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Hybrid Arithmetic-Walsh Transform

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Abstract

An idea of a combined Arithmetic-Walsh transform is introduced in this article. Recursive relationships between higher and lower matrix orders of hybrid Arithmetic-Walsh transform are developed. The new transform has a structure similar to that of the Walsh and Arithmetic transform matrices, so it combines the computational advantages of these two transforms and has similar fast forward and inverse transforms based on recursive Kronecker product equations.

1. Introduction

Walsh functions form an orthonormal system that can be applied in many situations [1, 3-8] and is very interesting both from theoretical and practical points of view. The Walsh system can perform all the usual applications of orthogonal system (e.g. data transmission, multiplexing, filtering, speech processing, image enhancement, pattern recognition, control theory, statistical analysis, solving differential equations, and the analysis, synthesis and classification and testing of logical circuits) and can perform them more efficiently [1, 3-8]. Arithmetic transform has been used in many applications of logic design and in reliability analysis [1, 2, 4-6]. As each of these transforms has some advantages and disadvantages based on the specific problem to be solved, it is interesting to create the new hybrid transform which combines properties of both Walsh and Arithmetic transform and which in special cases becomes either Arithmetic or Walsh transform and this is the main contribution of our article.

2. Basic Definitions of Walsh and Arithmetic Transform

Definition 1: The Walsh transform matrix of order 2^n is defined as [1, 3-8]

$$W(n)=\begin{bmatrix} W(n-1) & W(n-1) \\ W(n-1) & -W(n-1) \end{bmatrix}, \qquad (1)$$

$$W(0)=[1],\ n=1,2,3,\dots .$$

Also $$W(n)=W(1)\otimes W(n-1)=\bigotimes_{i=1}^{n} W(1) \qquad (2)$$

for $n=1,2,3\dots$.

Example 1: The Walsh matrix of 2^3 is [1, 3-8]

$$W(3)=\bigotimes_{i=1}^{3} W(1)=\bigotimes_{i=1}^{3}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

Definition 2: The matrix of order 2^n for the Arithmetic transform is defined as [2, 4-6]

$$A(n)=\begin{bmatrix} A(n-1) & 0 \\ -A(n-1) & A(n-1) \end{bmatrix}, \qquad (3)$$

$$A(0)=[1],\ n=1,2,3,\dots .$$

Also $$A(n)=A(1)\otimes A(n-1)=\bigotimes_{i=1}^{n} A(1) \qquad (4)$$

for $n=1,2,3\dots$.

Example 2: The Arithmetic transform matrix of 2^3 is [2, 4-6]

$$A(3)=\bigotimes_{i=1}^{3} A(1)=\bigotimes_{i=1}^{3}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

In all equations inside this paper, the symbols '$\overset{j}{\otimes}$' and '$\otimes$' represent the Kronecker product [1, 3-8] of j and two matrices, respectively.

3. Hybrid Arithmetic-Walsh Transform

In this section, a new hybrid transform based on Arithmetic and Walsh functions will be created.

Definition 3: For n-variable discrete binary function $f(x)$, the rth-order new Arithmetic-Walsh transform matrix $AW_r(n)$ and its inverse $AW_r^{-1}(n)$ are defined as:

$$AW_r(n) = W(r) \otimes A(n-r) = \left(\overset{r}{\otimes}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\right) \otimes \left(\overset{n-r}{\otimes}\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\right), \quad (5)$$

$$AW_r^{-1}(n) = W^{-1}(r) \otimes A^{-1}(n-r)$$

$$= \frac{1}{2^r}\left(\overset{r}{\otimes}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\right) \otimes \left(\overset{n-r}{\otimes}\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\right). \quad (6)$$

where $n = 1, 2, 3, \dots$ and $0 \le r \le n$.

When the Kronecker product of j matrices is carried out in equations 5 and 6 for $j = 0$, then the term '$\overset{j}{\otimes}$' disappears from them.

It can be noticed that the special cases of rth-order Arithmetic-Walsh transform are the Arithmetic transform for $r = 0$ and the Walsh transform for $r = n$.

Example 3: For $r = 2$ and $n = 3$, the Arithmetic-Walsh transform matrix $AW_2(3)$ and its inverse $AW_2^{-1}(3)$ are:

$$AW_2(3) = \left(\overset{2}{\otimes}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\right) \otimes \begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0\\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1\\
1 & 0 & -1 & 0 & 1 & 0 & -1 & 0\\
-1 & 1 & 1 & -1 & -1 & 1 & 1 & -1\\
1 & 0 & 1 & 0 & -1 & 0 & -1 & 0\\
-1 & 1 & -1 & 1 & 1 & -1 & 1 & -1\\
1 & 0 & -1 & 0 & -1 & 0 & 1 & 0\\
-1 & 1 & 1 & -1 & 1 & -1 & -1 & 1
\end{bmatrix},$$

$$AW_2^{-1}(3) = \frac{1}{4}\left(\overset{2}{\otimes}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\right) \otimes \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
1 & 0 & -1 & 0 & 1 & 0 & -1 & 0\\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1\\
1 & 0 & 1 & 0 & -1 & 0 & -1 & 0\\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1\\
1 & 0 & -1 & 0 & -1 & 0 & 1 & 0\\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1
\end{bmatrix}.$$

For a 3-variable function $f_1(x_1, x_2, x_3) = \sum(0, 2, 6, 7)$ the 2nd-order Arithmetic-Walsh spectrum can be calculated as follow:

$$AW_2(3) \times \begin{bmatrix}1\\0\\1\\0\\0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0\\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1\\
1 & 0 & -1 & 0 & 1 & 0 & -1 & 0\\
-1 & 1 & 1 & -1 & -1 & 1 & 1 & -1\\
1 & 0 & 1 & 0 & -1 & 0 & -1 & 0\\
-1 & 1 & -1 & 1 & 1 & -1 & 1 & -1\\
1 & 0 & -1 & 0 & -1 & 0 & 1 & 0\\
-1 & 1 & 1 & -1 & 1 & -1 & -1 & 1
\end{bmatrix} \times \begin{bmatrix}1\\0\\1\\0\\0\\0\\1\\1\end{bmatrix} = \begin{bmatrix}3\\-2\\-1\\0\\1\\-2\\1\\0\end{bmatrix},$$

Example 4: For $r = 1$ and $n = 4$, the Arithmetic-Walsh transform matrix $AW_1(4)$ and its inverse $AW_1^{-1}(4)$ are:

$$AW_1(4) = \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \otimes \left(\overset{3}{\otimes}\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\right)$$

$$= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0\\
1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0\\
-1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0\\
1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0\\
-1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1
\end{bmatrix},$$

For a 4-variable function $f_2(x_1, x_2, x_3, x_4) = \sum(1,2,5,8,11,12,14)$ the *1*st-order Arithmetic-Walsh spectrum can be calculated as follow:

$$
AW_1(4) \times \begin{bmatrix} 0\\1\\1\\0\\0\\1\\0\\0\\1\\0\\0\\1\\1\\0\\1\\0 \end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0\\
1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 & 0\\
-1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0\\
1 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0\\
-1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1
\end{bmatrix}
$$

$$
\times \begin{bmatrix} 0\\1\\1\\0\\0\\1\\0\\0\\1\\0\\0\\1\\1\\0\\1\\0 \end{bmatrix} = \begin{bmatrix} 1\\0\\0\\0\\0\\0\\0\\-1\\-1\\2\\2\\-4\\0\\0\\-2\\3 \end{bmatrix}
$$

As the hybrid Arithmetic-Walsh transform has recursive equations 5 and 6 that are based on the Kronecker product, it is possible to derive fast algorithm for the calculation of the Arithmetic-Walsh transform matrix. Hence $AW_2(3)$ and $AW_2^{-1}(3)$ can be factorized as follows:

$$
AW_2(3) = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1
\end{bmatrix} \times
$$

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1\\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix} \times
$$

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{bmatrix},
$$

$$
AW_2^{-1}(3) = \frac{1}{4} \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1
\end{bmatrix} \times
$$

$$
\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1\\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1
\end{bmatrix} \times
$$

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}.
$$

The corresponding fast flow diagrams for calculation of forward and inverse Arithmetic-Walsh transform matrices $AW_2(3)$ and $AW_2^{-1}(3)$ are shown in Fig. 1 and Fig. 2, accordingly. In both figures, the solid lines and dotted lines represent addition and subtraction respectively.

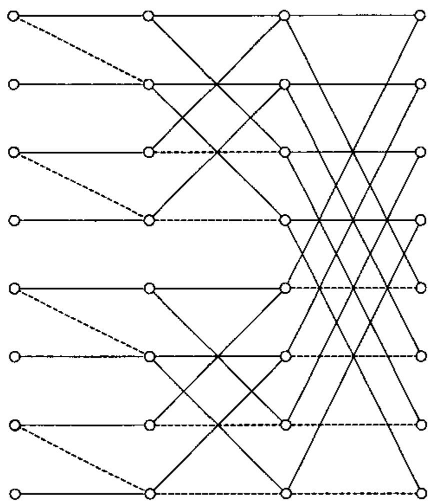

Figure 1. Fast butterfly diagram for forward Arithmetic-Walsh transform, $n = 3$ and $r = 2$.

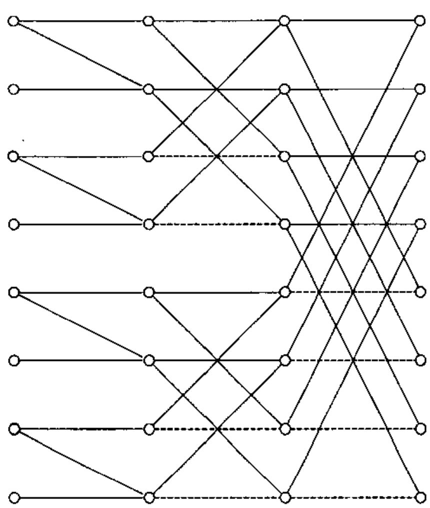

Figure 2. Fast butterfly diagram for inverse Arithmetic-Walsh transform, $n = 3$ and $r = 2$.

4. Conclusion

A novel hybrid Arithmetic-Walsh transform has been introduced in this article. By showing general recursive equations and their corresponding forward and inverse fast transforms it is obvious that the introduced transform can be also efficiently calculated for discrete functions using presented butterfly diagrams through Arithmetic-Walsh Spectral Decision Diagrams in a manner similar to the usage of the known Walsh and Arithmetic Spectral Decision Diagrams [4-6]. Hence the new transform can have efficient calculations of its spectra and their applications in CAD for VLSI Digital Signal Processors as well as in the other areas where the standard Walsh and Arithmetic transforms are currently applied [1-8]. In particular it would be interesting to use the introduced transform for analysis, synthesis and calculation of properties of discrete functions that represent internal high-level models of digital signal processing elements inside the CAD tools.

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