Claim 1: The Regular languages are closed under reverse.

Proof: Given a regular language A, show that the language A^R is regular.

Since A is regular, there is a DFA $M = \langle Q, \Sigma, \delta, q_0, F \rangle$ that recognizes it. From this construct an NFA M' that recognizes A^R.

Construction: $M' = \langle Q \cup \{q_0'\}, \Sigma, \delta', q_0', \{q_0\} \rangle$ where

$$
\begin{array}{rcllr}
\delta'(q_0', \epsilon) & = & F & & (\delta'.1) \\
\delta'(q_0', a) & = & \emptyset & \text{all } a \in \Sigma & (\delta'.2) \\
\delta'(p, a) & = & \{q | \delta(q, a) = p\} & \text{all } q \in Q, a \in \Sigma & (\delta'.3)
\end{array}
$$

Claim 2: $L(M') = A^R$. We prove this by showing (**2.1**) $w \in L(M) \rightarrow w^R \in L(M')$ and (**2.2**) $w^R \in L(M') \rightarrow w \in L(M)$ (or equivalently $w \in L(M') \rightarrow w^R \in L(M)$).

Proof of 2.1: Since $w \in L(M)$ we know that $w = w_1 w_2 \dots w_n$ and there exists states $r_0, r_1, \dots, r_n$ such that $r_0 = q_0$, $r_n \in F$, and $\forall i, 0 < i \leq n, r_i = \delta(r_{i-1}, w_i)$.

In this case M' will accept w^R, which will be rewritten equivalently as $\epsilon w_n w_{n-1} \dots w_1$, with the state sequence $q_0', r_n, r_{n-1}, \dots, r_1$. Note that q_0' and $r_1 = q_0$ are initial and final states for M', so to complete the argument that w^R is accepted we only need to show each transition is valid for M'.

The first transition satisfies $r_n \in \delta'(q_0', \epsilon)$ since by $(\delta'.1)$ this reduces to the previously established $r_n \in F$.

The remainder of the transitions are of the form: $r_{i-1} \in \delta'(r_i, w_i)$. By $(\delta'.3)$ this becomes $r_{i-1} \in \{q | \delta(q, w_i) = r_i\}$. This follows immediately from $\delta(r_{i-1}, w_i) = r_i$ which was established by $w \in L(M)$.

Proof of 2.2: Since $w \in L(M')$ we know that $w = w_1 w_2 \dots w_n$ and there are states $r_0, r_1, \dots, r_n$ such that $r_0 = q_0'$, $r_n \in \{q_0\}$, and $r_{i+1} \in \delta'(r_i, w_{i+1})$.

Furthermore, since clauses $(\delta'.1)$ and $(\delta'.2)$ define all transitions on q_0', we know that $w_1 = \epsilon$ and $r_1 \in F$. Since all other transitions are defined by clause $(\delta'.3)$ we know that states $r_1, r_2, \dots, r_n$ are in Q, the state space of the DFA M.

We need to show that $w^R \in L(M)$. We will do this by showing that M accepts $w_n w_{n-1} \dots w_2$ with the state sequence $r_n, r_{n-1}, \dots, r_1$. First note that r_n is q_0 and $r_1 \in F$. It remains to show that $r_{i-1} = \delta(r_i, w_i)$.

Since $w \in L(M')$, we know that $r_i \in \delta'(r_{i-1}, w_i)$. That is $r_i \in \{q | \delta(q, w_i) = r_{i-1}\}$. So, $\delta(r_i, w_i) = r_{i-1}$ as required.