2102427
Multimedia Compression Technology

Lecture 7
Wavelet Methods (I)

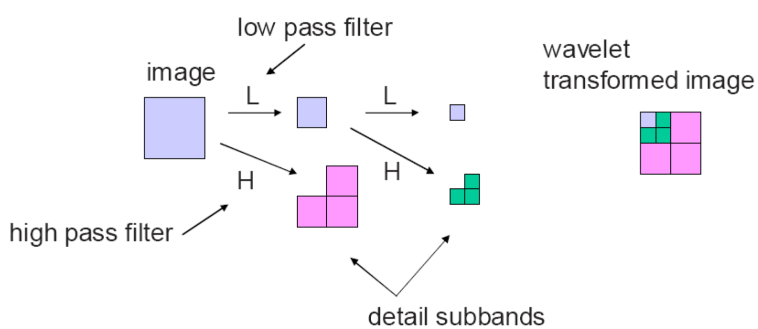
Dr. Supavadee Aramvith
Chulalongkorn University
Supavadee.A@chula.ac.th

Outline

- Averaging and Differencing
- The Haar Transform
- Subband Transforms

Wavelet Transform

- A family of transformations that filters the data into low resolution data plus detail data

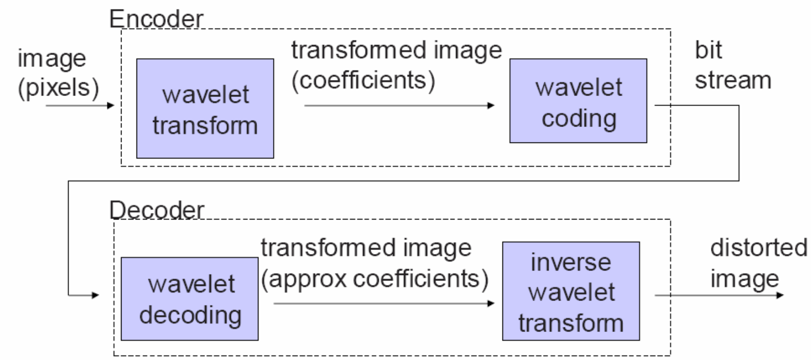


Wavelet Transform – Example (Enhanced)

Wavelet Transform – Example (Actual)

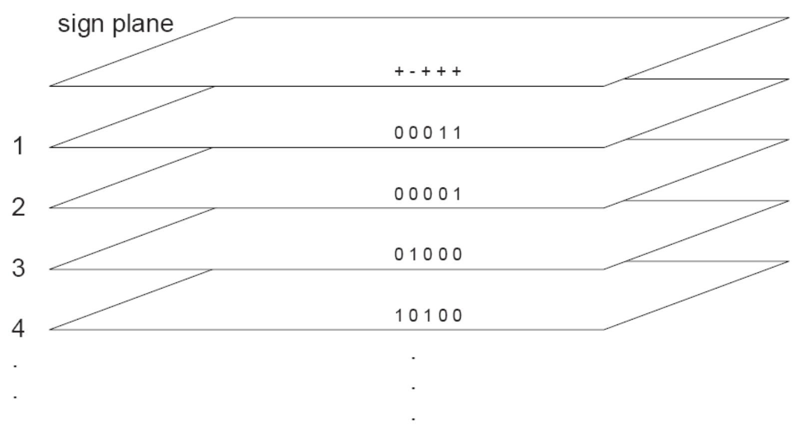
most of the details are small so they are very dark.

Wavelet Transform Compression



Wavelet coder transmits wavelet transformed image in bit plane order with the most significant bits first.

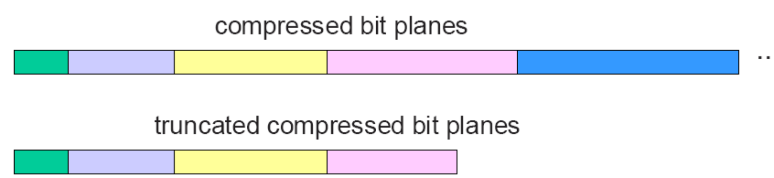
Bit planes of Coefficients



Coefficients are normalized between –1 and 1.

Why Wavelet Compression works

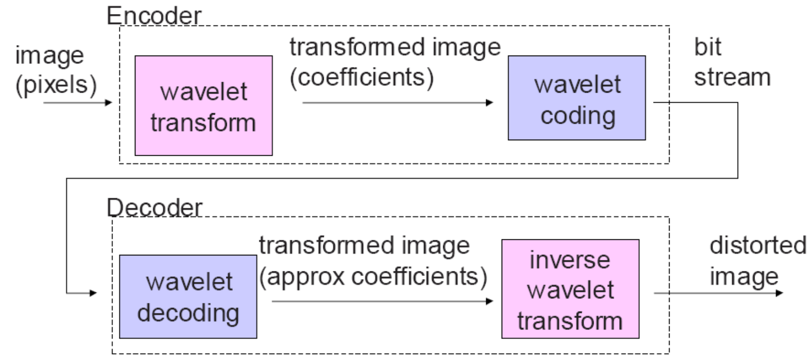
- Wavelet coefficients are transmitted in bit-plane order
  - In the most significant bit planes, many coefficients are zero so they can be coded efficiently.
  - Only some of the bit planes are transmitted (this is where quality is lost when doing lossy compression)
- Natural progressive transmission



Wavelet Coding Methods

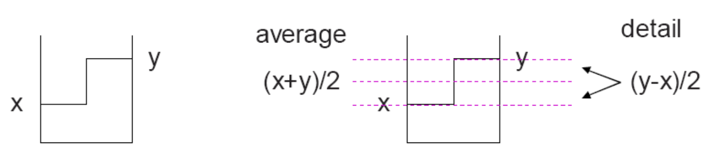
- EZW - Shapiro, 1993
  - Embedded Zerotree coding.
- SPIHT - Said and Pearlman, 1996
  - Set Partitioning in Hierarchical Trees coding. Also uses "zerotrees".
- ECECOW - Wu, 1997
  - Uses arithmetic coding with context.
- EBCOT – Taubman, 2000
  - Uses arithmetic coding with different context.
- JPEG 2000 – new standard based largely on EBCOT
- GTW – Hong, Ladner 2000
  - Uses group testing which is closely related to Golomb codes.

Wavelet Transform Compression



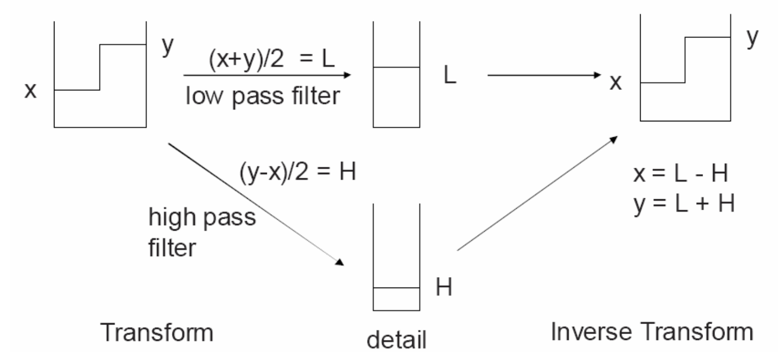
A wavelet transform decomposes the image into a low resolution version and details. The details are typically very small so they can be coded in very few bits.

One Dimensional Average Transform (I)

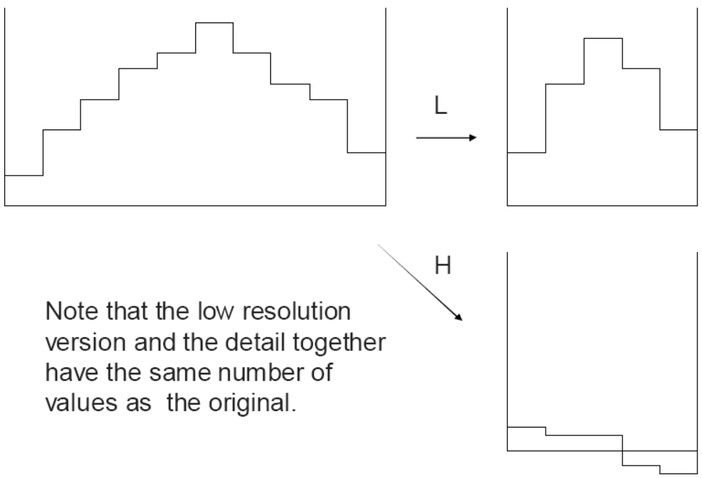


How do we represent
two data points at lower resolution?

One Dimensional Average Transform (II)

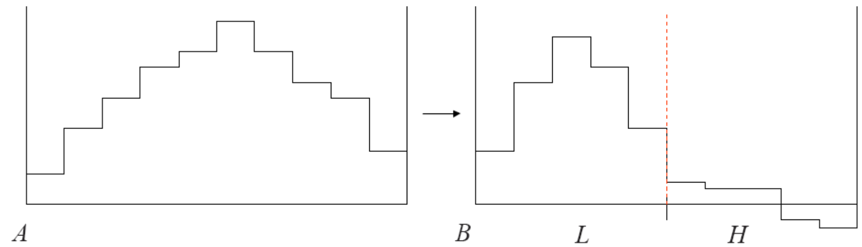


One Dimensional Average Transform (III)



Note that the low resolution version and the detail together have the same number of values as the original.

One Dimensional Average Transform (IV)

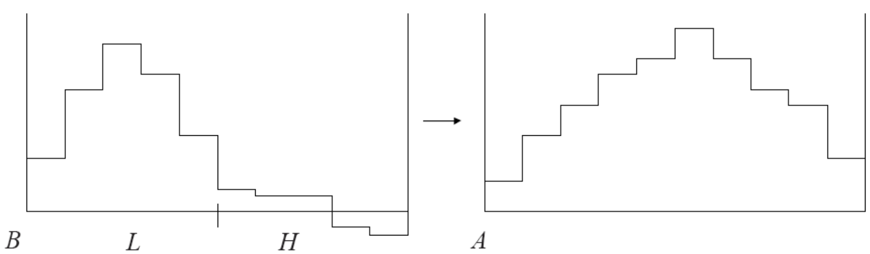


$$B[i] = \frac{1}{2}A[2i] + \frac{1}{2}A[2i+1], \quad 0 \le i < \frac{n}{2}$$

$$B[n/2+i] = -\frac{1}{2}A[2i] + \frac{1}{2}A[2i+1], \quad 0 \le i < \frac{n}{2}$$

L = B[0..n/2-1]
H = B[n/2..n-1]

One Dimensional Average Inverse Transform



$$A[2i] = B[i] - B[n/2+i], \quad 0 \le i < \frac{n}{2}$$

$$A[2i+1] = B[i] + B[n/2+i], \quad 0 \le i < \frac{n}{2}$$

Complexity of the transform

- The number of arithmetic operations as a function of the size of the data

$$\sum_{i=1}^{n} 2^i = \left(\sum_{i=0}^{n} 2^i\right) - 1 = \frac{1-2^{n+1}}{1-2} - 1 = 2^{n+1} - 2 = 2(2^n - 1) = 2(N-1)$$

Example

$$\left(\frac{36/8}{\sqrt{2^0}}, \frac{-16/8}{\sqrt{2^0}}, \frac{-4/4}{\sqrt{2^1}}, \frac{-4/4}{\sqrt{2^1}}, \frac{-1/2}{\sqrt{2^2}}, \frac{-1/2}{\sqrt{2^2}}, \frac{-1/2}{\sqrt{2^2}}, \frac{-1/2}{\sqrt{2^2}}\right).$$

$$\left(\frac{3}{\sqrt{2^4}}, \frac{7}{\sqrt{2^4}}, \frac{11}{\sqrt{2^4}}, \frac{15}{\sqrt{2^4}}, \frac{-1}{\sqrt{2^4}}, \frac{-1}{\sqrt{2^4}}, \frac{-1}{\sqrt{2^4}}, \frac{-1}{\sqrt{2^4}}\right),$$

$$\left(\frac{10}{\sqrt{2^5}}, \frac{26}{\sqrt{2^5}}, \frac{-4}{\sqrt{2^5}}, \frac{-4}{\sqrt{2^5}}, \frac{-1}{\sqrt{2^4}}, \frac{-1}{\sqrt{2^4}}, \frac{-1}{\sqrt{2^4}}, \frac{-1}{\sqrt{2^4}}\right),$$

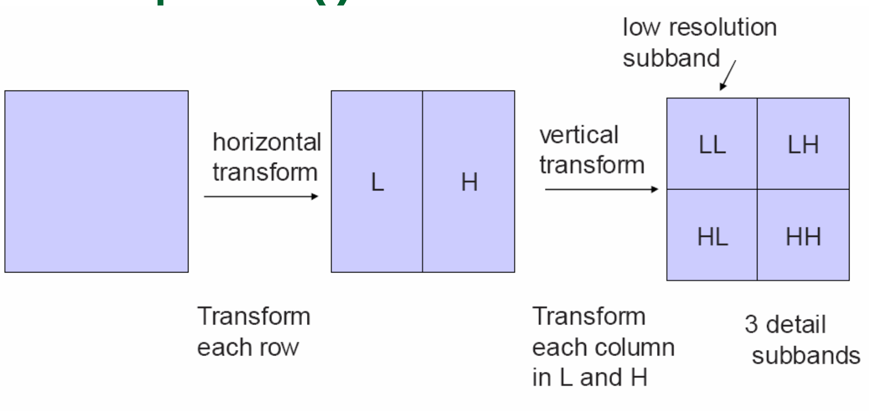
$$\left(\frac{36}{\sqrt{2^6}}, \frac{-16}{\sqrt{2^6}}, \frac{-4}{\sqrt{2^5}}, \frac{-4}{\sqrt{2^5}}, \frac{-1}{\sqrt{2^4}}, \frac{-1}{\sqrt{2^4}}, \frac{-1}{\sqrt{2^4}}, \frac{-1}{\sqrt{2^4}}\right),$$

$$\left(\frac{36/8}{\sqrt{2^0}}, \frac{-16/8}{\sqrt{2^0}}, \frac{-4/4}{\sqrt{2^1}}, \frac{-4/4}{\sqrt{2^1}}, \frac{-1/2}{\sqrt{2^2}}, \frac{-1/2}{\sqrt{2^2}}, \frac{-1/2}{\sqrt{2^2}}, \frac{-1/2}{\sqrt{2^2}}\right).$$

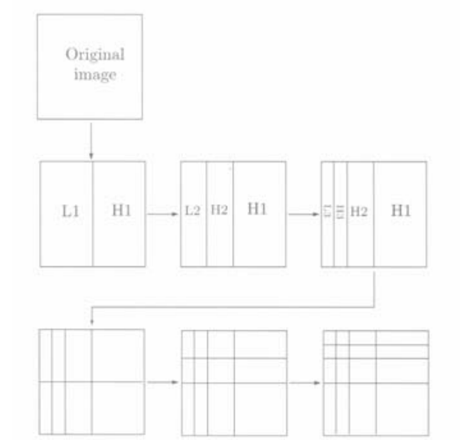
Two Dimensional Transform (I)

- 2 approaches
  - Standard decomposition
  - Pyramid decomposition

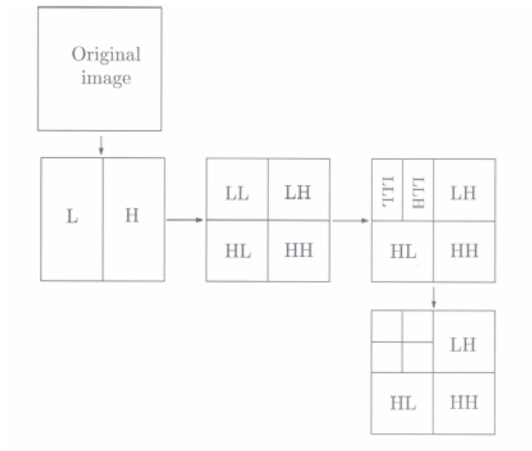
Standard Image Wavelet Transform and Decomposition (I)



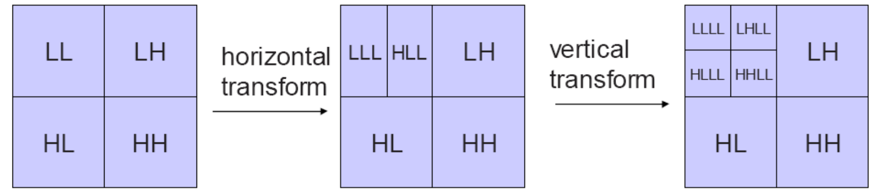
Standard Image Wavelet Transform and Decomposition (II)



Pyramid Image Wavelet Transform



Two Dimensional Transform



Transform each row in LL

Transform each column in LLL and HLL

2 levels of transform gives 7 subbands.
k levels of transform gives 3k + 1 subbands.

Wavelet Transformed Image

2 levels of wavelet transform

1 low resolution subband

6 detail subbands

Wavelet Transforms

- Technically wavelet transforms are special kinds of linear transformations. Easiest to think of them as filters
  - The filters depend only on a constant number of values (bounded support)
  - Preserve energy (norm of pixels = norm of the coefficients)
  - Inverse filters also have bounded support
- Well-known wavelet transforms
  - Haar transform – like the average but orthogonal to preserve energy. Not used in practice.
  - Daubechies 9/7 – biorthogonal (inverse is not the transpose). Most commonly used in practice.

An 8x8 Image and Its Subband Decompostion

12	12	12	12	14	12	12	12
12	12	12	12	14	12	12	12
12	12	12	12	14	12	12	12
12	12	12	12	14	12	12	12
12	12	12	12	14	12	12	12
16	16	16	16	14	16	16	16
12	12	12	12	14	12	12	12
12	12	12	12	14	12	12	12

(a)

12	12	13	12	0	0	2	0
12	12	13	12	0	0	2	0
12	12	13	12	0	0	2	0
12	12	13	12	0	0	2	0
12	12	13	12	0	0	2	0
16	16	15	16	0	0	2	0
12	12	13	12	0	0	2	0
12	12	13	12	0	0	2	0

(b)

12	12	13	12	0	0	2	0
12	12	13	12	0	0	2	0
14	14	14	14	0	0	0	0
12	12	13	12	0	0	2	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
4	4	2	4	0	0	4	0
0	0	0	0	0	0	0	0

(c)

The Subband Decomposition of a Diagonal Line

12	16	12	12	12	12	12	12
12	12	16	12	12	12	12	12
12	12	12	16	12	12	12	12
12	12	12	12	16	12	12	12
12	12	12	12	12	16	12	12
12	12	12	12	12	12	16	12
12	12	12	12	12	12	12	16
12	12	12	12	12	12	12	12

(a)

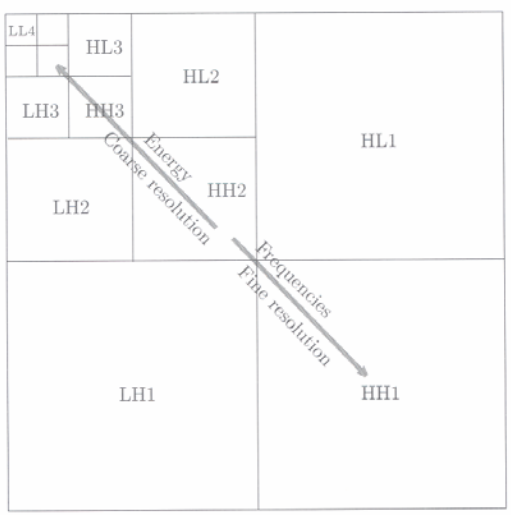
14	12	12	12	4	0	0	0
12	14	12	12	0	4	0	0
12	14	12	12	0	4	0	0
12	12	14	12	0	0	4	0
12	12	14	12	0	0	4	0
12	12	12	14	0	0	0	4
12	12	12	14	0	0	0	4
12	12	12	12	0	0	0	0

(b)

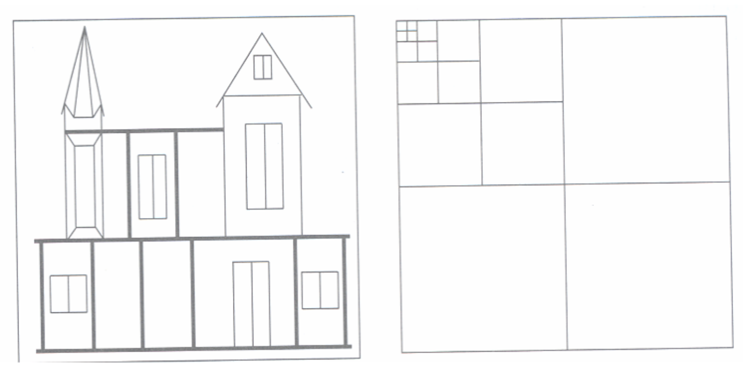
13	13	12	12	2	2	0	0
12	13	13	12	0	2	2	0
12	12	13	13	0	0	2	2
12	12	12	13	0	0	0	2
2	2	0	0	4	4	0	0
0	2	2	0	0	4	4	0
0	0	2	2	0	0	4	4
0	0	0	2	0	0	0	4

(c)

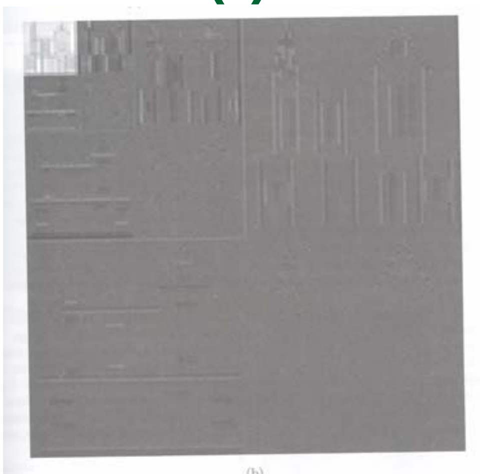
Subbands and Levels in Wavelet Decomposition



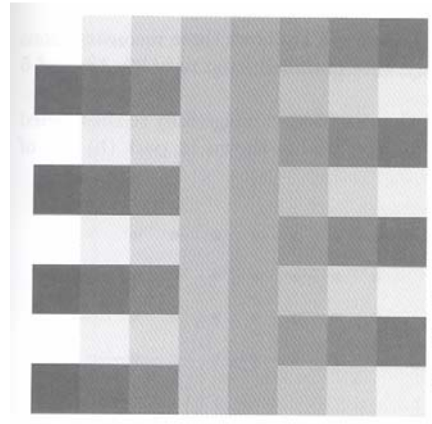
An Example of the Pyramid Image Wavelet Transform (I)



An Example of the Pyramid Image Wavelet Transform (II)

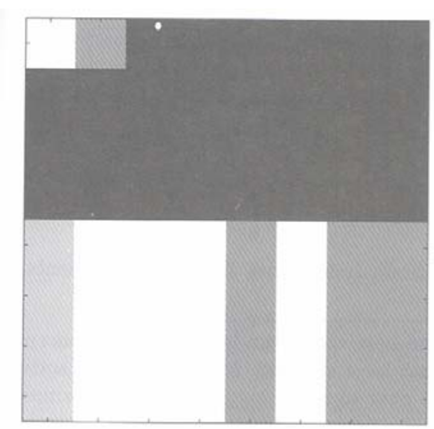


Highly Correlated Image (I)



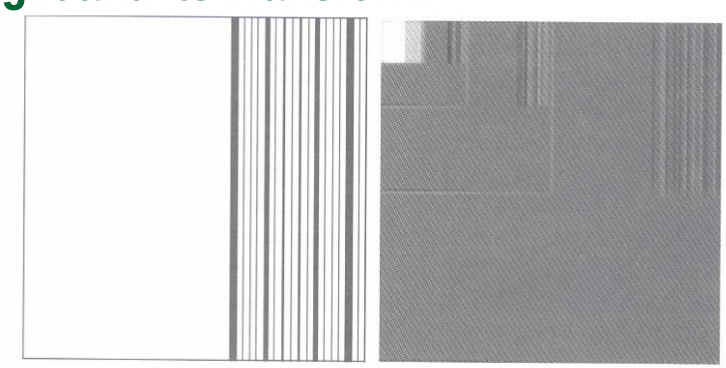
255	224	192	159	127	95	63	32
0	32	64	159	127	159	191	223
255	224	192	159	127	95	63	32
0	32	64	159	127	159	191	223
255	224	192	159	127	95	63	32
0	32	64	159	127	159	191	223
255	224	192	159	127	95	63	32
0	32	64	159	127	159	191	223

Its Haar Transform (II)

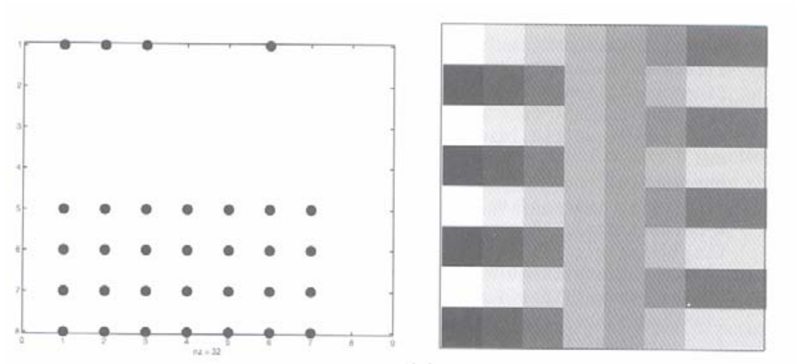


1051	34.0	-44.5	-0.7	-1.0	-62	0	-1.0
0	0.0	0.0	0.0	0.0	0	0	0.0
0	0.0	0.0	0.0	0.0	0	0	0.0
0	0.0	0.0	0.0	0.0	0	0	0.0
48	239.5	112.8	90.2	31.5	64	32	31.5
48	239.5	112.8	90.2	31.5	64	32	31.5
48	239.5	112.8	90.2	31.5	64	32	31.5
48	239.5	112.8	90.2	31.5	64	32	31.5

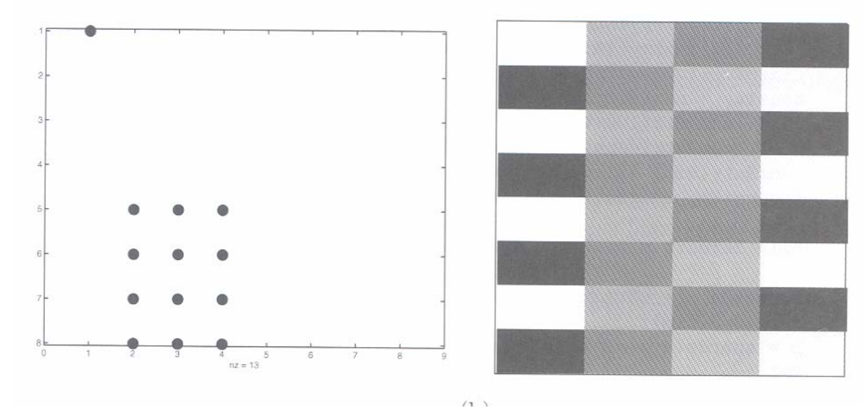
A 128x128 Image with Activity on the Right and Its Transform



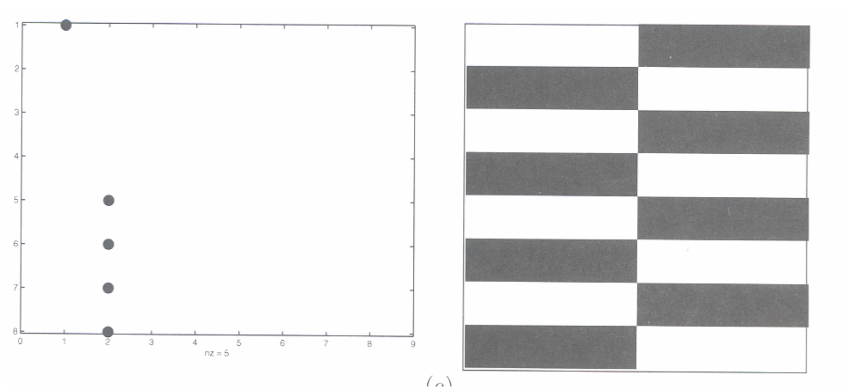
Three Lossy Reconstruction of an 8x8 Image (I)



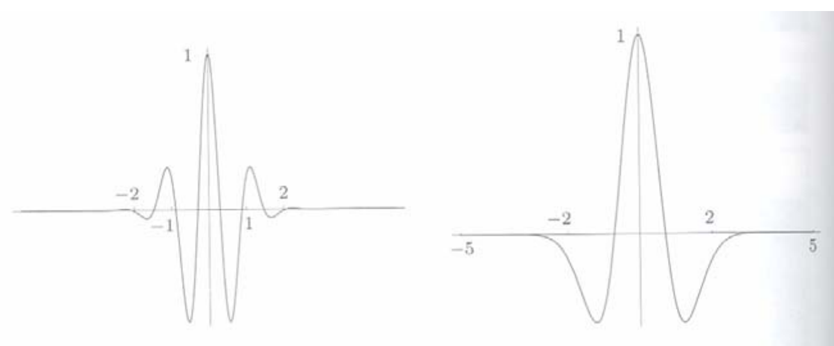
Three Lossy Reconstruction of an 8x8 Image (II)



Three Lossy Reconstruction of an 8x8 Image (III)



Reconstructing a 128x128 Simple Image from 4% of its Coefficients



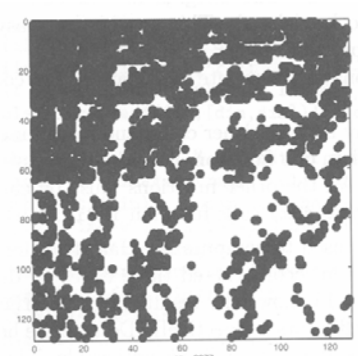
Matlab Code for the Haar Transform of an Image (I)

```
clear; % main program
filename='lena128'; dim=128;
fid=fopen(filename,'r');
if fid==-1 disp('file not found')
else img=fread(fid,[dim,dim])'; fclose(fid);
end
thresh=0.0;        % percent of transform coefficients deleted
figure(1), imagesc(img), colormap(gray), axis off, axis square
w=harmatt(dim);  % compute the Haar dim x dim transform matrix
timg=w*img*w';   % forward Haar transform
tsort=sort(abs(timg(:)));
tthresh=tsort(floor(max(thresh*dim*dim,1)));
cim=timg.*(abs(timg) > tthresh);
[i,j,s]=find(cim);
dimg=sparse(i,j,s,dim,dim);
% figure(2) displays the remaining transform coefficients
%figure(2), spy(dimg), colormap(gray), axis square
figure(2), image(dimg), colormap(gray), axis square
cimg=full(w'*sparse(dimg)*w);     % inverse Haar transform
density = nnz(dimg);
disp([num2str(100*thresh) '% of smallest coefficients deleted.'])
disp([num2str(density) ' coefficients remain out of ' ...
 num2str(dim) 'x' num2str(dim) '.'])
figure(3), imagesc(cimg), colormap(gray), axis off, axis square
```

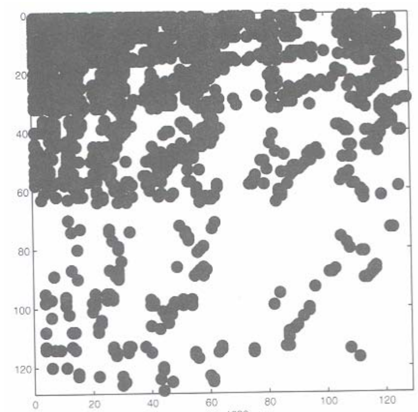

Matlab Code for the Haar Transform of an Image (II)

```
File harmatt.m with two functions

function x = harmatt(dim)
num=log2(dim);
p = sparse(eye(dim)); q = p;
i=1;
while i<=dim/2;
 q(1:2*i,1:2*i) = sparse(individ(2*i));
 p=p*q; i=2*i;
end
x=sparse(p);

function f=individ(n)
x=[1, 1]/sqrt(2);
y=[1,-1]/sqrt(2);
while min(size(x)) < n/2
 x=[x, zeros(min(size(x)),max(size(x)));...
   zeros(min(size(x)),max(size(x))), x];
end
while min(size(y)) < n/2
 y=[y, zeros(min(size(y)),max(size(y)));...
   zeros(min(size(y)),max(size(y))), y];
end
f=[x;y];
```


Three Lossy Reconstruction of the 128x128 Lena Image (I)

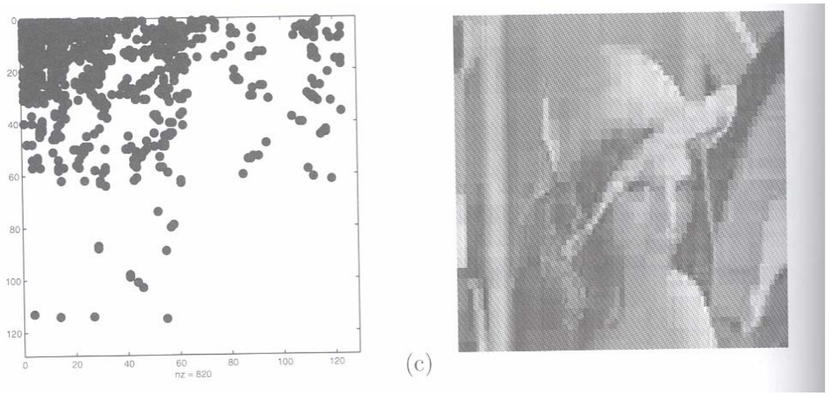



Three Lossy Reconstruction of the 128x128 Lena Image (II)



(b)

Three Lossy Reconstruction of the 128x128 Lena Image (III)



(c)

Lossy Wavelet Image Compression

- Lossy involves the discarding of coefficients
- Sparseness ratio
  - The measurement of number of coefficients discarded
  - The number of nonzero wavelet coefficients divided by number of coefficients left after some are discarded
  - Higher sparseness ratio -> fewer coefficients left
    - Better compression -> poorly reconstructed image

The Haar Transform

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \phi(t-k) + \sum_{k=-\infty}^{\infty} \sum_{j=0}^{\infty} d_{j,k} \psi(2^j t - k)$$

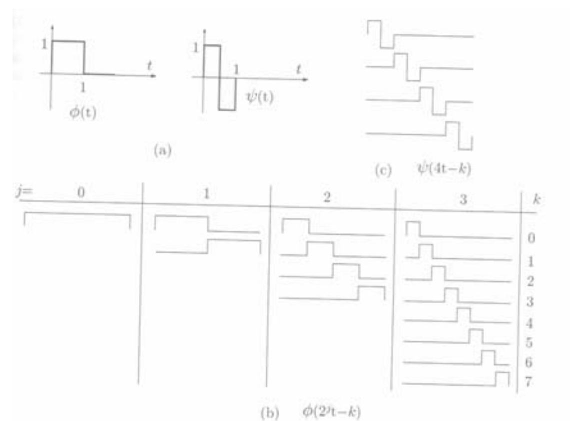
Basic scale function

$$\phi(t) = \begin{cases} 1 & ,0 \le t < 1 \\ 0 & ,otherwise \end{cases}$$

Basic Haar wavelet – step function

$$\psi(t) = \begin{cases} 1 & ,0 \le t < 0.5 \\ -1 & ,0.5 \le t < 1 \end{cases}$$

The Haar Basis Scale and Wavelet Functions



Haar Matrix Representation (I)

$$A_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \quad A_1 \begin{pmatrix} 255 \\ 224 \\ 192 \\ 159 \\ 127 \\ 95 \\ 63 \\ 32 \end{pmatrix} = \begin{pmatrix} 239.5 \\ 175.5 \\ 111.0 \\ 47.5 \\ 15.5 \\ 16.5 \\ 16.0 \\ 15.5 \end{pmatrix},$$

Haar Matrix Representation (II)

$$A_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

Haar Matrix Representation (III)

$$A_2 \begin{pmatrix} 239.5 \\ 175.5 \\ 111.0 \\ 47.5 \\ 15.5 \\ 16.5 \\ 16.0 \\ 15.5 \end{pmatrix} = \begin{pmatrix} 207.5 \\ 79.25 \\ 32.0 \\ 31.75 \\ 15.5 \\ 16.5 \\ 16.0 \\ 15.5 \end{pmatrix}, \quad A_3 \begin{pmatrix} 207.5 \\ 79.25 \\ 32.0 \\ 31.75 \\ 15.5 \\ 16.5 \\ 16.0 \\ 15.5 \end{pmatrix} = \begin{pmatrix} 143.375 \\ 64.125 \\ 32. \\ 31.75 \\ 15.5 \\ 16.5 \\ 16. \\ 15.5 \end{pmatrix}.$$

Haar Matrix Representation (IV)

$$W \begin{pmatrix} 255 \\ 224 \\ 192 \\ 159 \\ 127 \\ 95 \\ 63 \\ 32 \end{pmatrix} = \begin{pmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{-1}{8} & \frac{-1}{8} & \frac{-1}{8} & \frac{-1}{8} \\ \frac{1}{4} & \frac{1}{4} & \frac{-1}{4} & \frac{-1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{-1}{4} & \frac{-1}{4} \\ \frac{1}{2} & \frac{-1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{-1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{-1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{-1}{2} \end{pmatrix} \begin{pmatrix} 255 \\ 224 \\ 192 \\ 159 \\ 127 \\ 95 \\ 63 \\ 32 \end{pmatrix} = \begin{pmatrix} 143.375 \\ 64.125 \\ 32 \\ 31.75 \\ 15.5 \\ 16.5 \\ 16 \\ 15.5 \end{pmatrix}.$$

Example of Matrix Wavelet Transform

```
a1=[1/2 1/2 0 0 0 0 0 0; 0 0 1/2 1/2 0 0 0 0;
 0 0 0 0 1/2 1/2 0 0; 0 0 0 0 0 0 1/2 1/2;
 1/2 -1/2 0 0 0 0 0 0; 0 0 1/2 -1/2 0 0 0 0;
 0 0 0 0 1/2 -1/2 0 0; 0 0 0 0 0 0 1/2 -1/2];
% a1*[255; 224; 192; 159; 127; 95; 63; 32];
a2=[1/2 1/2 0 0 0 0 0 0; 0 0 1/2 1/2 0 0 0 0;
 1/2 -1/2 0 0 0 0 0 0; 0 0 1/2 -1/2 0 0 0 0;
 0 0 0 0 1 0 0 0; 0 0 0 0 0 1 0 0;
 0 0 0 0 0 0 1 0; 0 0 0 0 0 0 0 1];
a3=[1/2 1/2 0 0 0 0 0 0; 1/2 -1/2 0 0 0 0 0 0;
 0 0 1 0 0 0 0 0; 0 0 0 1 0 0 0 0;
 0 0 0 0 1 0 0 0; 0 0 0 0 0 1 0 0;
 0 0 0 0 0 0 1 0; 0 0 0 0 0 0 0 1];
w=a3*a2*a1;
dim=8; fid=fopen('8x8','r');
img=fread(fid,[dim,dim])'; fclose(fid);
w*img*w' % Result of the transform
```


Example of Matrix Wavelet Transform

131.375	4.250	−7.875	−0.125	−0.25	−15.5	0	−0.25
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
12.000	59.875	39.875	31.875	15.75	32.0	16	15.75
12.000	59.875	39.875	31.875	15.75	32.0	16	15.75
12.000	59.875	39.875	31.875	15.75	32.0	16	15.75
12.000	59.875	39.875	31.875	15.75	32.0	16	15.75

Subband Transforms

- Orthogonal transform
  - Inner product of the data with a set of basis functions
  - Output is set of transform coefficients
- Discrete inner product of the two vectors

$$\langle f, g \rangle = \sum_i f_i g_i$$

- Wavelet transform = subband transform
  - Compute a convolution of the data items with a set of bandpass filters
  - Each resulting subband encodes a particular portion of the frequency content of the data

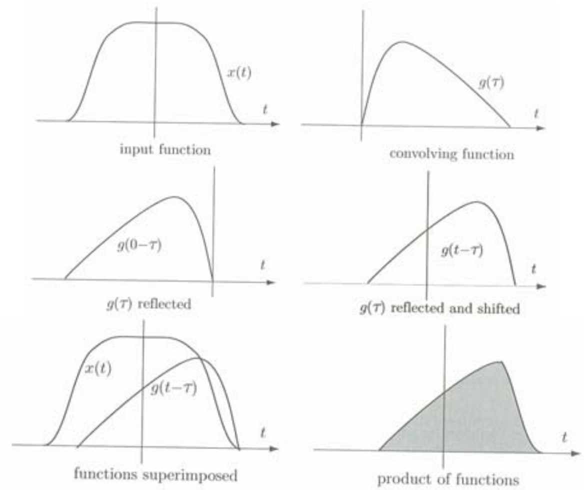
Subband Transforms

- Discrete convolution of two vectors

$$h(i) = f(i) * g(i) = \sum_i f(j)\ g(i-j)$$

- Wavelet transform = subband transform
  - Compute a convolution of the data items with a set of bandpass filters
  - Each resulting subband encodes a particular portion of the frequency content of the data

Convolution of x(t) and g(t)



Example of Convolution

$$h(0) = \sum_{j=0}^{0} f(j)g(0-j) = f(0)g(0)$$

$$h(1) = \sum_{j=0}^{1} f(j)g(1-j) = f(0)g(1) + f(1)g(0)$$

$$h(2) = \sum_{j=0}^{2} f(j)g(2-j) = f(0)g(2) + f(1)g(1) + f(2)g(0)$$

$$h(3) = \sum_{j=0}^{3} f(j)g(3-j) = f(0)g(3) + f(1)g(2) + f(2)g(1) + f(3)g(0)$$

$$h(4) = \sum_{j=0}^{4} f(j)g(4-j) = f(0)g(4) + f(1)g(3) + f(2)g(2) + f(3)g(1) + f(4)g(0)$$

$$h(5) = \sum^{5} f(j)g(5-j) = f(1)g(4) + f(2)g(3) + f(3)g(2) + f(4)g(1) + f(5)g(0)$$

$$h(6) = \sum_{j=2}^{5} f(j)g(6-j) = f(2)g(4) + f(3)g(3) + f(4)g(2) + f(5)g(1)$$

$$h(7) = \sum_{j=3}^{5} f(j)g(7-j) = f(3)g(4) + f(4)g(3) + f(5)g(2)$$

$$h(8) = \sum_{j=4}^{5} f(j)g(8-j) = f(4)g(4) + f(5)g(3)$$

$$h(9) = \sum_{j=5}^{5} f(j)g(9-j) = f(5)g(4)$$

Applying Convolution to Denoising a Function

$$g(t) = \begin{cases} 1, -a/2 < t < a/2 \\ \frac{1}{2}, t = \pm a/2 \\ 0, \textit{elsewhere} \end{cases}$$