI designers as well [7,10,11].

Local property of Haar transform makes it useful in those applications in computer-aided design systems where there are Boolean functions of many variables that have most of its values grouped locally. Such weakly specified and local functions frequently occur in logic design and can be extremely well described by few spectral coefficients from Haar transform while the application of Walsh, global transform would be quite cumbersome in such cases and the locally grouped minterms would be spread throughout all Walsh spectrum. To better deal with the mentioned cases, the idea of Paired Haar Transform was introduced [4]. In Paired Haar Transform, all the information about true and don't care minterms is kept separately, by what it is available in different stages of CAD process.

In this paper more properties allowing application of Paired Haar Transform in different problems of logic design are described. An incompletely specified function or group of such functions can be represented by nonpolynomial Haar expansions. To obtain Paired Haar spectrum in an efficient way, the methods are presented that calculate such spectra directly from disjoint cube representation of incompletely specified Boolean functions. It should be noted that efficient representation of incompletely specified functions is extremely important in algorithms used in many pattern recognition systems [2] and for such cases the Paired Haar transform can also be applied. Introduced properties and algorithms allow also on more efficient manipulation of different representations of Boolean functions during synthesis process since Paired Haar spectra and their expansions are available to the designer and either of them maybe used interchangeably dependent on the requirements of the design process.

2 Normalized Haar Spectrum

Definition 1 The normalized Haar transform H_N of order $N=2^n$ can be defined recursively as [1,13]:

$$H_N = \begin{bmatrix} H_{\frac{N}{2}} \otimes [1\ 1] \\ I_{\frac{N}{2}} \otimes [1\ -1] \end{bmatrix} \qquad \text{and } H_1 = 1 \tag{1}$$

where $I_{\frac{N}{2}}$ is an identity matrix of order $\frac{N}{2}$ and the symbol '$\otimes$' denotes the right-hand Kronecker product.

For an n-variable Boolean function $F(x_1, x_2, \cdots, x_n)$ Haar spectrum is given by $R = [H_N]F$, where R is Haar spectrum (a column vector of dimension $2^n \times 1$) and F is the R-coded truth vector of Boolean function $F(X)$ [4,5,7,8,13]. In R coding, the false minterms are coded as 0, true minterms as 1 and don't care (DC) minterms as 0.5.

Besides the first two Haar spectral coefficients r_{dc} (so called dc coefficient corresponding to dc function) and $r_0^{(0)}$, which are globally sensitive to $F(X)$, the remaining 2^n-2 Haar spectral coefficients are only locally sensitive. A spectral coefficient $r_l^{(k)}$ is characterized by its degree l and order k.

Property 1 For a Haar spectrum of an n-variable Boolean function F, there are 2^l spectral coefficients of degree l, each measures a correlation of a different set of 2^{n-l} neighboring minterms where $l=1,2,\cdots,n$. The dc coefficient r_{dc} and the zero degree coefficient $r_0^{(0)}$ measure a correlation of 2^n neighboring minterms (the whole Karnaugh map). The value of r_{dc} is equal to the number of minterms of F and the coefficient $r_0^{(0)}$ describes the difference between the number of minterms in the functions $\bar{x}_n$ and x_n.

Definition 2 A standard trivial function (STF), denoted by u_I, $I \in \{0, 1, \cdots, 2^n-1\}$, associated with each Haar spectral coefficient r_{dc} or $r_l^{(k)}$ describes some set of 2^{n-l} neighboring minterms on a Karnaugh map that has an influence on the value of a spectral coefficient r_{dc} or $r_l^{(k)}$ where $0 \leqslant l \leqslant n-1$ and $0 \leqslant k \leqslant 2^l-1$.

An index I of a STF u_I is equal to 2^l+k.

Property 2 The degree l of Haar coefficient indicates the number of literals present in a STF u_I for $I=1,2,\cdots,2^n-1$.

Property 3 The order k of Haar spectral coefficient $r_l^{(k)}$ is the decimal equivalence of the binary l-tuple formed by writing a 1 or 0 for each variable in a STF $u_I (I = 2, 3, \cdots, 2^n-1)$ according to whether this literal appears in affirmation or negation. When k is expressed as a binary l-tuple, the most significant bit (MSB) corresponds to the literal $\dot{x}_n$ and the least significant bit (LSB) corresponds to the literal $\dot{x}_{n-l+1}$.

Property 4 The positive standard trivial function (PSTF) and the negative standard trivial function (NSTF) are the cofactors of the Shannon's decomposition of the STF with respect to x_{n-l} and $\bar{x}_{n-l}$, respectively.

Example 1 All STFs of Haar spectrum for a four variable Boolean function are shown as areas filled with circles and triangles in Fig. 1. The area filled with circles is the PSTF and the area filled with triangles is the NSTF. For the STF u_9, index $I = 9$. For $0 \leqslant l \leqslant n-1$ and $0 \leqslant k \leqslant 2^l-1$, $2^l+k=9 \Rightarrow l=3, k=1$. This function describes $2^{4-3}= 2$ neighboring minterms for the spectral coefficient of degree 3.

3 Paired Haar Spectrum of Boolean Functiions

In efficient synthesis of incompletely specified Boolean functions there is a need for filling don't care minterms of the original function by '0' and '1' in such a way that the resulting completely specified Boolean function will be easily implemented by available basic gate structures and Programmable Logic Devices (PLDs). Different complexity criteria are used in order to make such a choice of minterms optimal [7,8]. In order to make easier to fulfill the above requirements on the final values of spectral coefficients the Paired Haar transform has been introduced [4].

Definition 3 A paired Haar transform for an incompletely specified n-variable Boolean function F is composed of 2^n vectors, each having four elements. The elements in the first vector are denoted by a_{dc}, b_{dc}, c_{dc}, d_{dc} and in the remaining vectors by $a_l^{(k)}$, $b_l^{(k)}$, $c_l^{(k)}$, $d_l^{(k)}$ where $0 \leqslant l \leqslant n-1$, $0 \leqslant k \leqslant 2^l-1$ and the elements $a_l^{(k)}$, $b_l^{(k)}$, $c_l^{(k)}$ and $d_l^{(k)}$ are defined as: $a_l^{(k)}$ is the number of true minterms in the NSTF; $b_l^{(k)}$ is the number of true minterms in the PSTF; $c_l^{(k)}$ is the number of don't care minterms in the NSTF; $d_l^{(k)}$ is the number of don't care minterms in the PSTF. a_{dc} and c_{dc} represents the total number of true and dc minterms respectively and b_{dc} and d_{dc} are always 0. In the case of a completely specified function, Paired Haar transform is described by 2^n vectors, each having only two elements: a_{dc} and b_{dc} for the first vector and $a_l^{(k)}$ and $b_l^{(k)}$ for the remaining vectors, since for completely specified Boolean functions $c_l^{(k)}$ and $d_l^{(k)}$ are always 0.

Example 2 Consider the Boolean function presented by Fig. 1. Paired Haar spectrum of this function is given by all the values $a_l^{(k)}$, $b_l^{(k)}$, $c_l^{(k)}$ and $d_l^{(k)}$ listed below each Karnaugh map.

Property 5 The Paired Haar spectrum of the complement ($\bar{F}$) of a Boolean function F is given by:

$$(a'_{dc}, b'_{dc}, c'_{dc}, d'_{dc}) = ((2^n - a_{dc}, b_{dc}, 2^n - c_{dc}, d_{dc})$$

$$(a_l^{(k)'}, b_l^{(k)'}, c_l^{(k)'}, d_l^{(k)'}) = (2^{n-l-1} - a_l^{(k)}, 2^{n-l-1} - b_l^{(k)}, 2^{n-l-1} - c_l^{(k)}, 2^{n-l-1} - d_l^{(k)}) \quad (2)$$

where the prime superscripts' are used to indicate the spectral coefficients of the complement function $\bar{F}$. $l = 0, 1, \cdots, n-1$ and $k = 0, 1, \cdots, 2^l-1$.

Property 6 The sum of all elements $a_l^{(k)}$ and $b_l^{(k)}$ with the same degree l is equal to a_{dc} and the sum of all elements $c_l^{(k)}$ and $d_l^{(k)}$ with the same degree l is equal to c_{dc}. i. e.,

$$\sum_{k=0}^{2^l-1}(a_l^{(k)} + b_l^{(k)}) = a_{dc} \text{ and } \sum_{k=0}^{2^l-1}(c_l^{(k)} + d_l^{(k)}) = c_{dc} \quad (3)$$

Property 7 If for some l and k, $a_l^{(k)}+c_l^{(k)}=2^{n-l-1}$ and $b_l^{(k)}+d_l^{(k)}=0$, then there exists an ON $(n-l-1)$-cube equal to the corresponding NSTF with some assignment of don't care minterms. Similarly, if for some l and k, $a_l^{(k)}+c_l^{(k)}=0$ and $b_l^{(k)}+d_l^{(k)}=2^{n-l-1}$, there exists an ON $(n-l-1)$-cube equal to the corresponding PSTF with some assignment of don't care minterms.

Property 8 If an n-variable Boolean function is independent of the variable x_i, then

$$\sum_{k=0}^{2^{n-i}-1} a_{n-i}^{(k)} = \sum_{k=0}^{2^{n-i}-1} b_{n-i}^{(k)} \tag{4}$$

By filling don't care minterms $c_l^{(k)}$ and $d_l^{(k)}$ by 0 or 1 such that (4) is fulfilled for some l, an optimal or near optimal assignment of such minterms can be obtained.

Property 9 With the consideration of all possible assignments of don't care minterms in an incompletely specified Boolean function, the size of the largest prime implicant has the upper and lower bounds given by: $[\, n-l\max -1 \leqslant p \leqslant \log_2(a_{dc}+c_{dc})]$ where p is the number of '–' in the cube representation of the largest prime implicant and $l$max is the maximum degree l that has at least one maximal valued (2^{n-l-1}) element $a_l^{(k)}, b_l^{(k)}, c_l^{(k)}$ or $d_l^{(k)}$ for some order k.

Example 3 As a numerical example, consider a four variable incompletely specified Boolean function for which all STFs and their respective Paired Haar spectral coefficients $(a_l^{(k)}, b_l^{(k)}, c_l^{(k)}, d_l^{(k)})$ are given in Fig. 1. According to the relationship given in [4], Haar spectral coefficients for this function are calculated as follows:

$r_{dc}=5+0.5\times 3=6.5$; $r_0^{(0)}=0-5+0.5(3-0)=-3.5$; $r_1^{(0)}=0-0+0.5(1-2)=-0.5$; $r_1^{(1)}=3-2+0.5(0-0)=1$; $r_2^{(0)}=0-0+0.5(1-0)=0.5$; $r_2^{(1)}=0-0+0.5(2-0)=1$; $r_2^{(2)}=2-1+0.5(0-0)=1$; $r_2^{(3)}=0-2+0.5(0-0)=-2$; $r_3^{(0)}=0-0+0.5(0-1)=-0.5$; $r_3^{(1)}=0-0+0.5(0-0)=0$; $r_3^{(2)}=0-0+0.5(1-1)=0$; $r_3^{(3)}=0-0+0.5(0-0)=0$; $r_3^{(4)}=1-1+0.5(0-0)=0$; $r_3^{(5)}=1-0+0.5(0-0)=1$; $r_3^{(6)}=0-0+0.5(0-0)=0$; $r_3^{(7)}=1-1+0.5(0-0)=0$.

From Property 9, since $l\max=2$, $4-2-1\leqslant p\leqslant \log_2(5+3)$, $1\leqslant p\leqslant 3$. In this example, $p=1$ since the size of the largest prime implicant is equal to 2.

4 Probabilistic Analysis By Paired Haar Spectrum

Lemma 1 *A nonpolynomial Haar expansion can be expressed as follows:*

$$F(X)=\frac{1}{2^n}\Big\{r_{dc}+(-1)^{x_n}r_0^{(0)}+\sum_{l=1}^{n-1}2^l(-1)^{x_{n-l}}\sum_{k=0}^{2^l-1}r_l^{(k)}\prod_{i=n-l+1}^{n}x_i^{k_{i-n+l}}\Big\} \tag{5}$$

where $k_i\in\{0,1\}$ is the i-th bit in the binary l-tuple of the order k; $x_i^j=x_i$ if $j=1$ and x_i^j

$=\bar{x}_i$ *if* $j=0$.

In order to apply Paired Haar transform to the optimization of digital circuits and the optimal assignment of don't care minterms, a probabilistic function $P_F(C)$ is introduced. The value of $P_F(C)$ lies between 0 and 1 which indicates the likelihood of the cube C being an implicant of the function F. $P_F(C)=1$ if all minterms covered by C are true minterms. $P_F(C)=0$ if C is an OFF cube.

Definition 4 Let m be any minterm covered by the cube C whose size is equal to $|C|$, then the probabilistic function $P_F(C)$ is given by:

$$P_F(C)=\frac{1}{|C|}\sum_{m\in C}F(m), \tag{6}$$

where $F(m)$ is the R-coded functional value of the minterm m.

Property 10 When $P_F(C)\geqslant 0.5$, C may be changed to an ON cube by assigning all dc minterms covered by it to 1. Conversely, when $P_F(C)<0.5$, it is impossible to make from C an ON cube even by assigning all dc minterms covered by C to 1.

Lemma 2 *Let $\mathscr{R}_i(C)$ be the cube resulting from rotating the cube C by i bits to the right and $\gamma_i(C)$ be the number of '—' in C between bit* 1 *to bit i.*

$$P_F(C)=\frac{1}{2^n|C|}\{|C|r_{dc}+\sum_{l=0}^{n-l}\delta(l)\},$$

$$\delta(l)=\begin{cases}0, & \text{if } x_{n-l}=\text{'—'}\\ 2^{l+\gamma_{n-l}(C)}(-1)^{x_{n-l}}\sum\limits_{M\in\mathscr{R}_{n-l}(C)}r_l^{(M)}, & \text{if } x_{n-l}\neq\text{'—'}\end{cases} \tag{7}$$

In Equation (7), $M\in\mathscr{R}_{n-l}(C)$ denotes the set of integers whose binary representations are equal to the minterm numbers covered by the cube $\mathscr{R}_{n-l}(C)$.

Example 4 Consider the cube $C=-1-0$ in Fig. 1, from (7), $\delta(0)=0$ since $x_4=-$; $\mathscr{R}_3(C)=000-$, $\gamma_3(C)=1$, $\delta(1)=2^{1+1}(-1)^1(r_1^{(0)}+r_1^{(1)})=-4(-0.5+1)=-2$; $\delta(2)=0$ since $x_2=-$; $\mathscr{R}_1(C)=0-1-$, $\gamma_1(C)=0$, $\delta(3)=2^{3+0}(-1)^0(r_3^{(2)}+r_3^{(3)}+r_3^{(6)}+r_3^{(7)})=8(0+0+0+0)=0$. since $|C|=4$ and $r_{dc}=6.5$, from (7),

$$P_F(C)=\frac{1}{16\times 4}\{4\times 6.5+(0-2+0+0)\}=\frac{1}{64}\times 24=0.375.$$

Since $P_k(C)<0.5$, it is impossible to make C an ON cube by any assignment of don't care minterms.

Tran [15] introduces a $\frac{3}{4}$ majority i-cube as an i-cube containing at least $\frac{3}{4}\times 2^i$ true minterms. He has used it successfully as a selection criterion in the decomposition method for minimization of Reed–Muller polynomials in mixed polarity for completely specified Boolean functions. By Definition 4, the $\frac{3}{4}$ majority cube can be detected by (7) from Haar spectrum since any cube C is a $\frac{3}{4}$ majority cube if and only if $P_F(C) \geqslant 0.75$ for a completely specified Boolean function. The concept of $\frac{3}{4}$ majority cube is further extended to the case of incompletely specified functions by the following lemma and definition. With the extended definition, Paired Haar spectrum can then be utilized as a selection criterion in decomposition method for the minimization of mixed polarity Reed – Muller expansions of incompletely specified Boolean functions.

Lemma 3

$$P_F(C) = P_{onF}(C) + 0.5 \times P_{dcF}(C), \tag{8}$$

where

$$P_{onF}(C) = \frac{1}{2^n|C|}\{|C|a_{dc} + \sum_{l=0}^{n-l}\delta_{on}(l)\},$$

$$\delta_{on}(l) = \begin{cases} 0, & \text{if } x_{n-l} = \text{‘}-\text{’}, \\ 2^{l+\tau_{n-l}(C)}(-1)^{x_{n-l}} \sum\limits_{M \in \Re_{n-l}(C)} [a_l^{(M)} - b_l^{(M)}], & \text{if } x_{n-l} \neq \text{‘}-\text{’}. \end{cases}$$

Definition 5 When

$$P_{onF}(C) + P_{dcF}(C) \geqslant 0.75,$$

C is known as a $\frac{3}{4}$ majority cube for any completely or incompletely specified Boolean function F.

Based on the above Lemmas and Properties, Paired Haar spectrum can be applied to multilevel multiplexer circuit synthesis using the heuristic approach from [12]. The basic principle is to minimize level by level the multiplexer modules by an appropriate choice of data select variables by using the three basic conditions for which the next module is redundant: the input function is a trivial function (constant 0, 1 or single variable x_i or $\bar{x}_i$), the input function is identical to or is the complement of another input function to a multiplexer in the same level [12]. When C_i is the cube formed by the conjunction of the k data select variables in some polarity i and F_i is the cofactor obtained by decomposing the function F around C_i, the latter problem is translated into finding a cube C_i that maximizes the possibility of the next level module being redundant. The following conditions lead to a trivial input function:

(1) If $P_{onF}(C_i) = 0$, all don't care minterms covered by C_i should be assigned to ‘0’.

(2) If

$$P_{on}(C_i) + P_{dcF}(C_i) = 1,$$

all don't care minterms covered by C_i should be assigned to '1'.

(3) If

$$P_{onF}(\text{Cube } x_i) + P_{dcF}(\text{Cube } x_i) = 1$$

or

$$P_{onF}(\text{Cube } \bar{x}_i) + P_{dcF}(\text{Cube } \bar{x}_i) = 1$$

where $\dot{x}_i$ has not been used as the data select variables.

Additionally, the data input functions that are complements of each other in the same level can be recognized by Property 5 and the input function that is independent of the data select variables can be recognized by Property 8.

If there is more than one set of data select variables that fulfill the above criteria, the supplementary condition (4) is taken into account:

(4) The set of k variables is selected for which the absolute value

$$\left|2^{k-1} - \sum_{w=0}^{2^k-1}\{P_{onF}(C_w) + P_{dcF}(C_w)\}\right|$$

is the greatest.

Condition 4 is based on the conjecture that a higher concentration of true or false minterms tends to reduce the complexity of the data input function and hence the number of modules in the next or subsequent levels. By examining the probabilistic functions of different classes of functions, additional criteria may be introduced to improve the quality of the results.

5 Calculation of Paired Haar Spectrum From and Array of Disjoint Cubes

To enhance the efficiency in the applications of Paired Haar spectrum, we extend the concepts used in [3,4,6] to calculate Paired Haar spectrum from an array of disjoint cubes. The advantages of the presented algorithm are that it allows on the independent calculation of only some selected coefficients and the partial spectral coefficients contributed by each disjoint ON or DC cube can be executed simultaneously in parallel dedicated processors.

Definition 6 The partial spectral coefficient of an ON or a DC $p-$cube of a Boolean function F is equal to the value of the spectral coefficient that corresponds to the contribution of this cube to the full $n-$space spectrum of the Boolean function F. The number of partial spectral coefficients $npsc$ describing the Boolean function F is equal to the number of ON and DC cubes describing this function.

Property 11 The partial dc coefficient $(a_{dc}, b_{dc}, c_{dc}, d_{dc})$ contributed by a $p-$cube C of a Boolean function F is equal to $(2^p, 0, 0, 0)$ if C is an ON cube and equal to $(0, 0, 2^p, 0)$

if C is a DC cube.

Property 12 Each ON (or DC) cube contributes a partial Paired Haar spectral coefficients ($a_l^{(k)}$, $b_l^{(k)}$) (or ($c_l^{(k)}$, $d_l^{(k)}$)) of degree l and order k depending on the logical value of the literal $x_{n-l}(0 \leqslant l \leqslant n-1$, x_n is the MSB and x_1 is the LSB). Each literal x_i of a p—cube C contributes a pair of values (v_1, v_2) to a degree $l = n-i$ of Paired Haar coefficient ($a_l^{(k)}$, $b_l^{(k)}$) if C is an ON cube and to ($c_l^{(k)}$, $d_l^{(k)}$) if C is a DC cube. The values of (v_1, v_2) which depend on the literal x_i and the order k of the spectral coefficient are given by:

$$(v_1, v_2) = \begin{cases} (2^{p-q-1}, 2^{p-q-1}) & \text{if } k \subseteq \mathscr{R}_i(C) \text{and } x_i = \text{'—'} \\ (2^{p-q}, 0) & \text{if } k \subseteq \mathscr{R}_i(C) \text{and } x_i = \text{'0'} \\ (0, 2^{p-q}) & \text{if } k \subseteq \mathscr{R}_i(C) \text{and } x_i = \text{'1'} \\ (0, 0) & \text{Otherwise.} \end{cases} \quad (9)$$

In the above Equation, $l = n-i$ for all $1 \leqslant i \leqslant n$ and q is the number of '—' in the cube $\mathscr{R}_i(C)$, i.e., $q = \log_2 \| \mathscr{R}_i(C) \|$. $v_1 = v_2 = 0$ iff the binary representation of the order k is not covered by the cube $\mathscr{R}_i(C)$.

Based on Property 12, the procedure to calculate the partial spectral coefficient ($a_l^{(k)}$, $b_l^{(k)}$) contributed by an ON p—cube of an n-variable Boolean function F is given below:

Procedure partial_coeft_oncube(ON p—cube C, degree l, order k)

```
{
  if (k⊆ℛ_{n-l}(C)){
    p=number of '—' in C;q=number of '—' in ℛ_{n-l}(C);
    Switch(logical value of literal x_{n-l} of C){
          case '—' :a_l^(k)=b_l^(k)=2^(p-q-1);
                    break;
       case '0' :a_l^(k)=2^(p-q);b_l^(k)=0;
                 break;
          case '1' :a_l^(k)=0;b_l^(k)=2^(p-q);
                    break;
    }
  }else a_l^(k)=b_l^(k)=0;
  return (a_l^(k),b_l^(k));
}
```

The procedure **partial_coeft_dccube** which is used to calculate the partial spectral coefficient ($c_l^{(k)}$, $d_l^{(k)}$) contributed by a DC p—cube is similar to the procedure **partial_coeft_on-**

cube except that the input argument ON p—cube C is replaced by DC p—cube C and every occurrence of the return values $(a_l^{(k)}, b_l^{(k)})$ are replaced by $(c_l^{(k)}, d_l^{(k)})$, accordingly.

By summing up the partial coefficients a_{dc}, $a_l^{(k)}$ and $b_l^{(k)}$ which are contributed by all the disjoint ON cubes and the partial coefficients c_{dc}, $c_l^{(k)}$ and $d_l^{(k)}$ contributed by all the disjoint DC cubes, the full spectrum composed of Paired Haar spectral coefficients $(a_l^{(k)}, b_l^{(k)}, c_l^{(k)}, d_l^{(k)})$ for the n—variable Boolean function F is obtained. The following algorithm describes the procedure of calculating the complete Paired Haar spectrum:

```
Procedure Paired_Haar(Array of disjoint ON and DC cubes D)
{
for (row =0 to 2^n−1)a[row]=b[row]=c[row]=d[row]=0;
b[0]=d[0]=0;
for (each cube C_j∈D, j=1 to npsc){
  p=number of '−' in C_j;
  for (l=0 to n−1){
    for (k=0 to 2^l−1){
      index =2^l+k;
      if (C_j is an ON cube){
           a[0]=a[0]+2^p;
           (a[index],b[index])=(a[index],b[index])+partial_coeft_oncube(C_j,l,k);
      }else{
           c[0]=c[0]+2^p;
           (c[index],d[index])=(c[index],d[index])+partial_coeft_dccube(C_j,l,k);
      }
    }
  }
}
return(arrays a,b,c,d);
}
```

The procedure **Paired_Haar** can be modified to include options to just calculate a selected Paired Haar coefficient or only spectral coefficients for a complete degree. In the former case, the desired degree l and order k are supplied as additional arguments to the procedure **Paired_Haar** and the two inner **for** loops with l and k are skipped. In the latter case, the degree l is supplied as an input argument to **Paired_Haar** and the inner **for** loop with l is omitted.

Example 5 An example for calculating Paired Haar spectrum by Procedure Paired_

Haar is shown in Table 1. The four-variable incompletely specified Boolean function used in this example is $ON(F) = x_2+x_4x_3+\bar{x}_4\bar{x}_2x_1$, $DC(F) = \bar{x}_4\bar{x}_3\bar{x}_2\bar{x}_1$. The disjoint ON and DC cubes describing F are generated by the algorithm in [3]. They are given in the first row of Table 1. Since there are four disjoint ON and DC cubes, $npsc = 4$. The column under each disjoint cube shows the partial spectral coefficients corresponding to that cube. It is obvious that the partial values $c_l^{(k)}$ and $d_l^{(k)}$ contributed by an ON cube and the partial values $a_l^{(k)}$ and $b_l^{(k)}$ contributed by a DC cube for any l and k are equal to zero. The total spectrum obtained by summing all partial coefficients is given in the last column. The calculation of Paired Haar spectral coefficient $(a_1^{(0)}, b_1^{(0)}, c_1^{(0)}, d_1^{(0)})$ is explained as follows:

For the ON cubes $--1-$, $110-$ and $0-01$, Procedure **partial_coeft_oncube** is applied to calculate their respective partial spectral coefficient. Since $l = 1$, $n-l = 3$. For the cube $--1-$, $p = 3$, $\mathcal{R}_3(--1-) = 000-$, $q = 1$. Since $k = 0 = 0000_2 \subseteq 000-$ and $x_3=$ '$-$', $a_1^{(0)} = b_1^{(0)} = 2^{3-1-1} = 2$. For the cube $110-$, $\mathcal{R}_3(110-) = 0001$. Since $k = 0 = 0000_2 \not\subseteq 0001$, $a_1^{(0)} = b_1^{(0)} = 0$. For the cube $0-01$, $p = 1$, $\mathcal{R}_3(0-01) = 0000$, $q = 0$. Since $k = 0 = 0000_2 \subseteq 0000$ and $x_3=$ '$-$', $a_1^{(0)} = b_1^{(0)} = 2^{1-0-1} = 1$. The contribution to $c_1^{(0)}$ and $d_1^{(0)}$ by the DC cube 0000 is calculated by Procedure **partial_coeft_dccube**. $\mathcal{R}_3(0000) = 0000$, $p = q = 0$. Since $k = 0 = 0000_2 \subseteq 0000$ and $x_3=$ '0', $c_1^{(0)} = 2^{0-0} = 1$, $d_1^{(0)} = 0$. Hence $(a_1^{(0)}, b_1^{(0)}, c_1^{(0)}, d_1^{(0)}) = (2\ 2\ 0\ 0) + (0\ 0\ 0\ 0) + (1\ 1\ 0\ 0) + (0\ 0\ 1\ 0) = (3\ 3\ 1\ 0)$.

6 Conclusion

The essential relationships between classical (Karnaugh maps and disjoint cubes) and spectral (Paired Haar Spectra and Expansions) representations of Boolean functions used in the design of VLSI digital circuits have been stated. Paired Haar Transform based on local basis functions is especially well suited to spectral processing of weakly defined locally grouped multi-variable incompletely specified Boolean functions. It can also be applied to optimal don't care assignment for mixed polarity Reed-Muller expansions of incompletely specified Boolean functions and optimization in the multiplexer synthesis.

In order to calculate Paired Haar transform in an efficient way, the procedure to convert disjoint cube representation of incompletely specified Boolean functions has been introduced. It is also possible to modify recently introduced method of conversion of Binary Decision Diagram representation of Boolean functions to/from standard Haar spectrum [5] to the case of Paired Haar spectrum.

The research summarized here will have not only impact on the more efficient applications of Paired Haar transform in logic design but should also influence the application of such a transform in the areas where standard Haar transform is used outside logic design. Some of such possible areas are pattern recognition and image processing, especially when some locally mosaic types of patterns and images are considered [2].

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Table 1. Calculation of Paired Harr Spectrum From an Array of Disjoint Cubes

Cube	--1-	110-	0-01	0000	Total
Type	ON	ON	ON	DC	spectrum
$(a_{dc},b_{dc},c_{dc},d_{dc})$	(8,0,0,0)	(2,0,0,0)	(2,0,0,0)	(0,0,1,0)	(12,0,1,0)
$(a_0^{(0)},b_0^{(0)},c_0^{(0)},d_0^{(0)})$	(4,4,0,0)	(0,2,0,0)	(2,0,0,0)	(0,0,1,0)	(6,6,1,0)
$(a_1^{(0)},b_1^{(0)},c_1^{(0)},d_1^{(0)})$	(2,2,0,0)	(0,0,0,0)	(1,1,0,0)	(0,0,1,0)	(3,3,1,0)
$(a_1^{(1)},b_1^{(1)},c_1^{(1)},d_1^{(1)})$	(2,2,0,0)	(0,2,0,0)	(0,0,0,0)	(0,0,0,0)	(2,4,0,0)
$(a_2^{(0)},b_2^{(0)},c_2^{(0)},d_2^{(0)})$	(0,2,0,0)	(0,0,0,0)	(1,0,0,0)	(0,0,1,0)	(1,2,1,0)
$(a_2^{(1)},b_2^{(1)},c_2^{(1)},d_2^{(1)})$	(0,2,0,0)	(0,0,0,0)	(1,0,0,0)	(0,0,0,0)	(1,2,0,0)
$(a_2^{(2)},b_2^{(2)},c_2^{(2)},d_2^{(2)})$	(0,2,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,2,0,0)
$(a_2^{(3)},b_2^{(3)},c_2^{(3)},d_2^{(3)})$	(0,2,0,0)	(2,0,0,0)	(0,0,0,0)	(0,0,0,0)	(2,2,0,0)
$(a_3^{(0)},b_3^{(0)},c_3^{(0)},d_3^{(0)})$	(0,0,0,0)	(0,0,0,0)	(0,1,0,0)	(0,0,1,0)	(0,1,1,0)
$(a_3^{(1)},b_3^{(1)},c_3^{(1)},d_3^{(1)})$	(1,1,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(1,1,0,0)
$(a_3^{(2)},b_3^{(2)},c_3^{(2)},d_3^{(2)})$	(0,0,0,0)	(0,0,0,0)	(0,1,0,0)	(0,0,0,0)	(0,1,0,0)
$(a_3^{(3)},b_3^{(3)},c_3^{(3)},d_3^{(3)})$	(1,1,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(1,1,0,0)
$(a_3^{(4)},b_3^{(4)},c_3^{(4)},d_3^{(4)})$	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)
$(a_3^{(5)},b_3^{(5)},c_3^{(5)},d_3^{(5)})$	(1,1,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(1,1,0,0)
$(a_3^{(6)},b_3^{(6)},c_3^{(6)},d_3^{(6)})$	(0,0,0,0)	(1,1,0,0)	(0,0,0,0)	(0,0,0,0)	(1,1,0,0)
$(a_3^{(7)},b_3^{(7)},c_3^{(7)},d_3^{(7)})$	(1,1,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(1,1,0,0)

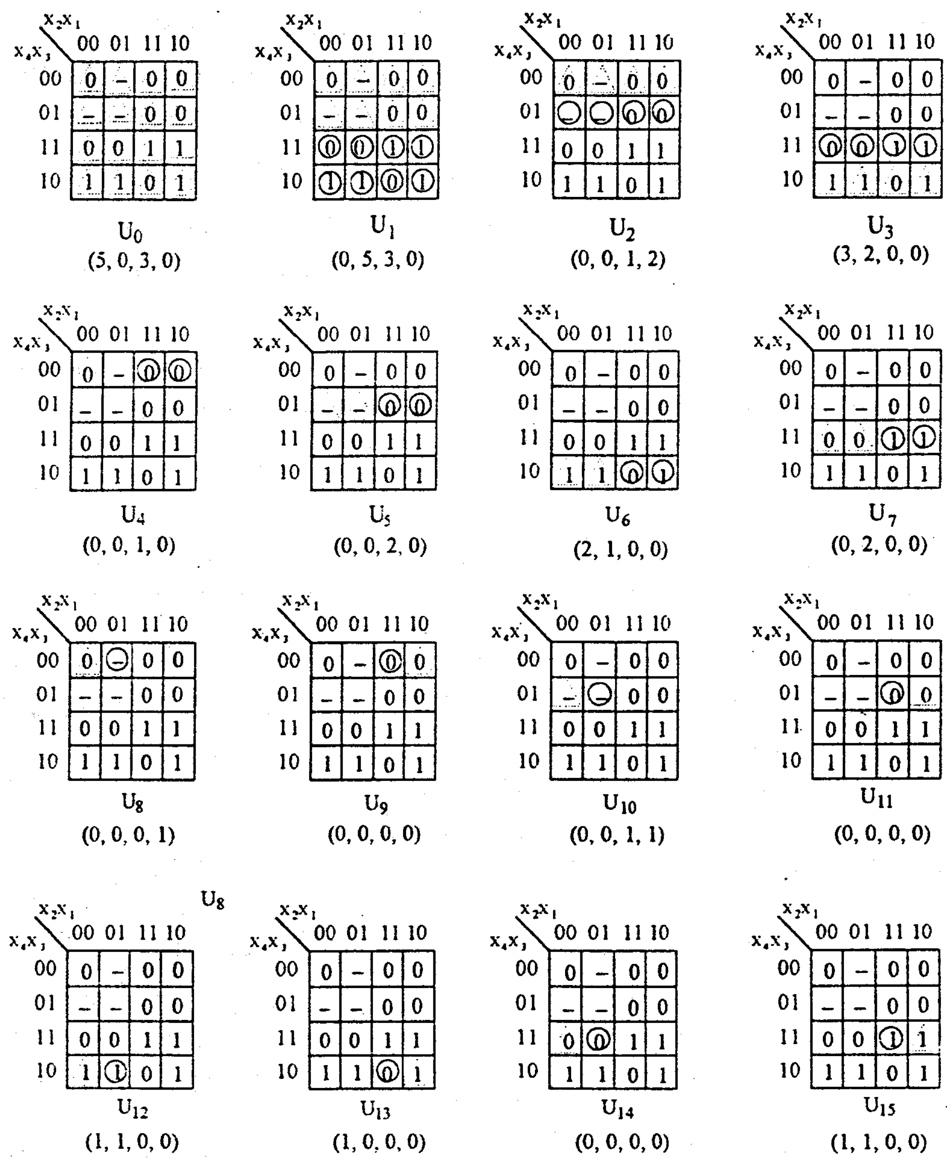

Fig. 1. Standard trivial functions u_l and Paired Haar spectrum of a four-variable incompletely specified Boolean function.

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