Fourier Transform

Decomposition of signals in terms of sinusoids

$$A^{i\omega x} = A\cos(\omega x) + iA\sin(\omega x)$$

Using complex exponential as an input

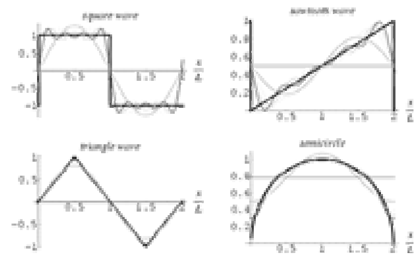
$$g[x] = e^{i\omega x} * h[x] = \sum_{k=-\infty}^{\infty} h[k]e^{i\omega(x-k)} = e^{i\omega x}\sum_{k=-\infty}^{\infty} h[k]e^{i\omega k}$$

$$g[x] = e^{i\omega x}H(\omega)$$

Given complex exponential as input, output is again complex exponential scaled by H(ω)

H(ω) is the frequency response of linear-time Invariant systems

Fourier Series



Fourier Series and Transform

Decomposition of signals in terms of sinusoids – Fourier series

$$f[x] = \frac{1}{2\pi}\sum_{j=-\pi}^{\pi} c_k cos[kx + \phi_k]$$

$$f[x] = \frac{1}{2\pi}\sum_{k=-\pi}^{\pi} a_k \cos[kx] + b_k \sin[kx] \qquad a_k = \sum_{j=-\infty}^{\infty} f[j]\cos[kj] \quad b_k = \sum_{j=-\infty}^{\infty} f[j]\sin[kj]$$

Fourier series and transform $\qquad \sum_j f_j b_j$

$$f[x] = \frac{1}{2\pi}\sum_{k=-\pi}^{\pi} c_k e^{ikx} \quad \text{and} \quad c_k = \sum_{j=-\infty}^{\infty} f[j]e^{-ijk}$$

Alternative form

$$f[x] = \frac{1}{2\pi}\sum_{\omega=-\pi}^{\pi} F[\omega]e^{i\omega x} \quad \text{and} \quad F[\omega] = \sum_{k=-\infty}^{\infty} f[k]e^{-i\omega k}$$

Central theme – approximate a function with given a family of basis functions

Fourier Transform

$$|F[\omega]| = \sqrt{F[\omega]_r^2 + F[\omega]_i^2}$$

$$F[\omega] = \sum_{k=-\infty}^{\infty} f[k]e^{-i\omega k}$$

$$\prec F[\omega]| = tan^{-1}(F_i(\omega)/F_r(\omega))$$

- Connecting back to LTI
- Frequency response of a linear time invariant system is the Fourier transform of the unit-impulse response
- i.e. LTI are uniquely characterized by their impulse response and equivalently by their frequency response
- Relationship between frequency and space domains

$$g[x] = f[x] * h[x] \qquad G[\omega] = F[\omega]H[\omega]$$

$$F = Mf \qquad M = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ e^{-0i} & e^{i} & e^{-2i} & \dots & e^{-Ti} \\ e^{-0i} & e^{2i} & e^{-4i} & \dots & e^{-2Ti} \\ \vdots & & & \vdots & \\ e^{-0i} & e^{Ti} & e^{-2Ti} & \dots & e^{-T^2 i} \end{bmatrix}$$

Fourier transform as representation

- The spatial information is lost – spectral information is gained
- Can we achieve simultaneous localization in space (time) and frequency ? -> only within some bounds
- Windowed Fourier Transform – balance between the two

Filtering - Fourier Transform

- Examples FT phase and frequency information

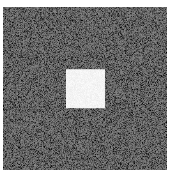
$$f \longrightarrow \boxed{\text{filter}} \longrightarrow g$$

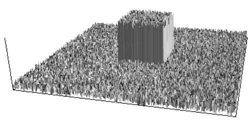
$$g[x] = f[x] * h[x] \qquad G[\omega] = F[\omega]H[\omega]$$

- Low pass filtering
- High pass filtering
- Band pass filtering
- Differentiation

Fourier transform and filtering

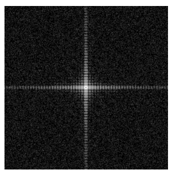
- http://astronomy.swin.edu.au/~pbourke/analysis/imagefilter/

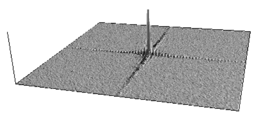




Fourier transform and denoising

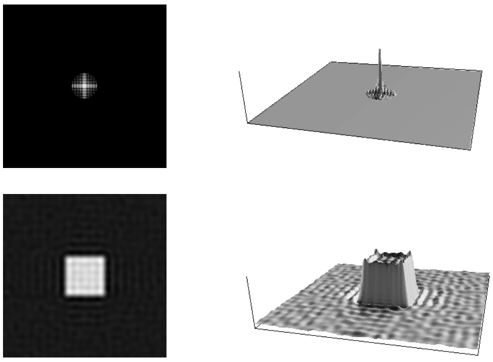
- http://astronomy.swin.edu.au/~pbourke/analysis/imagefilter/





Fourier transform of the image

Low Pass filtering



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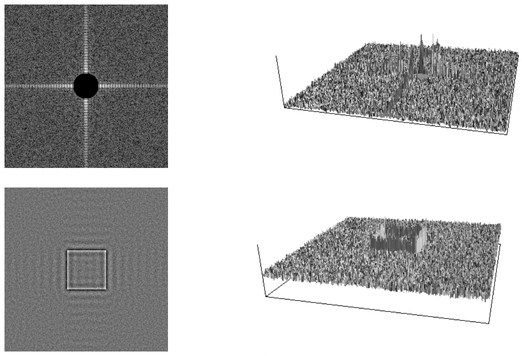
High pass filtering



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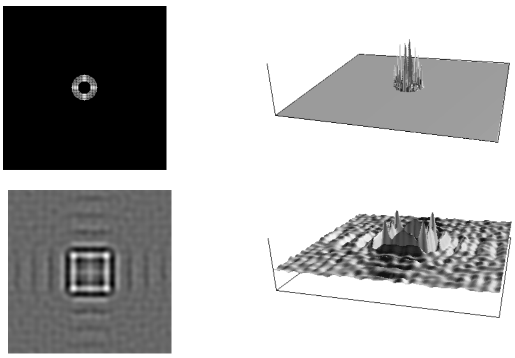
Band-pass filtering



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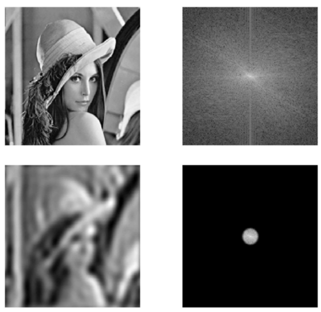
FFT by example



original

low pass

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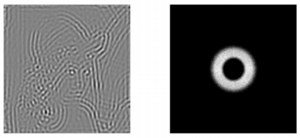
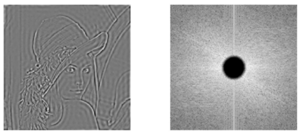
FFT Example



bandpass



highpass

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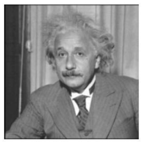
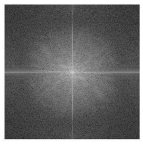
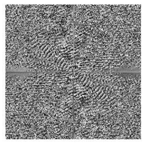
Importance of phase



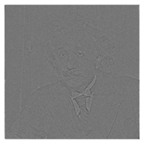
original



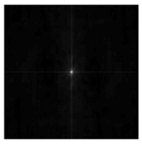
power



phase



constant power + original phase



original power + zero phase

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Scale Space Representations

- Gaussian Pyramids

  enable to extract different structures in the image – since different structures are more apparent at different scales

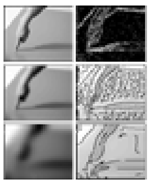
  (useful for search over scale - detection, spatial search, feature tracking)

  $I(x, y, \sigma) = I_0(x, y) * G(x, y, \sigma)$

  (Witkin'83, Koenderink, van Doorn'86)

  Family of functions – solutions to diffusion eg.

  $$\frac{\partial I}{\partial \sigma} = \frac{\partial I}{\partial x^2} + \frac{\partial I}{\partial y^2} = \nabla^2 I$$



Courtesy T. Lindenberg, KTH

- Causality (features at coarse level – have cause at fine level)
- Homogeneity and Isotropy (blurring is space-invariant)

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Anisotropic Diffusion

Problem with traditional scale space models

Gaussian is symmetric – smoothes over edges
Does not preserve localization of edges
Idea – edge preserving smoothing

$$\frac{\partial I}{\partial \sigma} = \frac{\partial I}{\partial x^2} + \frac{\partial I}{\partial y^2} = \nabla^2 I \qquad I(x, y, 0)$$

With spatially varying term (Perona & Malik '90)

$$\frac{\partial I}{\partial \sigma} = \nabla.(c(x, y, \sigma)\nabla I = c(x, y, \sigma)\nabla^2 I + (\nabla c(x, y, \sigma)).(\nabla I)$$

If we knew where are the edges – we can
Create a mask $c(x, y)$ (assuming that $c(x, y, \sigma)$) is independent of σ

We get noise free image and smooth where there are no edges

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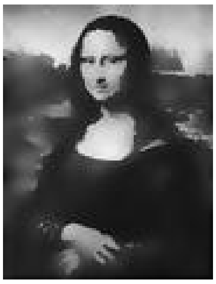
Anisotropic Diffusion - Example



Images courtesy P. Kovesi (www.csee.unwa.au/~pk)